Stein's method, logarithmic and transport inequalities

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joint work with

I. Nourdin, G. Peccati (Luxemburg)

new connections between

Stein's method

logarithmic Sobolev inequalities

transportation cost inequalities

I. Nourdin, G. Peccati, Y. Swan (2013)

classical logarithmic Sobolev inequality

L. Gross (1975)

γ standard Gaussian (probability) measure on $\mathbb{R}^d$

$$d\gamma(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{d/2}}$$

$h > 0$ smooth, $\int_{\mathbb{R}^d} h \, d\gamma = 1$

entropy $$\int_{\mathbb{R}^d} h \log h \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \, d\gamma$$ Fisher information

$h \to h^2$ $$\int_{\mathbb{R}^d} h^2 \log h^2 \, d\gamma \leq 2 \int_{\mathbb{R}^d} |\nabla h|^2 d\gamma$$

classical logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} h \log h \, d\gamma \; \leq \; \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \, d\gamma, \qquad \int_{\mathbb{R}^d} h \, d\gamma = 1$$

$$\nu << \gamma \qquad d\nu \; = \; h \, d\gamma$$

$$\mathrm{H}\big(\nu \,|\, \gamma\big) \; \leq \; \frac{1}{2} \, \mathrm{I}\big(\nu \,|\, \gamma\big)$$

(relative) H-entropy $\qquad \mathrm{H}\big(\nu \,|\, \gamma\big) \; = \; \int_{\mathbb{R}^d} h \log h \, d\gamma$

(relative) Fisher Information $\qquad \mathrm{I}\big(\nu \,|\, \gamma\big) \; = \; \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \, d\gamma$

hypercontractivity (integrability of Wiener chaos),

convergence to equilibrium, concentration inequalities

logarithmic Sobolev inequality and concentration

Herbst argument (1975)

$$\int_{\mathbb{R}^d} h \log h \, d\gamma \;\leq\; \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \, d\gamma, \qquad \int_{\mathbb{R}^d} h \, d\gamma = 1$$

$\varphi : \mathbb{R}^d \to \mathbb{R}$ 1-Lipschitz $\int_{\mathbb{R}^d} \varphi \, d\gamma = 0$

$$h \;=\; \frac{e^{\lambda\varphi}}{\int_{\mathbb{R}^d} e^{\lambda\varphi} d\gamma}\,, \qquad \lambda \in \mathbb{R}$$

$$Z(\lambda) \;=\; \int_{\mathbb{R}^d} e^{\lambda\varphi} d\gamma$$

logarithmic Sobolev inequality and concentration

Herbst argument (1975)

$$\lambda Z'(\lambda) - Z(\lambda) \log Z(\lambda) \,\leq\, \frac{\lambda^2}{2} Z(\lambda)$$

integrate

$$Z(\lambda) = \int_{\mathbb{R}^d} e^{\lambda\varphi} d\gamma \,\leq\, e^{\lambda^2/2}$$

Chebyshev's inequality

$$\gamma(\varphi \geq r) \,\leq\, e^{-r^2/2}, \quad r \geq 0$$

Gaussian concentration

logarithmic Sobolev inequality and concentration

$\varphi : \mathbb{R}^d \to \mathbb{R}$ 1-Lipschitz $\int_{\mathbb{R}^d} \varphi \, d\gamma = 0$

$$\gamma(\varphi \geq r) \leq e^{-r^2/2}, \quad r \geq 0$$

Gaussian concentration

equivalent (up to numerical constants)

$$\left(\int_{\mathbb{R}^d} |\varphi|^p d\gamma \right)^{1/p} \leq C \sqrt{p}, \qquad p \geq 1$$

moment growth: concentration rate

$\mathcal{F}$ collection of functions $f : S \to \mathbb{R}$

$G(f),\ f \in \mathcal{F}$ centered Gaussian process

$$M = \sup_{f \in \mathcal{F}} G(f), \qquad M \ \text{Lipschitz}$$

Gaussian concentration

$$\mathbb{P}\big(|M - m| \geq r\big) \leq 2\, e^{-r^2/2\sigma^2}, \quad r \geq 0$$

m mean or median, $\sigma^2 = \sup_{f \in \mathcal{F}} \mathbb{E}\big(G(f)^2\big)$

Gaussian isoperimetric inequality

C. Borell, V. Sudakov, B. Tsirel'son, I. Ibragimov (1975)

M. Talagrand (1996)

$X_1, \ldots, X_n$ independent in $(S, \mathcal{S})$

$\mathcal{F}$ collection of functions $f : S \to \mathbb{R}$

$$M = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

M Lipschitz and convex

concentration inequalities on

$$\mathbb{P}\big(|M - m| \geq r\big), \quad r \geq 0$$

$$M = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

$$|f| \leq 1, \quad \mathbb{E}\big(f(X_i)\big) = 0, \quad f \in \mathcal{F}$$

$$\mathbb{P}\big(|M - m| \geq r\big) \leq C \exp\left(-\frac{r}{C} \log\left(1 + \frac{r}{\sigma^2 + m}\right)\right), \quad r \geq 0$$

m mean or median, $\quad \sigma^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}\big(f^2(X_i)\big)$

M. Talagrand (1996) isoperimetric methods for product measures

entropy method – Herbst argument

P. Massart (2000)

S. Boucheron, G. Lugosi, P. Massart (2005, 2013)

C. Stein (1972)

γ standard normal on $\mathbb{R}$

$$\int_{\mathbb{R}} x\,\phi\,d\gamma = \int_{\mathbb{R}} \phi'\,d\gamma, \qquad \phi : \mathbb{R} \to \mathbb{R} \quad \text{smooth}$$

characterizes γ

Stein's inequality

ν probability measure on $\mathbb{R}$

$$\|\nu - \gamma\|_{\mathrm{TV}} \leq \sup_{\|\phi\|_\infty \leq \sqrt{\pi/2},\, \|\phi'\|_\infty \leq 2} \left[\int_{\mathbb{R}} x\,\phi\,d\nu - \int_{\mathbb{R}} \phi'\,d\nu \right]$$

ν (centered) probability measure on $\mathbb{R}$

Stein factor for ν : $\quad x \mapsto \tau_\nu(x)$

$$\int_{\mathbb{R}} x\,\phi\,d\nu = \int_{\mathbb{R}} \tau_\nu\,\phi'\,d\nu, \qquad \phi : \mathbb{R} \to \mathbb{R} \text{ smooth}$$

γ standard normal $\qquad \tau_\gamma = 1$

Stein discrepancy $\mathrm{S}(\nu\,|\,\gamma)$

$$\mathrm{S}^2(\nu\,|\,\gamma) = \int_{\mathbb{R}} |\tau_\nu - 1|^2 d\nu$$

Stein's inequality

$$\|\nu - \gamma\|_{\mathrm{TV}} \leq 2\,\mathrm{S}(\nu\,|\,\gamma)$$

Stein factor and discrepancy: examples I

Stein factor for ν : $\quad x \mapsto \tau_\nu(x)$

$$\int_{\mathbb{R}} x\,\phi\,d\nu = \int_{\mathbb{R}} \tau_\nu\,\phi'\,d\nu$$

γ standard normal $\quad \tau_\gamma = 1$

$$d\nu = f\,dx$$

$$\tau_\nu(x) = \big[f(x)\big]^{-1} \int_x^\infty y\,f(y)dy, \qquad x \in \mathrm{supp}(f)$$

(τ_ν polynomial: Pearson class)

Stein factor and discrepancy: examples II

central limit theorem

$X, X_1, \ldots, X_n$ iid random variables

mean zero, variance one

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$$\mathrm{S}^2\big(\mathcal{L}(S_n) \,|\, \gamma\big) \leq \frac{1}{n} \mathrm{S}^2\big(\mathcal{L}(X) \,|\, \gamma\big) = \frac{1}{n} \mathrm{Var}\big(\tau_{\mathcal{L}(X)}(X)\big)$$

$$\mathrm{S}^2\big(\mathcal{L}(S_n) \,|\, \gamma\big) = O\Big(\frac{1}{n}\Big)$$

Stein factor and discrepancy: examples III

Wiener multiple integrals (chaos)

multilinear Gaussian polynomial

$$F = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}$$

$X_1, \ldots, X_N$ independent standard normal

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

$$\mathbb{E}(F^2) = 1$$

Stein factor and discrepancy: examples III

D. Nualart, G. Peccati (2005)

$F = F_n, \quad n \in \mathbb{N}$ $\quad$ k-chaos $\quad$ (fixed degree k)

$$N = N_n \to \infty$$

$\mathbb{E}(F_n^2) = 1$ $\quad$ (or $\to 1$)

F_n converges to a standard normal

if and only if

$$\mathbb{E}(F_n^4) \to 3 \qquad \left(= \int_{\mathbb{R}} x^4 d\gamma \right)$$

Stein factor and discrepancy: examples III

F Wiener chaos or multilinear polynomial

$$\tau_F(x) = \mathbb{E}\big(\langle DF, -D\,\mathrm{L}^{-1}F\rangle \,|\, F = x\big)$$

L Ornstein-Uhlenbeck operator, D Malliavin derivative

$$\mathrm{S}^2\big(\mathcal{L}(F)\,|\,\gamma\big) \le \frac{k-1}{3k}\,\big[\mathbb{E}(F^4) - 3\big]$$

multidimensional versions

I. Nourdin, G. Peccati (2009), I. Nourdin, J. Rosinski (2012)

ν (centered) probability measure on $\mathbb{R}^d$

Stein matrix for ν : $x \mapsto \tau_\nu(x) = \big(\tau_\nu^{ij}(x)\big)_{1 \leq i,j \leq d}$

$$\int_{\mathbb{R}^d} x \, \phi \, d\nu = \int_{\mathbb{R}^d} \tau_\nu \, \nabla \phi \, d\nu, \qquad \phi : \mathbb{R}^d \to \mathbb{R} \quad \text{smooth}$$

Stein discrepancy $\mathrm{S}(\nu \,|\, \gamma)$

$$\mathrm{S}^2(\nu \,|\, \gamma) = \int_{\mathbb{R}^d} \|\tau_\nu - \mathrm{Id}\|_{\mathrm{HS}}^2 \, d\nu$$

no Stein inequality in general

Stein's inequality (on $\mathbb{R}$)

$$\|\nu - \gamma\|_{\mathrm{TV}} \leq 2\,\mathrm{S}\big(\nu \,|\, \gamma\big)$$

stronger convergence in entropy

ν probability measure on $\mathbb{R}^d$, $d\nu = h\,d\gamma$ density h

(relative) H-entropy $\mathrm{H}\big(\nu \,|\, \gamma\big) = \int_{\mathbb{R}^d} h \log h \, d\gamma$

Pinsker's inequality

$$\|\nu - \gamma\|_{\mathrm{TV}}^2 \leq \frac{1}{2}\,\mathrm{H}\big(\nu \,|\, \gamma\big)$$

γ standard Gaussian measure on $\mathbb{R}^d$

logarithmic Sobolev inequality $\nu << \gamma$ $d\nu = h\, d\gamma$

$$\mathrm{H}\big(\nu \,|\, \gamma\big) \leq \frac{1}{2}\, \mathrm{I}\big(\nu \,|\, \gamma\big)$$

(relative) H-entropy $\mathrm{H}\big(\nu \,|\, \gamma\big) = \int_{\mathbb{R}^d} h \log h \, d\gamma$

(relative) Fisher Information $\mathrm{I}\big(\nu \,|\, \gamma\big) = \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \, d\gamma$

(relative) Stein discrepancy

$$\mathrm{S}^2\big(\nu \,|\, \gamma\big) = \int_{\mathbb{R}^d} \|\tau_\nu - \mathrm{Id}\|^2_{\mathrm{HS}} \, d\nu$$

new HSI (H-entropy-Stein-Information) inequality

$$\mathrm{H}\big(\nu\,|\,\gamma\big) \;\leq\; \frac{1}{2}\,\mathrm{S}^2\big(\nu\,|\,\gamma\big)\log\left(1+\frac{\mathrm{I}(\nu\,|\,\gamma)}{\mathrm{S}^2(\nu\,|\,\gamma)}\right)$$

$\log(1+x) \leq x$ improves upon the logarithmic Sobolev inequality

entropic convergence

if $\mathrm{S}(\nu_n\,|\,\gamma) \to 0$ and $\mathrm{I}(\nu_n\,|\,\gamma)$ bounded, then

$$\mathrm{H}\big(\nu_n\,|\,\gamma\big) \to 0$$

entropic central limit theorem

$X, X_1, \dots, X_n$ iid random variables, mean zero, variance one

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$$\mathrm{S}^2\big(\mathcal{L}(S_n) \,|\, \gamma\big) \leq \frac{1}{n} \mathrm{Var}\big(\tau_{\mathcal{L}(X)}(X)\big)$$

Stam's inequality $\quad \mathrm{I}\big(\mathcal{L}(S_n) \,|\, \gamma\big) \leq \mathrm{I}\big(\mathcal{L}(X) \,|\, \gamma\big) < \infty$

HSI inequality $\quad \mathrm{H}\big(\mathcal{L}(S_n) \,|\, \gamma\big) = O\Big(\frac{\log n}{n}\Big)$

optimal $O(\frac{1}{n})$ under fourth moment on X

S. Bobkov, G. Chistyakov, F. Götze (2013-14)

ν probability measure on $\mathbb{R}^d$

$\varphi : \mathbb{R}^d \to \mathbb{R}$ 1-Lipschitz $\int_{\mathbb{R}^d} \varphi \, d\nu = 0$

moment growth in $p \geq 2$, $C > 0$ numerical

$$\left(\int_{\mathbb{R}^d} |\varphi|^p d\nu \right)^{1/p} \leq C \left[\mathrm{S}_p(\nu \,|\, \gamma) + \sqrt{p} \left(\int_{\mathbb{R}^d} \|\tau_\nu\|_{\mathrm{Op}}^{p/2} \, d\nu \right)^{1/p} \right]$$

$$\mathrm{S}_p(\nu \,|\, \gamma) = \left(\int_{\mathbb{R}^d} \|\tau_\nu - \mathrm{Id}\|_{\mathrm{HS}}^p \, d\nu \right)^{1/p}$$

HSI and concentration inequalities

$X, X_1, \ldots, X_n$ iid random variables in $\mathbb{R}^d$

mean zero, covariance identity

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$\varphi : \mathbb{R}^d \to \mathbb{R}$ 1-Lipschitz

$$\mathbb{P}\Big(\Big| \varphi(S_n) - \mathbb{E}\big(\varphi(S_n)\big) \Big| \geq r \Big) \leq C \, e^{-r^2/C}$$

$$0 \leq r \leq r_n \to \infty$$

according to the growth in p of $\displaystyle\int_{\mathbb{R}^d} \|\tau_\nu - \mathrm{Id}\|_{\mathrm{HS}}^p \, d\nu$

HSI inequality

$$\mathrm{H}(\nu \,|\, \gamma) \;\leq\; \frac{1}{2}\,\mathrm{S}^2(\nu \,|\, \gamma)\,\log\left(1 + \frac{\mathrm{I}(\nu \,|\, \gamma)}{\mathrm{S}^2(\nu \,|\, \gamma)}\right)$$

H-entropy	$\mathrm{H}(\nu \,\vert\, \gamma)$
Fisher Information	$\mathrm{I}(\nu \,\vert\, \gamma)$
Stein discrepancy	$\mathrm{S}(\nu \,\vert\, \gamma)$

HSI inequality: elements of proof

Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$

$$P_t f(x) = \int_{\mathbb{R}^d} f\big(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\big)\, d\gamma(y)$$

$$d\nu = h\, d\gamma, \qquad d\nu_t = P_t h\, d\gamma \qquad (\nu_0 = \nu, \ \ \nu_\infty = \gamma)$$

$$\mathrm{H}\big(\nu \,|\, \gamma\big) = \int_0^\infty \mathrm{I}\big(\nu_t \,|\, \gamma\big)\, dt$$

classical $\quad \mathrm{I}\big(\nu_t \,|\, \gamma\big) \leq e^{-2t}\, \mathrm{I}\big(\nu \,|\, \gamma\big)$

new main ingredient $\quad \mathrm{I}\big(\nu_t \,|\, \gamma\big) \leq \dfrac{e^{-4t}}{1 - e^{-2t}}\, \mathrm{S}^2\big(\nu \,|\, \gamma\big)$

HSI inequality: elements of proof

$$\mathrm{H}(\nu \,|\, \gamma) \;=\; \int_0^\infty \mathrm{I}(\nu_t \,|\, \gamma)\, dt$$

classical $\quad \mathrm{I}(\nu_t \,|\, \gamma) \;\le\; e^{-2t}\, \mathrm{I}(\nu \,|\, \gamma)$

new main ingredient $\quad \mathrm{I}(\nu_t \,|\, \gamma) \;\le\; \dfrac{e^{-4t}}{1 - e^{-2t}}\, \mathrm{S}^2(\nu \,|\, \gamma)$

representation of $\mathrm{I}(\nu_t \,|\, \gamma) \qquad (v_t = \log P_t h)$

$$\frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big[\big(\tau_\nu(x) - \mathrm{Id}\big) y \cdot \nabla v_t\big(e^{-t} x + \sqrt{1 - e^{-2t}}\, y\big) \Big]\, d\nu(x)\, d\gamma(y)$$

optimize small $t > 0$ and large $t > 0$

HSI inequalities for other distributions

$$\mathrm{H}(\nu \,|\, \mu) \;\leq\; \frac{1}{2}\, \mathrm{S}^2(\nu \,|\, \mu) \log \left(1 + \frac{\mathrm{I}(\nu \,|\, \mu)}{\mathrm{S}^2(\nu \,|\, \mu)} \right)$$

μ gamma, beta distributions

multidimensional

families of log-concave distributions μ

Markov Triple (E, μ, Γ)

(typically abstract Wiener space)

HSI inequalities for other distributions

$$\mathrm{H}\big(\nu \,|\, \mu\big) \,\leq\, C\,\mathrm{S}^2\big(\nu \,|\, \mu\big)\Psi\bigg(\frac{C\,\mathrm{I}(\nu \,|\, \mu)}{\mathrm{S}^2(\nu \,|\, \mu)}\bigg)$$

$$\Psi(r) \,=\, 1 + \log r, \quad r \geq 1$$

μ gamma, beta distributions

multidimensional

families of log-concave distributions μ

Markov Triple (E, μ, Γ)

(typically abstract Wiener space)

ν (centered) probability measure on $\mathbb{R}^d$

Stein matrix for ν : $\quad x \mapsto \tau_\nu(x) = \big(\tau_\nu^{ij}(x)\big)_{1 \leq i,j \leq d}$

$$\int_{\mathbb{R}} x\,\phi\,d\nu = \int_{\mathbb{R}} \tau_\nu\,\nabla\phi\,d\nu, \qquad \phi : \mathbb{R}^d \to \mathbb{R} \ \text{ smooth}$$

weak form

$$\int_{\mathbb{R}^d} x \cdot \nabla\phi\,d\nu = \int_{\mathbb{R}^d} \langle \tau_\nu, \mathrm{Hess}(\phi) \rangle_{\mathrm{HS}}\,d\nu, \qquad \phi : \mathbb{R}^d \to \mathbb{R} \ \text{ smooth}$$

Stein matrix for diffusion operator

second order differential operator

$$\mathcal{L}f = \langle a, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} + b \cdot \nabla f = \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial f}{\partial x_i}$$

μ invariant measure

example: Ornstein-Uhlenbeck operator

$$\mathcal{L}f = \Delta f - x \cdot \nabla f = \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i}$$

γ invariant measure

Stein matrix for diffusion operator

second order differential operator

$$\mathcal{L}f = \langle a, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} + b \cdot \nabla f = \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial f}{\partial x_i}$$

μ invariant measure

Stein matrix for ν

$$-\int_{\mathbb{R}^d} b \cdot \nabla f \, d\nu = \int_{\mathbb{R}^d} \langle \tau_\nu, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} \, d\nu$$

Stein discrepancy

$$\mathrm{S}(\nu \,|\, \mu) = \left(\int_{\mathbb{R}^d} \left\| a^{-\frac{1}{2}} \tau_\nu a^{-\frac{1}{2}} - \mathrm{Id} \right\|_{\mathrm{HS}}^2 d\nu \right)^{1/2}$$

Stein matrix for diffusion operator

second order differential operator

$$\mathcal{L}f = \langle a, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} + b \cdot \nabla f = \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial f}{\partial x_i}$$

μ invariant measure

Stein matrix for ν $\quad (\tau_\mu = a)$

$$- \int_{\mathbb{R}^d} b \cdot \nabla f \, d\nu = \int_{\mathbb{R}^d} \langle \tau_\nu, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} \, d\nu$$

Stein discrepancy

$$\mathrm{S}(\nu \,|\, \mu) = \left(\int_{\mathbb{R}^d} \left\| a^{-\frac{1}{2}} \tau_\nu a^{-\frac{1}{2}} - \mathrm{Id} \right\|_{\mathrm{HS}}^2 d\nu \right)^{1/2}$$

Laguerre operator $\mathcal{L}f = \sum_{i=1}^{d} x_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{i=1}^{d} (p_i - x_i) \frac{\partial f}{\partial x_i}$ on $\mathbb{R}_+^d$

μ product of gamma distributions $\Gamma(p_i)^{-1} x_i^{p_i - 1} e^{-x_i} dx_i$

Stein matrix $p = (p_1, \ldots, p_d)$

$$- \int_{\mathbb{R}_+^d} (p - x) \cdot \nabla f \, d\nu = \int_{\mathbb{R}_+^d} \langle \tau_\nu, \mathrm{Hess}(f) \rangle_{\mathrm{HS}} \, d\nu$$

HSI inequality $(p_i \geq \frac{3}{2})$

$$\mathrm{H}(\nu \,|\, \mu) \leq \mathrm{S}^2(\nu \,|\, \mu) \, \Psi\left(\frac{\mathrm{I}(\nu \,|\, \mu)}{\mathrm{S}^2(\nu \,|\, \mu)} \right)$$

towards entropic convergence via HSI

$\mathrm{I}(\nu \,|\, \gamma)$ difficult to control in general

Wiener chaos or multilinear polynomial

$$F = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}$$

$X_1, \ldots, X_N$ independent standard normal

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

law $\mathcal{L}(F)$ of F? Fisher information $\mathrm{I}\big(\mathcal{L}(F) \,|\, \gamma\big)$?

I. Nourdin, G. Peccati, Y. Swan (2013)

$(F_n)_{n \in \mathbb{N}}$ sequence of Wiener chaos, fixed degree

$$\mathrm{H}\big(\mathcal{L}(F_n)\,|\,\gamma\big) \to 0 \quad \text{as} \quad \mathrm{S}\big(\mathcal{L}(F_n)\,|\,\gamma\big) \to 0$$

(fourth moment theorem $\mathrm{S}(\mathcal{L}(F_n)\,|\,\gamma) \to 0$)

Markov operator L with state space E

μ invariant and symmetric probability measure

Γ bilinear gradient operator (carré du champ)

$$\Gamma(f,g) = \frac{1}{2}\left[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\right], \qquad f,g \in \mathcal{A}$$

$$\int_E f(-\mathrm{L}g)\,d\mu = \int_E \Gamma(f,g)d\mu$$

$$\mathrm{L} = \sum_{i,j=1}^{d} a^{ij}\,\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i\,\frac{\partial f}{\partial x_i} \qquad \text{on} \quad E = \mathbb{R}^d$$

$$\Gamma(f,g) = \sum_{i,j=1}^{d} a^{ij}\,\frac{\partial f}{\partial x_i}\,\frac{\partial g}{\partial x_i}$$

Markov Triple (E, μ, Γ)

(typically abstract Wiener space)

$F : E \to \mathbb{R}^d$ with law $\mathcal{L}(F)$

$$\mathrm{H}\big(\mathcal{L}(F)\,|\,\gamma\big) \leq C_F \, \mathrm{S}^2\big(\mathcal{L}(F)\,|\,\gamma\big)\, \Psi\bigg(\frac{C_F}{\mathrm{S}^2(\mathcal{L}(F)\,|\,\gamma)}\bigg)$$

$$\Psi(r) = 1 + \log r, \quad r \geq 1$$

$C_F > 0$ depend on integrability of $F, \ \Gamma(F_i, F_j)$

and inverse of the determinant of $(\Gamma(F_i, F_j))_{1 \leq i,j \leq d}$

(Malliavin calculus)

$$\mathrm{H}\big(\mathcal{L}(F)\,|\,\gamma\big) \;\leq\; \frac{\mathrm{S}^2(\mathcal{L}(F)\,|\,\gamma)}{2(1-4\kappa)}\,\Psi\bigg(\frac{2(A_F + d(B_F+1))}{\mathrm{S}^2(\mathcal{L}(F)\,|\,\gamma)}\bigg)$$

$$\kappa = \tfrac{2+\alpha}{2(4+3\alpha)} \quad (< \tfrac{1}{4}\,)$$

$A_F < \infty$ under moment assumptions

$$B_F \;=\; \int_E \frac{1}{\det(\widetilde{\Gamma})^\alpha}\,d\mu, \qquad \alpha > 0$$

$$\widetilde{\Gamma} \;=\; \big(\Gamma(F_i,F_j)\big)_{1\leq i,j\leq d}$$

abstract HSI inequality

$$B_F = \int_E \frac{1}{\det(\widetilde{\Gamma})^\alpha} \, d\mu, \qquad \alpha > 0$$

Gaussian vector chaos $F = (F_1, \ldots, F_d)$

$$\Gamma(F_i, F_j) = \langle DF_i, DF_j \rangle_{\mathfrak{H}}$$

$\mathcal{L}(F)$ density: $\mathbb{E}(\det(\widetilde{\Gamma})) > 0$

$$\mathbb{P}\big(\det(\widetilde{\Gamma}) \leq \lambda\big) \leq cN\lambda^{1/N} \, \mathbb{E}\big(\det(\widetilde{\Gamma})\big)^{-1/N}, \quad \lambda > 0$$

N degrees of the F_i's

A. Carbery, J. Wright (2001)

logconcave models

Kantorovich-Rubinstein-Wasserstein distance

$$\mathrm{W}_2^2(\nu,\mu) = \inf_{\nu \leftarrow \pi \rightarrow \mu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^2 d\pi(x,y)$$

$\nu << \gamma$ probability measure on $\mathbb{R}^d$

Talagrand inequality

$$\mathrm{W}_2^2(\nu,\gamma) \leq 2\,\mathrm{H}\big(\nu\,|\,\gamma\big)$$

(relative) H-entropy $\quad \mathrm{H}\big(\nu\,|\,\gamma\big) = \int_{\mathbb{R}^d} h \log h \, d\gamma$

Talagrand inequality

$$\mathrm{W}_2^2(\nu, \gamma) \leq 2\,\mathrm{H}\big(\nu\,|\,\gamma\big)$$

$\nu << \gamma$ (centered) probability measure on $\mathbb{R}^d$

WSH inequality

$$\mathrm{W}_2(\nu, \gamma) \leq \mathrm{S}\big(\nu\,|\,\gamma\big)\,\arccos\left(e^{-\frac{\mathrm{H}(\nu\,|\,\gamma)}{\mathrm{S}^2(\nu\,|\,\gamma)}}\right)$$

$$\arccos(e^{-r}) \leq \sqrt{2r}$$

$$\big(\mathrm{W}_2(\nu, \gamma) \leq \mathrm{S}\big(\nu\,|\,\gamma\big)\big)$$

WSH inequality: elements of proof

$$\mathrm{W}_2(\nu,\gamma) \,\leq\, \mathrm{S}\big(\nu\,|\,\gamma\big)\,\mathrm{arccos}\left(e^{-\frac{\mathrm{H}(\nu\,|\,\gamma)}{\mathrm{S}^2(\nu\,|\,\gamma)}}\right)$$

$$d\nu = h d\gamma, \qquad d\nu_t = P_t h d\gamma, \qquad v_t = \log P_t h$$

F. Otto, C. Villani (2000)

$$\frac{d^+}{dt}\,\mathrm{W}_2(\nu,\nu_t) \,\leq\, \left(\int_{\mathbb{R}^d} |\nabla v_t|^2\, d\nu_t\right)^{1/2} = \,\mathrm{I}\big(\nu_t\,|\,\gamma\big)^{1/2}$$

new main ingredient $\qquad \mathrm{I}\big(\nu_t\,|\,\gamma\big) \,\leq\, \dfrac{e^{-4t}}{1-e^{-2t}}\,\mathrm{S}^2\big(\nu\,|\,\gamma\big)$

p-WSH inequality

$$\tau_\nu = (\tau_\nu^{ij})_{1\le i,j\le d}$$

$$\|\tau_\nu - \mathrm{Id}\|_{p,\nu} = \left(\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left|\tau_\nu^{ij} - \delta_{ij}\right|^p d\nu \right)^{1/p}$$

$$p \in [1,2)$$

$$\mathrm{W}_p(\nu,\gamma) \le C_p\, d^{1-1/p} \|\tau_\nu - \mathrm{Id}\|_{p,\nu}$$

$$p \in [2,\infty)$$

$$\mathrm{W}_p(\nu,\gamma) \le C_p\, d^{1-2/p}\, \|\tau_\nu - \mathrm{Id}\|_{p,\nu}$$

Thank you for your attention