Degree distributions in preferential attachment random graphs and connections to urns, walks and trees

Part III: Urns, walks and trees

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(joint with E. Peköz and N. Ross)

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There once was a coupling from France

That was used to hold up some pants,

But the coupling probability is low

And there's lots of room to grow,

So it's best to keep a wide stance.

(E. Pekoz)

There once was a coupling from France

Whose bounds were so tight theorems danced,

With errors so small,

You'll publish it all,

And never again not get grants.

(L. Goldstein)

There once was a coupling from France

That wanted to help us enhance

Our grasp of an urn,

And resolve our concern

That all's just a mere game of chance.

(A. R.)

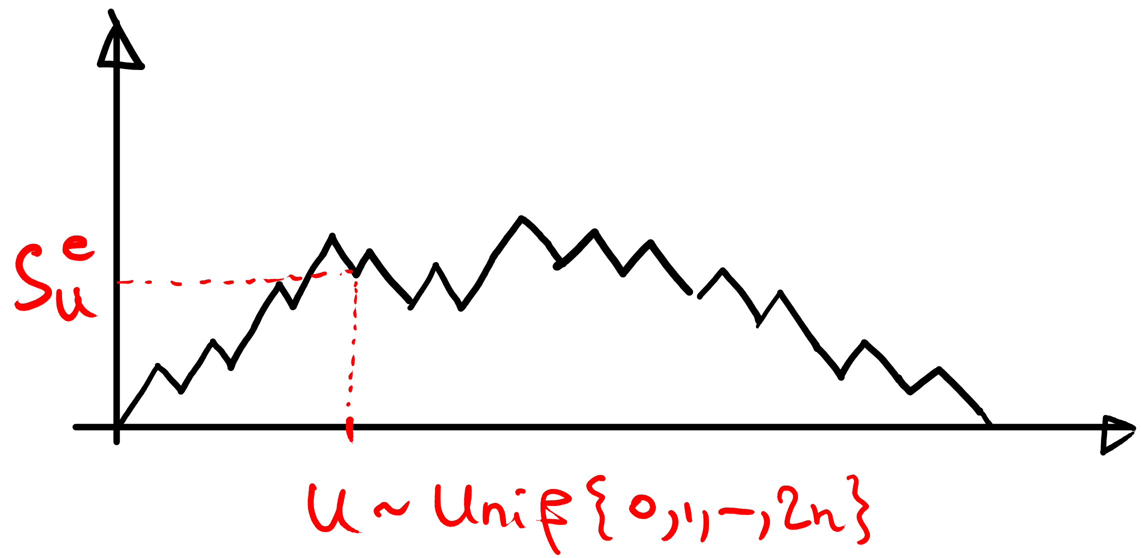
Couplings instead of

Characterisic functions?

It's called Stein's method.

(N. Ross)

Motivation Simple random walk excursion



$$\boxed{\frac{S_U^e}{n^{1/2}} \longrightarrow \chi_2}$$

(Rayleigh)

Proof Donsker's Theorem

and $B_U^e \sim \chi_2$ $\square$

Generalized Gamma distribution $GG(k,r)$

- Density $\boxed{r \cdot x^{k-1} e^{-x^r} / \Gamma(k/r)}$ $\quad k, r, x > 0$
- If $Z \sim GG(k,r)$ then $\mathbb{E} Z^r = \frac{k}{r}$ $\to$ scaling
- Special cases (up to scaling)

– half normal: $GG(1,2)$ – χ_k : $GG(k,2)$

– Rayleigh: $GG(2,2)$ – χ_k^2 : $GG(k/2, 1)$

– Weibull (k): $GG(k,k)$

$$GG(k,r) \overset{\mathcal{D}}{=} \left(\mathrm{Gamma}\left(\frac{k}{r}\right)\right)^{1/r}$$

Polya urn with immigration

$P^{\ell}_{n}(i,j)$

- start with i black balls and j white balls

classic Polya urn

- draw a ball, replace it and add another ball of same colour

immigration

- after every ℓ-th draw, add a black ball
- $P^{\ell}_{n}(i,j)$ is the no. of white balls!

Thm

If $W_n \sim P_n^{\ell}(1,j)$, $\mu_n^{\ell+1} = \frac{\ell+1}{j} \mathbb{E} W_n^{\ell+1}$, then

$$\sup_{x \geq 0} \left| \mathbb{P}[W_n/\mu_n \leq x] - \mathbb{P}[Z \leq x] \right| \leq \frac{C}{n^{\ell/\ell+1}},$$

where $Z \sim GG(j, \ell+1)$

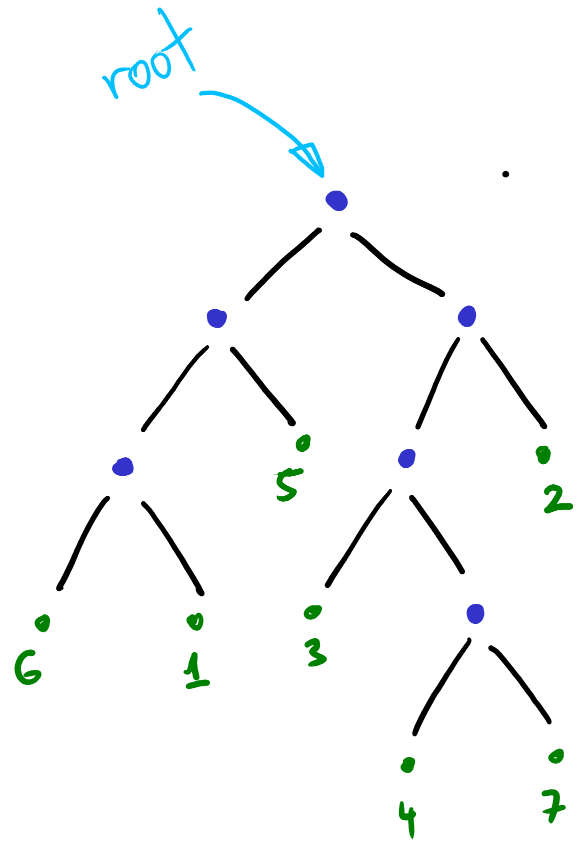
- Our interest now : $P^1_{n-k}(0, 2k-1)$

- Note : $P^1_{n-k}(0, 2k-1) = P^1_{n-k-1}(1, 2k)$

- Limits are $\boxed{GG(2k, 2) = \chi_k}$

- $\ell = 1$: general Polya urn $\begin{array}{ccc} & b & w \\ b & 2 & 0 \\ w & 1 & 1 \end{array}$

Rooted, decorated, (full) binary plane trees



$n = 7$ leafs

$n-1 = 6$ internal nodes

There are $n!\, C_{n-1}$ such trees

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{"Catalan numbers"}$$

Rémy's algorithm (1985)

A way to choose uniformly at random among such trees with n leafs : T_n^b

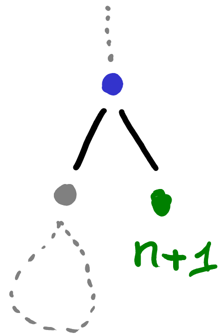
- To obtain an undecorated tree, just remove the labels.

- $\boxed{\text{Pick leaf at random in the undecorated tree} \Leftrightarrow \text{pick leaf } 1.}$

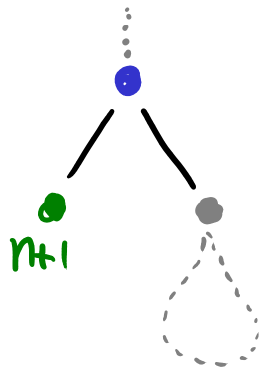
Algorithm : $\boxed{T_n^b \longrightarrow T_{n+1}^b}$

(1) Pick node at random

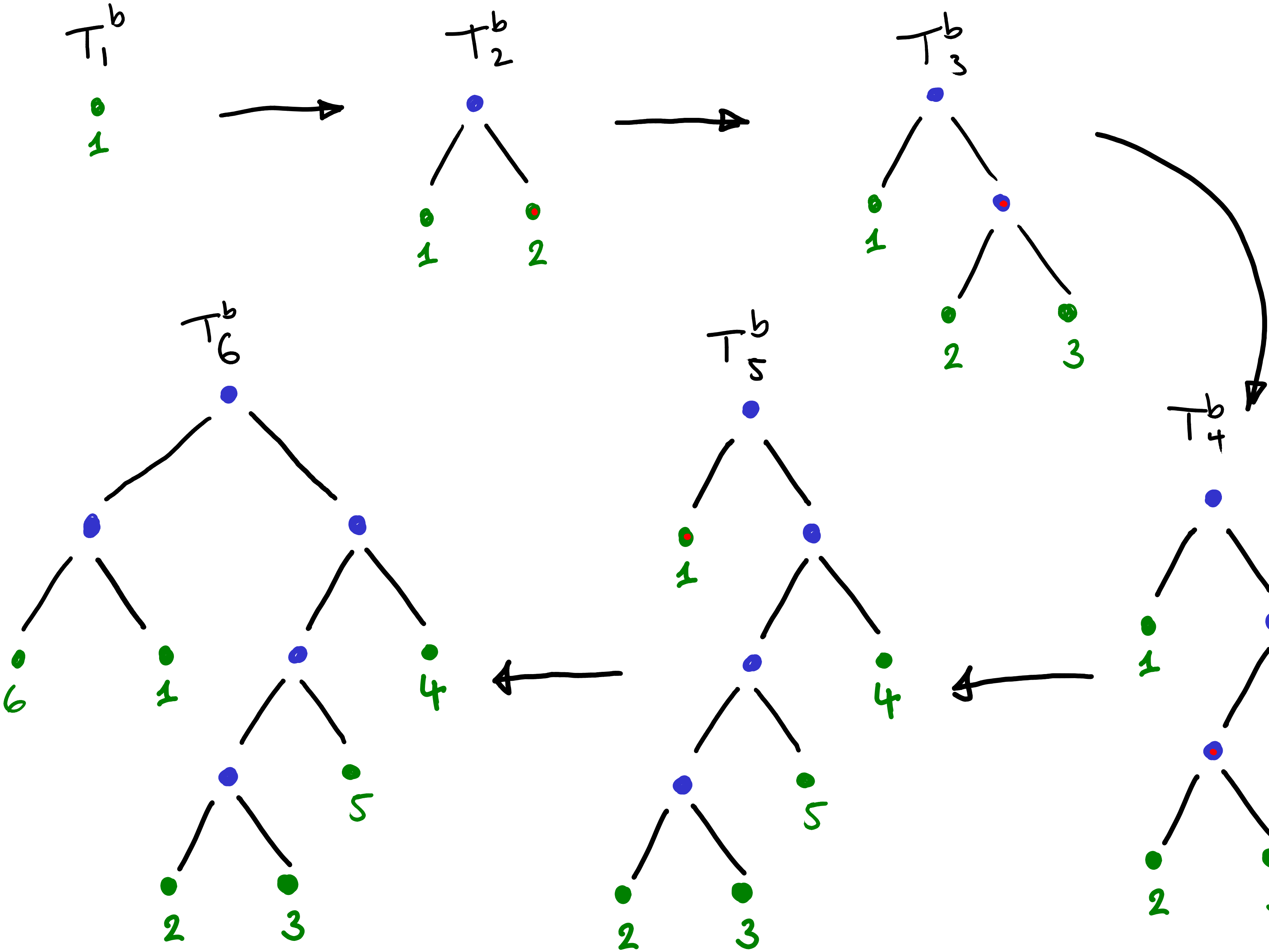
(2) Insert a "cherry" like this



or like this



(probability $\frac{1}{2}$ each choice)



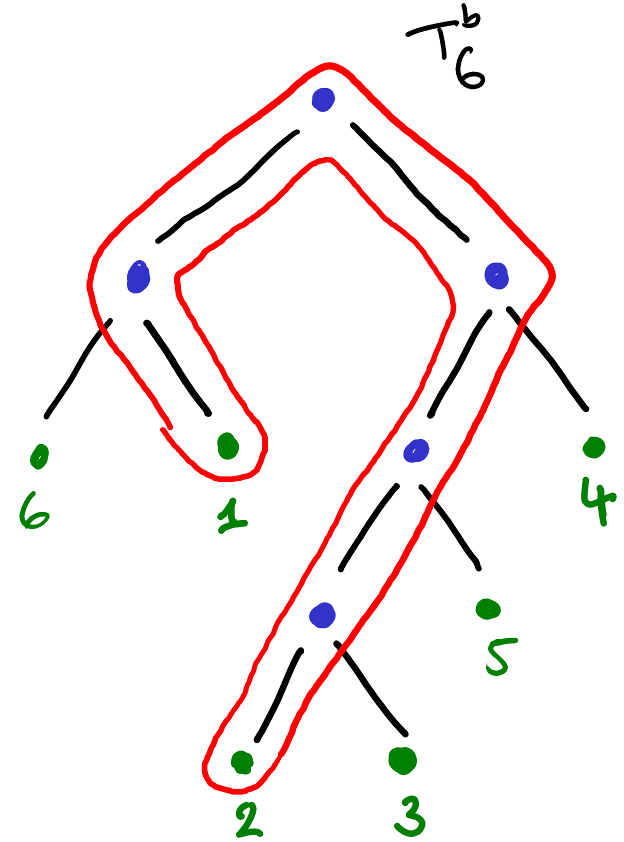
T_1^b
T_2^b
T_3^b
T_4^b
T_5^b
T_6^b
1
1 2
1 2 3
1 2 3 4
1 2 3 4 5
6 1 2 3 5 4

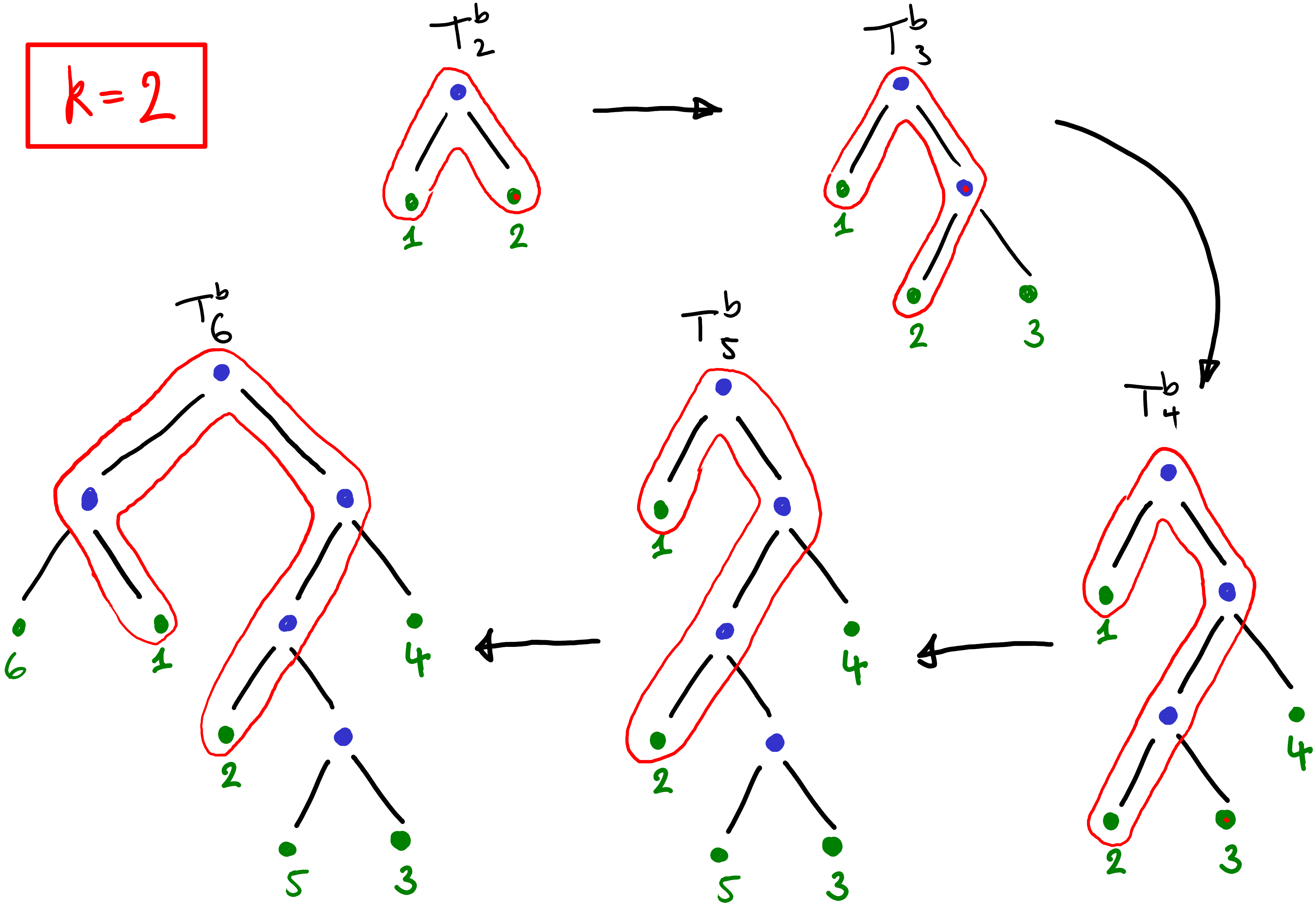
- The resulting tree is choosen uniformly because
    - every tree can be obtained in excatly one way,
    - choices are made with equal probabilities,
    - the number of choices at stage n is $(2n-1) \times 2$.

Spanning subtree

- Pick k distinct leafs (nodes) at random

- Consider the # of nodes in the subtree spanned by the k leafs (nodes) and the root





k=2
T^b_2
1
2
T^b_3
1
2
3
T^b_4
1
2
3
4
T^b_5
1
2
3
4
5
T^b_6
6
1
2
3
4
5

$$\Rightarrow \quad k\text{-leaf-span}(T_n^b) \sim P^1_{n-k}(0, 2k-1)$$

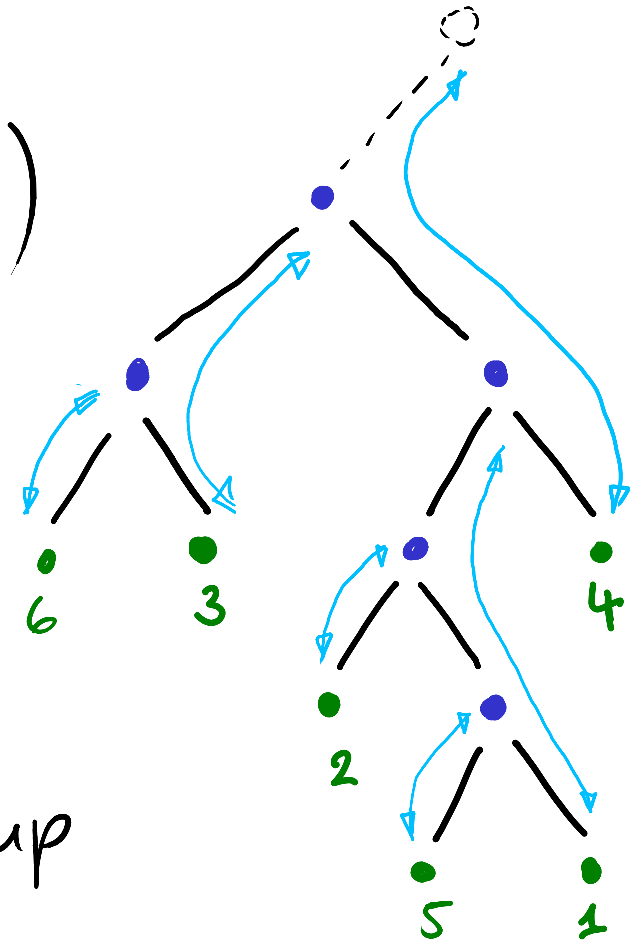
$$\Rightarrow \quad \boxed{\frac{k\text{-span}(T_n^b)}{\mu_n} \longrightarrow \chi_k \qquad \left(\text{rate } \frac{1}{n^{1/2}}\right)}$$

- Whenever something is "close" to k-leaf-span of T_n^b, we can hope for similar rates.

Spanning subtree of 1 node in T_n^b

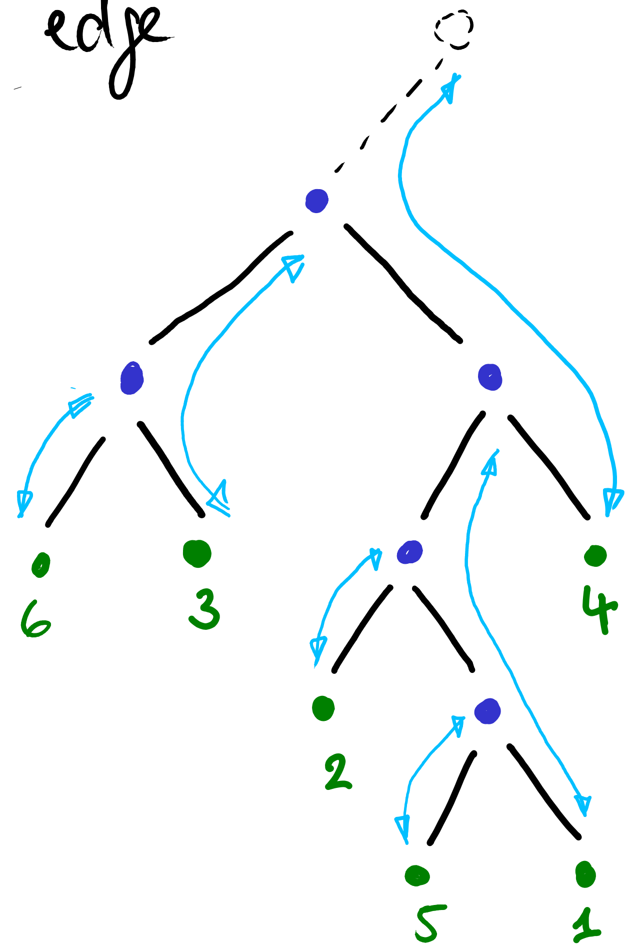
- We can pair up internal nodes and leafs (if we "plant" the tree)

- Each leaf is paired with the parental node of the first "right turn" on the way up to the root.



- Each edge in the way up to the root is a left/right edge with prob $\frac{1}{2}$.

- The directions of the edges along the path are independent



- $N_{n-1} \sim P^1_{n-1}(0,1)$, $Y \sim Ge(1/2)$ (started in 0)

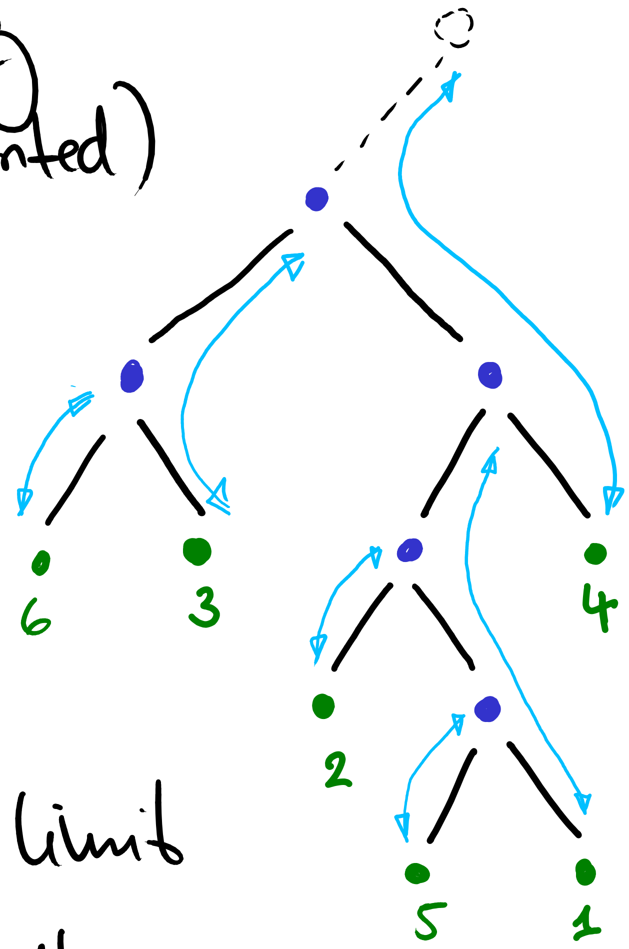
- No. of vertices in 1-spanning subtree of random node (planted)

$$\boxed{X_n = (N_{n-1} - Y) \vee 0}$$

- Unplanted: $\boxed{\mathcal{L}(X_n \mid X_n > 0)}$

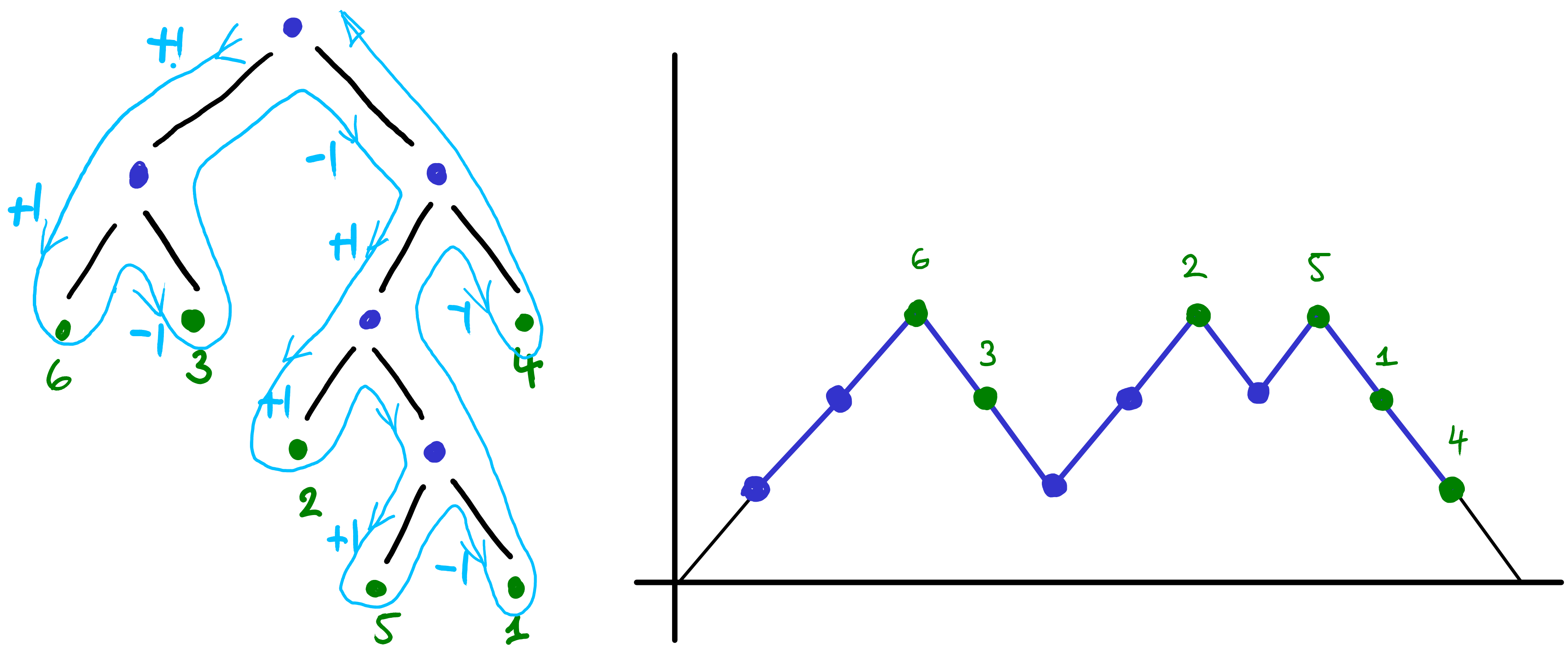
$\Rightarrow$ rates of convergence to χ_1 limit

- We can handle plane tree as well.

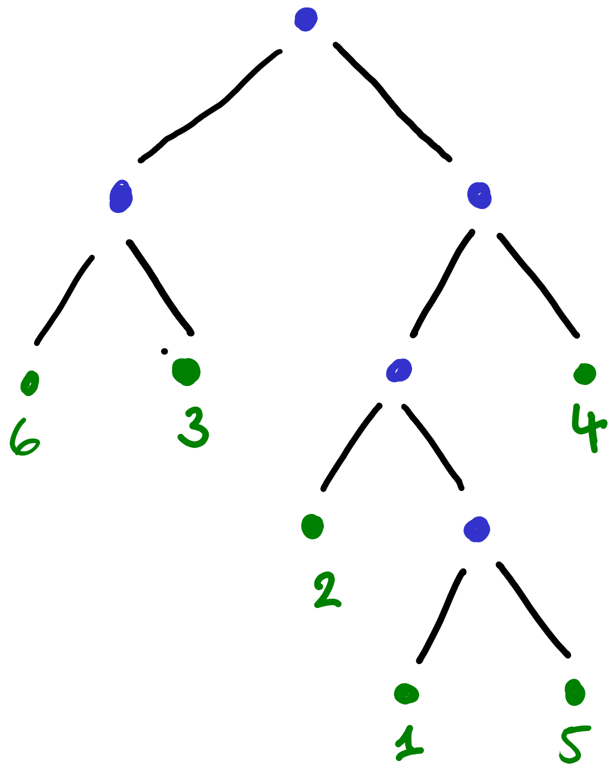


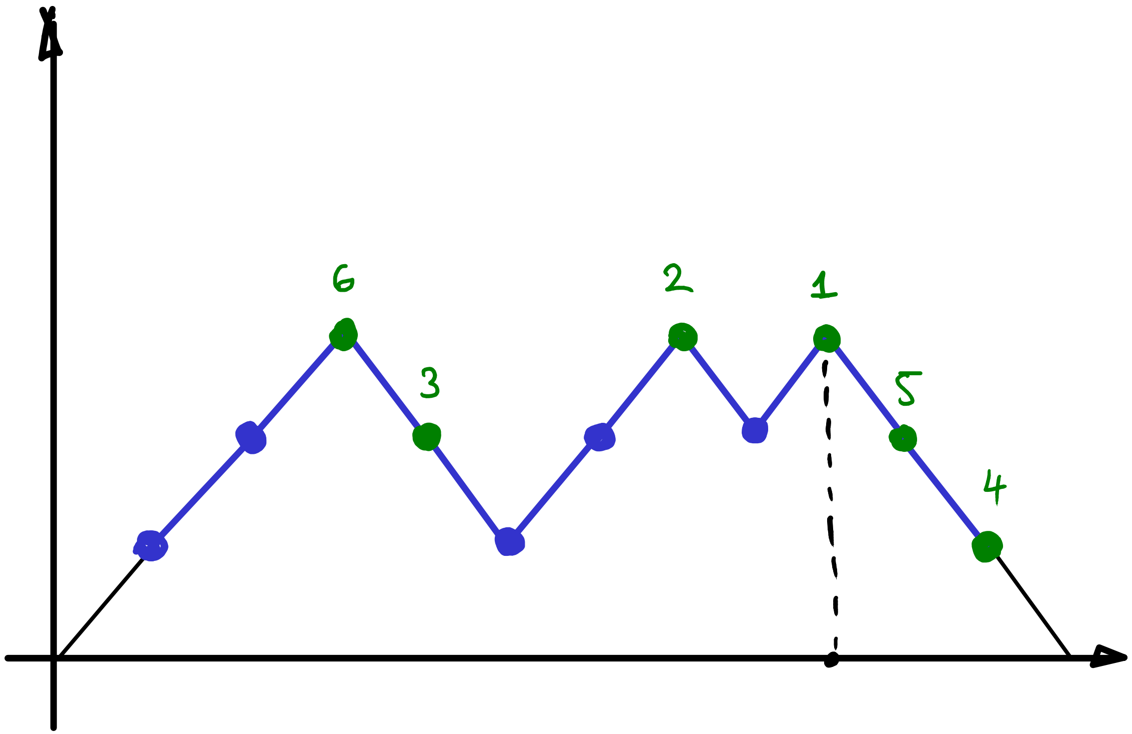
Simple random walk excursion

- Use bijection between binary tree and excursion



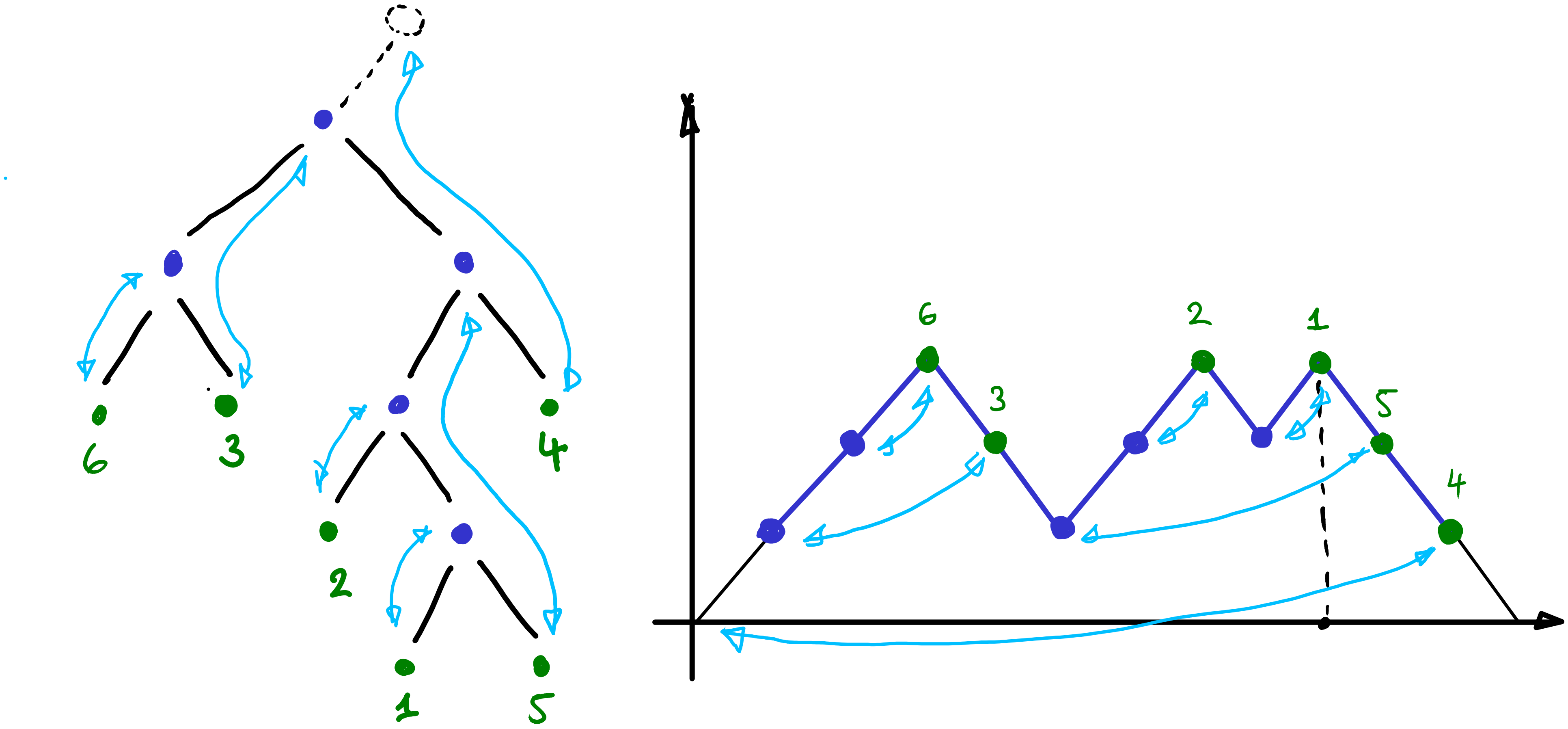
Height at random time point





- Height of leaf 1

  = 1 + #"+1" edges in 1-spanning subtree



- $N_{n-1} \sim P^1_{n-1}(0,1)$, $H_{\text{Leaf }1} \sim 1 + Bi\left(N_{n-1} - 1, 1/2\right)$

$H_{\text{Partner of }1} \sim Bi\left(N_{n-1} - 1, 1/2\right)$

$\Rightarrow$ Height at random time from $\{0, 1, \dots, 2n-1\}$

$$\sim \mathrm{Bi}(N_{n-1}, 1/2)$$

- Can also do bridge, meander, time at origin of unconditional r.w.