Robust feedback switching control

Huyên PHAM*

*University Paris Diderot, LPMA

Based on joint works with
Erhan BAYRAKTAR, University of Michigan
Andrea COSSO, University Paris Diderot

USC, April 20, 2015
Switching control

- **Switching control**: sequence of interventions \((\tau_n)_n\) that occur at random times \((\tau_n)_n\) due to switching costs, and naturally arises in investment problems with fixed transaction costs or in real options.

- **Standard approach**:
  - **open-loop** (\(\neq\) closed-loop) control
  - give the evolution for the controlled state process, with *assigned* drift and diffusion coefficients.

- In practice, the coefficients are obtained through estimation procedures and are unlikely to coincide with the *real* coefficients.

- **Robust approach**: switching control problem *robust* to a misspecification of the model for the controlled state process.
We formulate the problem as a game: switcher vs nature (model uncertainty).

We consider the two-step optimization problem

$$\sup_{\alpha} \left( \inf_{\nu} J(\alpha, \nu) \right).$$

What definition for the switching control $\alpha$ and for $\nu$?

- Tractable for deriving typically DPP
- Consistent with modeling concern
Feedback formulation

- **Elliott-Kalton formulation** (Fleming-Souganidis 89):
  - $\alpha$ non-anticipative strategy and $\nu$ open-loop control, i.e. the switcher knows the current and past choices made by nature $\rightarrow$ Suitable for proving dynamic programming principle (DPP)
  - In practice, the switcher only knows the evolution of the state process.

- **Feedback formulation**
  - $\alpha$ feedback switching control (closed-loop control) $\iff$ feedback formulation of the switching control problem.
  - $\nu$ open-loop control (nature is aware of the all information at disposal) $\leftrightarrow$ Knightian uncertainty
  $\rightarrow$ zero-sum control/control game but **not symmetric**
Outline

1. Model setup
2. Stochastic Perron’s method and Hamilton-Jacobi-Bellman-Isaacs equation
3. Ergodicity
Outline

1. Model setup

2. Stochastic Perron's method and Hamilton-Jacobi-Bellman-Isaacs equation

3. Ergodicity
Robust feedback switching system

- Fixed \((\Omega, \mathcal{F}, \mathbb{P}), T > 0\), and \(W\) a \(d\)-dimensional Brownian motion.

For any \((s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m\), consider the system on \(\mathbb{R}^d \times \mathbb{I}_m\), with \(\mathbb{I}_m = \{1, \ldots, m\}\) the set of regimes:

\[
\begin{align*}
X_t &= x + \int_s^t b(X_r, l_r, \nu_r)dr + \int_s^t \sigma(X_r, l_r, \nu_r)dW_r, \quad s \leq t \leq T, \\
l_t &= i1\{s \leq t < \tau_0(X_., l_.)\} \\
&\quad + \sum_{n \in \mathbb{N}} l_n(X_., l_-)1\{\tau_n(X_., l_-) \leq t < \tau_{n+1}(X_., l_-)\}, \quad s \leq t < T, \\
l_{s-} &= l_s, \quad l_T = l_{T-}.
\end{align*}
\]

\(\nu: [s, T] \times \Omega \rightarrow U\) is an open-loop control adapted to a filtration \(\mathbb{F}^s = (\mathcal{F}^s_t t \geq s)\) satisfying the usual conditions.

- \(U\) compact metric space.

\(\mathcal{U}_{s, s}\) : class of all open-loop controls starting at \(s\).
Feedback switching controls

- $\mathcal{L}([s, T]; \mathbb{I}_m)$ space of càglàd paths valued in $\mathbb{I}_m$.
- $\mathbb{B}^s = (\mathcal{B}^s_t)_{t \in [s, T]}$ natural filtration of $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$.
- $\mathcal{T}^s$ family of all $\mathbb{B}^s$-stopping times valued in $[s, T]$.

▸ Feedback switching control $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$ where:
  - Switching times: $\tau_n \in \mathcal{T}^s$ and
    
    $$s \leq \tau_0 \leq \cdots \leq \tau_n \leq \cdots \leq T.$$  
  - Interventions: $\iota_n: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \to \mathbb{I}_m$ is $\mathcal{B}_{\tau_n}$-measurable, for any $n \in \mathbb{N}$.

▸ $\mathcal{A}_{s,s}$ : class of all feedback switching controls starting at $s$. 

Huyên PHAM
Robust feedback switching control
Existence and uniqueness result

(H1) $b$ and $\sigma$ jointly continuous on $\mathbb{R}^d \times I_m \times U$ and

$$|b(x, i, u) - b(x', i, u)| + \|\sigma(x, i, u) - \sigma(x', i, u)\| \leq L|x - x'|.$$
Value function of robust switching control problem

Feedback control/open-loop control game:

\[ V(s, x, i) := \sup_{\alpha \in \mathcal{A}_{s, s}} \inf_{\nu \in \mathcal{U}_{s, s}} J(s, x, i; \alpha, \nu), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m, \]

with

\[ J(s, x, i; \alpha, \nu) := \mathbb{E} \left[ \int_s^T f(X^s_r, x, i; \alpha, \nu, I^s_r, x, i; \alpha, \nu, \nu_r) dr \right. \]
\[ \left. + g(X^s_T, x, i; \alpha, \nu, I^s_T, x, i; \alpha, \nu) \right. \]
\[ \left. - \sum_{n \in \mathbb{N}} c(X^{s, \tau_n}_n, x, i; \alpha, \nu, I^{s, \tau_n}_n, x, i; \alpha, \nu, I^{s, \tau_n}_n, x, i; \alpha, \nu) 1\{s \leq \tau_n < T\} \right], \]

where \( \tau^n \) stands for \( \tau^n(X^{s, x, i; \alpha, \nu}, I^{-}_{s, x, i; \alpha, \nu}) \).
Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation

\[
\begin{aligned}
&\min \left\{ -\frac{\partial V}{\partial t}(s, x, i) - \inf_{u \in U} \left[ \mathcal{L}^{i,u} V(s, x, i) + f(x, i, u) \right],
\right.

\left. V(s, x, i) - \max_{j \neq i} \left[ V(s, x, j) - c(x, i, j) \right] \right\} = 0, \\
&V(T, x, i) = g(x, i), \quad (x, i) \in \mathbb{R}^d \times \mathbb{I}_m,
\end{aligned}
\]

where

\[
\mathcal{L}^{i,u} V(s, x, i) = b(x, i, u).D_x V(s, x, i) + \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top(x, i, u) D_x^2 V(s, x, i) \right].
\]

First aim: prove that $V$ is a viscosity solution to the dynamic programming HJBI equation:

- by stochastic Perron method: avoiding the direct proof of Dynamic Programming Principle (DPP)
Outline

1 Model setup

2 Stochastic Perron’s method and Hamilton-Jacobi-Bellman-Isaacs equation

3 Ergodicity
Stochastic Perron : main idea

Developed in a series of papers by Bayraktar and Sirbu

- Define **stochastic sub and super-solutions** as functions that satisfy (roughly) half of the DPP

  ▶ with these definitions, sub and super-solutions envelope the value function

- Consider sup of sub-solutions and inf of super-solutions (Perron) :

  $v^- := \sup \text{ of sub-solutions} \leq V \leq v^+ := \inf \text{ of super-solutions}$

  ▶ Show that $v^-$ is a viscosity super-solution and $v^+$ is a viscosity sub-solution.

- Comparison principle $\rightarrow$

  \[ v^- = V = v^+ \]

  is the unique continuous viscosity solution.

and (as a byproduct) $V$ satisfies the DPP
• Stochastic semi-solutions have to be carefully defined (depending on the control problem) → constructive proof for the existence of a viscosity solution comparing with the value function ($\neq$ from Perron’s method)
  - linear, control, optimal stopping problems (Bayraktar-Sirbu, 12, 13)
  - game problems: delicate issues, no symmetry of players.
    recent work by Sirbu (2014)
Stochastic semisolutions

Definition (Stochastic subsolutions $V^-$)

A stochastic subsolution to the HJBI equation if:
1. $v$ is continuous, $v(T, x, i) \leq g(x, i)$ for any $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$, and
2. \[ \sup_{(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|v(s, x, i)|}{1 + |x|^q} < \infty, \text{ for some } q \geq 1. \]

Half-DPP property. For any $s \in [0, T]$ and $\tau, \rho \in T^s$ with $\tau \leq \rho \leq T$, there exists $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_s, \tau^+$ such that, for any $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_s, s$, $\nu \in \mathcal{U}_s, s$, and $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$, we have

\[
 v(\tau', X_{\tau'}, I_{\tau'}) \leq \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, l_t, \nu_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\
 \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n}) 1\{\tau' \leq \tilde{\tau}'_n < \rho'\} \right| \mathcal{F}^s_{\tau'} \]

with the shorthands $X = X^{s, x, i; \alpha \otimes \tau, \tilde{\alpha}, \nu}$, $I = I^{s, x, i; \alpha \otimes \tau, \tilde{\alpha}, \nu}$.

The set of stochastic supersolutions $V^+$ is defined similarly.
Stochastic Perron’s method : assumptions

(H2)

(i) $g, f, c$ are jointly continuous on their domains.

(ii) $c$ is nonnegative.

(iii) $g, f, c$ satisfy the polynomial growth condition:

$$|g(x, i)| + |f(x, i, u)| + |c(x, i, j)| \leq M(1 + |x|^p),$$

$\forall x \in \mathbb{R}^d, i, j \in I_m, u \in U$, for some positive constants $M$ and $p \geq 1$.

(iv) $g$ satisfies

$$g(x, i) \geq \max_{j \neq i} [g(x, j) - c(x, i, j)],$$

for any $x \in \mathbb{R}^d$ and $i \in I_m$. 
Stochastic Perron’s method

Proposition

Let Assumptions (H1) and (H2) hold.

(i) \( \mathcal{V}^- \neq \emptyset \) and \( \mathcal{V}^+ \neq \emptyset \).

(ii) \( \sup_{\nu \in \mathcal{V}^-} \nu =: \nu^- \leq V \leq \nu^+ := \inf_{\nu \in \mathcal{V}^+} \nu \).

(iii) If \( \nu^1, \nu^2 \in \mathcal{V}^- \) then \( \nu := \nu^1 \lor \nu^2 \in \mathcal{V}^- \). Moreover, there exists a nondecreasing sequence \( (\nu_n)_n \subset \mathcal{V}^- \) such that \( \nu_n \uparrow \nu^- \).

(iv) If \( \nu^1, \nu^2 \in \mathcal{V}^+ \) then \( \nu := \nu^1 \land \nu^2 \in \mathcal{V}^+ \). Moreover, there exists a nonincreasing sequence \( (\nu_n)_n \subset \mathcal{V}^+ \) such that \( \nu_n \downarrow \nu^+ \).

Theorem [Stochastic Perron’s method]

Let Assumptions (H1) and (H2) hold. Then, \( \nu^- \) is a viscosity supersolution to the HJBI equation and \( \nu^+ \) is a viscosity subsolution to the HJB equation.
Comparison principle

\[(H3)\] \[c\] satisfies the no free loop property: for any sequence of indices \(i_1, \ldots, i_k \in \mathbb{I}_m\), with \(k \in \mathbb{N} \setminus \{0, 1, 2\}\), \(i_1 = i_k\), and \(\text{card}\{i_1, \ldots, i_k\} = k - 1\), we have

\[c(x, i_1, i_2) + c(x, i_2, i_3) + \cdots + c(x, i_{k-1}, i_k) + c(x, i_k, i_1) > 0.\]

We also assume: \(c(x, i, i) = 0, \forall (x, i) \in \mathbb{R}^d \times \mathbb{I}_m\).

**Theorem [Comparison principle]**

Let Assumptions \((H1), (H2), (H3)\) hold and consider a viscosity subsolution \(u\) (resp. supersolution \(v\)) to the HJB equation. Suppose that, for some \(q \geq 1\),

\[
\sup_{(t,x,i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|u(t, x, i)| + |v(t, x, i)|}{1 + |x|^q} < \infty.
\]

Then, \(u \leq v\) on \([0, T] \times \mathbb{R}^d \times \mathbb{I}_m\).
Dynamic programming and viscosity properties

**Theorem**

Let Assumptions *(H1), (H2), (H3)* hold. Then, the value function $V$ is the unique viscosity solution to the HJB equation and satisfies the dynamic programming principle: for any $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ and $\rho \in \mathbb{T}^s$,

$$V(s, x, i) = \sup_{\alpha \in \mathcal{A}_s, s} \inf_{\nu \in \mathcal{U}_s, s} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, l_t, \nu_t) dt + V(\rho', X_{\rho'}, I_{\rho'}) \right]$$

$$- \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I(\tau'_n)^-; I_{\tau'_n}) \mathbb{1}_{\{s \leq \tau'_n < \rho'\}},$$

with the shorthands $X = X^{s, x, i; \alpha, \nu}$, $l = l^{s, x, i; \alpha, \nu}$, $\rho = \rho(X, l_-)$, $\tau'_n = \tau_n(X, l_-)$, and $\nu'_t = \nu(t, X, l_-)$.
Comparison with the Elliott-Kalton formulation

• In general: $V \leq V^{Kalton}$.
  - If comparison principle holds, then $V = V^{Kalton}$ unique solution to the HJBI equation

• One can find a counterexample with $c \equiv 0$ (no-free loop property is not satisfied) such that
  - $V$ is solution to the lower Bellman Isaacs equation
  - $V^{Kalton}$ is solution to the upper Bellman Isaacs equation
  - $V < V^{Kalton}$: the Isaacs equation does not hold.
Outline

1 Model setup

2 Stochastic Perron’s method and Hamilton-Jacobi-Bellman-Isaacs equation

3 Ergodicity

Huyêń PHAM

Robust feedback switching control
Problem

**Forward** parabolic system of variational inequalities:

\[
\begin{aligned}
\min \left\{ \frac{\partial V}{\partial T} - \inf_{u \in U} \left[ L^{i,u} V + f(x, i, u) \right] , \\
V(T, x, i) - \max_{j \neq i} \left[ V(T, x, j) - c(x, i, j) \right] \right\} &= 0 , \\
V(0, x, i) &= g(x, i),
\end{aligned}
\]

- Long time asymptotics of \( V(T, \cdot, \cdot) \) as \( T \to \infty \):

  - Stationary solution of robust feedback switching control
  - Literature on ergodic stochastic control: Bensoussan, Frehse (92); Arisawa, P.L. Lions (98), Kaise and Sheu (06), Barles, Porretta and Tchamba (10), Nagai (12), Ichihara and Sheu (13), Hu, Madec and Richou (13), Cosso, Fuhrman and P. (14), ... **but often under non degeneracy condition and/or regularity of value function and very few on games!**
Some heuristics and principles

- We expect to prove (under suitable conditions) that
  \[
  \frac{V(T, x, i)}{T} \to \lambda \text{ (const. independent of } x, i) \quad \text{as } T \to \infty.
  \]

- **Tauberian Meta theorem**: ergodic \sim infinite horizon with vanishing discount factor, i.e.
  \[
  \lim_{T \to \infty} \frac{V(T, \cdot)}{T} = \lim_{\beta \to 0} \beta V^\beta
  \]
  where
  \[
  V^\beta(x, i) = \sup_{\alpha \in A_{0,0}} \inf_{\nu \in \mathcal{U}_{0,0}} \mathbb{E} \left[ \int_0^\infty e^{-\beta t} f(X_t^{x, i; \alpha, \nu}, I_t^{x, i; \alpha, \nu}, \nu_t) dt \right] \\
  - \sum_{n \in \mathbb{N}} e^{-\beta \tau_n} c(X_{\tau_n}^{x, i; \alpha, u}, I_{\tau_n}^{x, i; \alpha, \nu}, I_{\tau_n}^{x, i; \alpha, u}) 1\{\tau_n < \infty\}
  \]

\leftrightarrow **Elliptic** system of variational inequalities:

\[
\min \left\{ \beta V^\beta - \inf_{u \in \mathcal{U}} [L^{i,u} V^\beta + f(x, i, u)] ; V^\beta(x, i) - \max_{j \neq i} [V^\beta(x, j) - c(x, i, j)] \right\} = 0.
\]
Ergodic system of variational inequalities

- Formally, by setting $V(T, x, i) \sim \lambda T + \phi(x, i)$ as $T \to \infty$, we get the ergodic HJBI equation:

$$\min \left\{ \lambda - \inf_{u \in U} \left[ \mathcal{L}^i, u \phi + f(x, i, u) \right], \phi(x, i) - \max_{j \neq i} \left[ \phi(x, j) - c(x, i, j) \right] \right\} = 0.$$  

\[\blacktriangleright\] The pair $(\lambda, \phi)$ is the unknown.

- Aim:
  - Prove existence (and uniqueness) of a solution to the ergodic HJBI
  - Show:

$$\lim_{T \to \infty} \frac{V(T, x, i)}{T} = \lambda = \lim_{\beta \to 0} \beta V^\beta(x, i).$$
Main issues for asymptotic analysis

- Prove equicontinuity of the family \((V^\beta)_\beta\): for all \(\beta > 0\),

\[
|V^\beta(x, i) - V^\beta(x', i)| \leq C|x - x'|,
\]

\[
\beta|V^\beta(x, i)| \leq C(1 + |x|), \quad \forall (x, i).
\]

- by PDE methods from the elliptic HJBI system?
- from the robust feedback switching control representation, which would rely on an estimate of the form:

\[
\sup_{\alpha \in \mathcal{A}_{0,0}, \nu \in \mathcal{U}_{0,0}} \mathbb{E}\left|X^x_{t,i;\alpha,\nu} - X^{x'}_{t,i;\alpha,\nu}\right| \leq C_t|x - x'|, \quad \forall x, x', i.
\]

Not clear due to the feedback form of the switching control!
Randomization of the control

Following idea of Kharroubi and P. (13):

\[
\begin{align*}
X_t &= x + \int_0^t b(X_s, I_s, \Gamma_s)ds + \int_0^t \sigma(X_s, I_s, \Gamma_s)dW_s, \\
l_t &= i + \int_0^t \int_{I_m} (j - I_{s^-})\pi(ds, dj), \\
\Gamma_t &= u + \int_0^t \int_U (u' - \Gamma_{s^-})\mu(ds, du'), 
\end{align*}
\]

\(\pi\) Poisson random measure on \(\mathbb{R}_+ \times I_m\), \(\mu\) Poisson random measure on \(\mathbb{R}_+ \times U\). \(W, \pi, \text{and } \mu\) are independent.

\( (X^{x,i,u}, I^i, \Gamma^u) \) exogenous (uncontrolled) Markov process
Control of intensity measures:

- $\Xi$ (resp. $\mathcal{V}$) class of essentially bounded predictable maps
- $\xi: [0, \infty) \times \Omega \times \mathbb{I}_m \rightarrow (0, \infty)$ (resp. $\nu: [0, \infty) \times \Omega \times \mathbb{U} \rightarrow [1, \infty)$)

$$
\frac{dP_{\xi,\nu}}{dp} \bigg|_{\mathcal{F}_T} = \mathcal{E}_T \left( \int_0^T \int_{\mathbb{I}_m} (\xi_t(j) - 1) \tilde{\pi} (dt, dj) \right) \cdot \mathcal{E}_T \left( \int_0^T \int_{\mathbb{U}} (\nu_t(u') - 1) \tilde{\mu} (dt, du') \right)
$$

- **Under $P^{\xi,\nu}$:**
  - $W$ remains a Brownian motion.
  - $P$-compensator $\vartheta_{\pi} (di) dt$ of $\pi \rightarrow \xi_t(i) \vartheta_{\pi} (di) dt$.
  - $P$-compensator $\vartheta_{\mu} (du) dt$ of $\mu \rightarrow \nu_t(u) \vartheta_{\mu} (du) dt$.

$\rightarrow$ Easy to derive moment and Lipschitz estimates on $X^{x,i,u}$ under $P^{\xi,\nu}$!
Dual robust switching control

\[ v^\beta(x, i, u) := \sup_{\xi \in \Xi} \inf_{\nu \in \mathcal{V}} \mathbb{E}^{\xi, \nu} \left[ \int_0^\infty e^{-\beta t} f(X_t^{x, i, u}, I_t^i, \Gamma_t^u) dt \right. \]

\[ - \left. \int_0^\infty \int_{\mathbb{I}_m} e^{-\beta t} c(X_t^{-x, i, u}, I_t^{-i}, j) \pi(dt, dj) \right], \]

for all \((x, i, u) \in \mathbb{R}^d \times \mathbb{I}_m \times U\).

The dual problem is a symmetric game: control vs control (as in T. Pham, J. Zhang 14)

Theorem

For any \(\beta > 0\) and \((x, i) \in \mathbb{R}^d \times \mathbb{I}_m\),

\[ v^\beta(x, i, u) = v^\beta(x, i, u'), \quad \forall u, u' \in U \]

and for any \(u \in U\),

\[ V^\beta(x, i) = v^\beta(x, i, u), \quad \forall (x, i) \in \mathbb{R}^d \times \mathbb{I}_m. \]
Ergodicity under dissipativity condition

- **Dissipativity condition (DC)**: for all $x, x' \in \mathbb{R}^d$, $i \in \mathbb{I}_m$, $u \in U$,

\[
(x - x').(b(x, i, u) - b(x', i, u)) + \frac{1}{2} \|\sigma(x, i, u) - \sigma(x', i, u)\|^2 \\
\leq -\gamma |x - x'|^2
\]

for some constant $\gamma > 0$.

\[
\sup_{\xi, \nu} \mathbb{E}^\xi,\nu [ |X_t^{x, i, u} - X_t^{x', i, u}|^2 ] \leq e^{-2\gamma t} |x - x'|^2
\]

\[
\sup_{t \geq 0} \sup_{\xi, \nu} \mathbb{E}^\xi,\nu |X_t^{x, i, u}| \leq C(1 + |x|).
\]
Main steps of proof for existence to ergodic system

- **Equicontinuity:**
  
  \[ |V^\beta(x, i) - V^\beta(x', i)| \leq \sup_{\xi \in \Xi, \nu \in \mathcal{V}} \mathbb{E}^{\xi, \nu} \left[ \int_0^\infty e^{-\beta t} \left| f(X_t^{x,i,u}, l_t^i, \Gamma_t^u) - f(X_t^{x',i,u}, l_t^i, \Gamma_t^u) \right| dt \right] \]
  
  \[ \leq L|x - x'| \int_0^\infty e^{-(\beta + \gamma)t} dt = \frac{L}{\beta + \gamma} |x - x'| \leq \frac{L}{\gamma} |x - x'|. \]

- **Convergence of** \(V^\beta\). Define
  
  \[ \lambda_i^\beta := \beta V^\beta(0, i), \quad \phi^\beta(x, i) := V^\beta(x, i) - V^\beta(0, i_0), \]

  By Bolzano-Weierstrass and Ascoli-Arzelà theorems, we can find a sequence \((\beta_k)_{k \in \mathbb{N}}, \) with \(\beta_k \downarrow 0^+\), such that
  
  \[ \lambda_i^{\beta_k} \xrightarrow{k \to \infty} \lambda_i, \quad \phi^{\beta_k}(\cdot, i) \xrightarrow{k \to \infty} \phi(\cdot, i). \]

  \(\triangleright\) \(\lambda := \lambda_i \) does not depend on \(i \in \mathbb{I}_m.\)
  
  Finally, stability results of viscosity solutions \(\Rightarrow\) \((\lambda, \phi)\) is a viscosity solution to the ergodic system.
A simple argument for large time convergence

Let \((\lambda, \phi)\) be a solution to the ergodic HJBI:

- \(\phi\) is the unique viscosity solution to the parabolic HJBI equation with unknown \(\psi\) and terminal condition \(\phi\):

\[
\begin{align*}
\min \left\{ -\frac{\partial \psi}{\partial t}(t, x, i) - \inf_{u \in U} \left[ \mathcal{L}^{i,u}(t, x, i) + f(x, i, u) - \lambda \right] \right. \\
\psi(t, x, i) - \max_{j \neq i} \left[ \psi(t, x, j) - c(x, i, j) \right] \right\} &= 0, \quad (t, x, i) \in [0, T) \times \mathbb{R}^d \times \mathbb{I}_m, \\
\psi(T, x, i) &= \phi(x, i),
\end{align*}
\]

- For any \(T > 0\), \(\phi(x, i)\) admits the dual game representation:

\[
\phi(x, i) = \sup_{\xi \in \Xi} \inf_{\nu \in \mathcal{V}} \mathbb{E}^{\xi, \nu} \left[ \int_0^T \left( f(X_t^{x,i,u}, l_t^i, \Gamma_t^u) - \lambda \right) dt + \phi(X_T^{x,i,u}, l_T^i) \right. \\
- \int_0^T \int_{\mathbb{I}_m} e^{-\beta t} c(X_t^{x,i,u}, l_t^i, j) \pi(dt, dj) \right]\]
Large time convergence (Ctd and end)

From the dual game representation for $V(T, .)$ :

$$|V(T, x, i) - \lambda T - \phi(x, i)|$$

$$\leq \sup_{\xi \in \Xi, \nu \in \mathcal{V}} \mathbb{E}^{\xi, \nu} \left[ |g(X_T, \nu^i_T)| + \max_j |\phi(X_T, j)| \right]$$

$$\leq C(1 + |x|^2),$$

from growth condition of $g$, $\phi$, and estimate of $X$ under dissipativity condition.

$$\Rightarrow$$

$$\frac{V(T, x, i)}{T} \to \lambda, \quad \text{as } T \to \infty.$$  

**Remark.** This probabilistic argument does not require any non degeneracy condition on $\sigma$, hence any regularity on value functions.
Concluding remarks

- Robust (model uncertainty) feedback switching control:
  - Non symmetric zero-sum control/control game
  - $\neq$ Elliott-Kalton game formulation

- Stochastic Perron method
  - HJBI equation and DPP

- Ergodicity of HJBI
  - Randomization method $\rightarrow$ dual symmetric (open loop) control/control game representation
  - No non-degeneracy condition