On the Error Bound in the Normal Approximation for Jack Measures

(Joint work with Le Van Thanh)

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International Colloquium on Stein's Method, Concentration Inequalities, and Malliavin Calculus

June 29 - July 2 2014

Château de la Bretesche

Missillac, Loire-Atlantique, France

Outline

- Partitions of positive integers
- Young diagram and Young Tableau
- Plancherel measure
- Normal approximation for Character Ratios
- Jack measures
- Normal Approximation for Jack Measures
- Main Theorems
- Zero-bias Coupling
- Rosenthal Inequality for Zero-bias Coupling
- Normal Approximation for Zero-bias Coupling
- Zero-bias Coupling for Jack Measures
- Sketch of Proof of Main Theorems
- Summary

Partitions of Positive Integers

- A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ such that $\sum_{i=1}^{l} \lambda_i = n$. Write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.
- The λ_i are called the parts of the partition λ and the number l of parts called the the length of λ.
- We write $\lambda \vdash n$ to denote "λ is a partition of n".
- Denote that set of all partitions of n by $\mathcal{P}_n$ and the set of all partitions by $\mathcal{P}$, that is, $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. By convention, the empty sequence forms the only partition of zero.

Partitions of Positive Integers

- Let $p(n)$ be the partition function, that is, the number of partitions of n.
- An important and fundamental question is to evaluate $p(n)$.
- Euler started the analytic theory of partitions by providing an explicit formula for the generating function of $p(n)$:

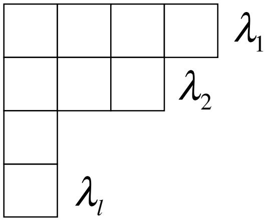
$$\mathcal{F}(q) := \sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

- In a celebrated series of memoirs published in 1917 and 1918, Hardy and Ramanujan established:

$$p(n) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}n}} (1 + O(\frac{1}{\sqrt{n}})).$$

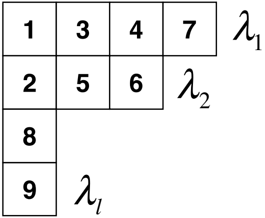
Young Diagram

- To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is associated its Young diagram (shape).



Young Tableau

- A standard Young tableau T with the shape $\lambda \vdash n$ is a one-to-one assignment of the numbers $1, 2, \ldots, n$ to the squares of λ in such a way that the numbers increase along the rows and down the columns. See, for example, $n = 9$.



- Let d_λ denote the total number of standard Young tableaux associated with a given shape λ.

Plancherel Measure

- The set of irreducible representations of the symmetric group S_n of permutations of $1, 2, \ldots, n$ can be parameterized by $\lambda \in \mathcal{P}_n$.
- The degree (dimension) of the irreducible representation indexed by λ is equal to d_λ.
- The Burnside identity is:

$$\sum_{\lambda \vdash n} d_\lambda^2 = n! \qquad \text{(that is,} \quad \sum_{\lambda \vdash n} \frac{d_\lambda^2}{n!} = 1\text{).}$$

- The Plancherel measure is a probability measure on $\lambda \vdash n$ (also on the irreducible representations of S_n, parameterized by λ) given by:

$$P(\{\lambda\}) = \frac{d_\lambda^2}{n!}.$$

Plancherel Measure

- The first row of a random partition distributed according to the Plancheral measure has the same distribution as the longest increasing subsequence of a random permutation distributed according to the uniform measure.

- Let $l(\pi)$ be the length of the longest increasing subsequence of the random permutation π. It is knwon that $(l(\pi) - 2\sqrt{n})/n^{1/6}$ converges to the Tracy-Widom distribution. (Baik, Deift and Johansson (1999), *J. Amer. Math. Soc.)*

Character Ratio

- The character of a group representation is a function on the group that associates to each group element the trace of the corresponding matrix. It is called irreducible if it is the character of an irreducible representation.
- Let $\chi^\lambda(12)$ be the irreducible character parametrized by λ evaluated on the transposition (12).
- The quantity $\dfrac{\chi^\lambda(12)}{d_\lambda}$ is called a character ratio.
- The eigenvalues for the random walk on the symmetric group generated by transpositions are the character ratios $\chi^\lambda(12)/d_\lambda$, each occcuring with multipicity d_λ^2. Diaconis and Shahshahani (1981), *Z. Wahr. Verw. Gebiete*.
- Character ratios also play an essential role in work on the moduli spaces of curves (see Eskin and Okounkov (2001), *Invent. Math.* and Okounkov and Pandharipande (2005), *Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc.*).

Normal Approximation for Character Ratios

Let

$$W_n = \sqrt{\binom{n}{2}} \frac{\chi^\lambda(12)}{d_\lambda}$$

and let Φ be the $\mathcal{N}(0,1)$ distribution function. Assume $n \geq 2$ and let $x \in \mathbb{R}$.

- Kerov (1993), *Compt. Rend. Acad. Sci. Paris.*

$$W_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \text{ as } n \longrightarrow \infty.$$

- Fulman (2005), *Trans. AMS* (using Stein's method)

$$|P(W_n \leq x) - \Phi(x)| \leq 40.1 n^{-1/4}.$$

Normal Approximation for Character Ratios

- Fulman (2006) *Trans. AMS* (using martingales)

$$|P(W_n \leq x) - \Phi(x)| \leq Cn^{-s} \text{ for any } s < 1/2.$$

- Shao and Su (2006), *Proc. AMS* (using Stein's method)

$$|P(W_n \leq x) - \Phi(x)| \leq Cn^{-1/2}.$$

Jack Measures

The Jack_α measure, $\alpha > 0$, is a probaility measure on $\lambda \vdash n$ given by:

$$\mathrm{Jack}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_{x\in\lambda}(\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)},$$

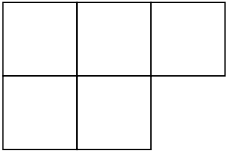
where in the product over all boxes x in the partition λ,

(i) $a(x)$ denotes the number of boxes in the same row of x and to the right of x (the "arm" of x),

(ii) $l(x)$ denotes the number of boxes in the same column of x and below x (the "leg" of x).

Jack Measures

For example, take $n = 5$ and λ as shown below.

$\lambda =$ 

$$\mathrm{Jack}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_{x\in\lambda}(\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)}$$

$$= \frac{60\alpha^2}{(2\alpha+2)(3\alpha+1)(\alpha+2)(2\alpha+1)(\alpha+1)}.$$

Jack Measures

- The Jack_α measure with $\alpha = 2/\beta$ is a discrete analog of Dyson's β ensembles in random matrix theory, which are tractable for $\beta = 1, 2, 4$.
- The joint probability density for the eigenvalues $x_1 \geq x_2 \geq \cdots \geq x_n$ of the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble(GSE) is given by

$$\frac{1}{Z_\beta} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{2}\right) \Pi_{1\leq i<j\leq n}(x_i - x_j)^\beta$$

  for $\beta = 1, 2, 4$ respectively.
- The Jack_α measure with $\alpha (= 2/\beta) = 2, 1, 1/2$ has group theoretical interpretation.

Jack Measures

- In the case $\alpha = 1$,

$$\mathrm{Jack}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_{x\in\lambda}(\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)}$$
$$= \frac{n!}{\prod_{x\in\lambda} h^2(x)},$$

  where $h(x) = a(x) + l(x) + 1$ is the hook length of the box x.
- The hook-length formula states that

$$d_\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)}.$$

- Hence the Plancherel measure can be expressed as

$$P(\{\lambda\}) = \frac{d_\lambda^2}{n!} = \frac{n!}{\prod_{x\in\lambda} h^2(x)},$$

  which agrees with the Jack_α measure for $\alpha = 1$.

Normal Approximation for Jack Measures

Let

$$W_{n,\alpha} = W_{n,\alpha}(\lambda) = \frac{\sum_i \left(\alpha\binom{\lambda_i}{2} - \binom{\lambda_i'}{2}\right)}{\sqrt{\alpha\binom{n}{2}}},$$

where the partition $\lambda \vdash n$ is chosen according to the Jack_α measure, λ_i is the length of the ith row of λ and λ_i' the length of the ith column of λ.

If $\alpha = 1$,

$$W_{n,\alpha} = \sqrt{\binom{n}{2}} \frac{\chi^\lambda(12)}{d_\lambda}$$

by the Frobenius formula.

Normal Approximation for Jack Measures

Assume $n \geq 2$ and let $x \in \mathbb{R}$.

- Fulman (2004), *J. Comb. Theory Ser. A*
  For $\alpha \geq 1$,
  $$|P(W_{n,\alpha} \leq x) - \Phi(x)| \leq \frac{C_\alpha}{n^{1/4}}.$$
  He conjectured that for $\alpha \geq 1$, the optimal bound is a univeral constant multiplied by $\max\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}}{n}\}$.
- Fulman (2006) *Trans. AMS* (using martingales)
  For $\alpha \geq 1$,
  $$|P(W_{n,\alpha} \leq x) - \Phi(x)| \leq \frac{C_\alpha}{n^{1/2-\epsilon}} \quad \text{for any } \epsilon > 0.$$

Normal Approximation for Jack Measures

- Fulman (2006), *Ann. Comb.* (using Stein's method)
  For $\alpha \geq 1$,
  $$|P(W_{n,\alpha} \leq x) - \Phi(x)| \leq \frac{C_\alpha}{n^{1/2}}.$$
- Fulman and Goldstein (2011), *Comb. Probab. Comput.* (using Stein's method and zero-bias coupling)
  For $\alpha > 0$,
  $$\|F - \Phi\|_1 \leq \sqrt{\frac{2}{n}} \left(2 + \sqrt{2 + \frac{\max(\alpha, 1/\alpha)}{n-1}} \right),$$
  where $F(x) = P(W_{n,\alpha} \leq x)$.

Main Theorems

Chen and Thanh (2014), *Preprint*

Theorem 1

For $\alpha > 0$,

$$\sup_{x\in\mathbb{R}} |P(W_{n,\alpha} \le x) - \Phi(x)| \le 9\max\Big\{\frac{1}{\sqrt{n}}, \frac{\max\{\sqrt{\alpha}, 1/\sqrt{\alpha}\}\log n}{n}\Big\}.$$

Remarks.

1. For $\alpha = 1$, the theorem reduces to one for character ratios under the Plancherel measure with the bound $\dfrac{9}{\sqrt{n}}$, where the constant is explicit.

2. For $\alpha \ge 1$, the bound becomes $9\max\Big\{\dfrac{1}{\sqrt{n}}, \dfrac{\sqrt{\alpha}\log n}{n}\Big\}$, which is close to that conjectured by Fulman (2004).

Main Theorems

Chen and Thanh (2014), *Preprint*

Theorem 2

For $\alpha > 0$ *and* $p \geq 2$, *and for* $x \in \mathbb{R}$,

$$|P(W_{n,\alpha} \leq x) - \Phi(x)| \leq \frac{C_p}{1+|x|^p} \max\left\{ \frac{1}{\sqrt{n}}, \frac{\max\{\sqrt{\alpha}, 1/\sqrt{\alpha}\} \log n}{n} \right\}$$

where C_p *is a constant depending only on* p.

Remarks.

1. For $\alpha = 1$, the theorem reduces to one for character ratios under the Plancherel measure with the bound $\frac{C_p}{1+|x|^p} \frac{1}{\sqrt{n}}$.
2. For $\alpha \geq 1$, the bound becomes $\frac{C_p}{1+|x|^p} \max\left\{ \frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha} \log n}{n} \right\}$.

Zero-bias Coupling

Goldstein and Reinert (1997), *Ann. Appl. Probab.*

- For W with $EW = 0$ and $\mathrm{Var}(W) = B^2$, there always exisits W^* such that $EWf(W) = B^2 Ef'(W^*)$ for absolutely continuous functions f for which the expectations exist.
- The distribution of W^* is called W-zero-biased.
- W^* must necessarily be absolutely continuous and its density function is given by $B^{-2} EWI(W > x)$.
- Not easy to find couplings of W with W^* which are effective for normal approximation.
- Effective couplings are known for $W = \sum_{i=1}^n X_i$, where the X_i are independent and for $W = \sum_{i=1}^n a_{i\pi(i)}$ where π is a random permutation (Goldstein (2005), *J. Appl. Probab.*).

Rosenthal Inequality for Zero-bias Coupling

Chen and Thanh (2014), *Preprint*

Proposition 3

Let W *be such that* $EW = 0$ *and* $\mathrm{Var}(W) = B^2 > 0$*. Suppose W-zero-biased* W^* *is defined on the same probability space as* W*. Then for* $p \geq 2$*,*

$$E|W|^p \leq \kappa_p \{B^p + B^2 E|W^* - W|^{p-2}\}$$

where $\kappa_p = 2^{(p-2)+(p-4)+\cdots}(p-1)(p-3)\ldots.$

If $W = \sum_{i=1}^n X_i$, where the X_i are independent with zero mean, then

$$E|W|^p \leq \kappa_p B^p + \overline{\kappa}_p \sum_{i=1}^n E|X_i|^p$$

where $\overline{\kappa}_p = 2\max\{1, 2^{p-3}\}\kappa_p$.

Normal Approximation for Zero-bias Coupling

Chen and Thanh (2014), *Preprint*

Theorem 4

Let $EW = 0$ *and* $\mathrm{Var}(W) = 1$. *Suppose the zero-biased* W^* *is defined on the same probability space as* W. *Let* $T = W^* - W$.

1. Then

$$\sup_{x \in \mathbb{R}} |P(W^* \leq x) - \Phi(x)| \leq \sqrt{ET^2} + \frac{\sqrt{2\pi}}{4} E|T|$$

2. Assume $E|T|^{2p} \leq 1$ *for some* $p \geq 2$. *Then for all* $x \in \mathbb{R}$,

$$|P(W^* \leq x) - \Phi(x)| \leq \frac{C_p(\sqrt{ET^2} + \sqrt{ET^4} + E|T|^{p+1} + E|T|^{p+2})}{1 + |x|^p}.$$

Normal Approximation for Zero-bias Coupling

Sketch of proof of Theorem 4 part 2

Since $(-W)^* = -W^*$ and in view of part 1, it suffices to assume $x \geq 2$. Using the properties of the solution of the Stein equation,

$$|P(W^* \leq x) - \Phi(x)| \leq \frac{C_p(E|T| + E|T|^{p+1} + E|T|^{p+2})}{1 + x^p}$$

$$+\frac{C_p(\sqrt{E|W|^{2p}E|T|^2} + \sqrt{E|W|^{2p+2}E|T|^2} + \sqrt{E|W|^{2p}E|T|^4})}{1 + x^p}.$$

Since $E|T|^{2p} \leq 1$, by the Rosenthal inequality,

$$E|W|^{p+2} \leq C_p(1 + E|T|^{2p}) \leq C_p.$$

Similarly,

$$E|W|^{2p} \leq C_p.$$

Zero-bias Coupling for Jack Measures

Kerov's growth process (Kerov (2000), *Funct. Anal. Appl.*) gives a sequence of partitions $(\lambda(1), \lambda(2), \ldots, \lambda(n))$, where for each j, $\lambda(j)$ is a partition of j distributed according to the Jack_α measure. Using this process, one can show that

$$W_{n,\alpha}(\lambda) = \frac{\sum_{x\in\lambda} c_\alpha(x)}{\sqrt{\alpha\binom{n}{2}}}$$

where $c_\alpha(x)$ denotes the "α-content" of x, which is defined as

$$c_\alpha(x) = \alpha[(\text{column number of } x) - 1] - [(\text{row number of } x) - 1)].$$

Zero-bias Coupling for Jack Measures

In the diagram below, representing a partition of 7, each box is filled with its α-content.

Recall that

$c_\alpha(x) = \alpha[(\text{column number of } x) - 1] - [(\text{row number of } x) - 1)]$.

0	1α	2α	3α
-1	α-1		
-2			

Zero-bias Coupling for Jack Measures

Fulman and Goldstein (2011), *Comb. Probab. Comput.*,
The statistic $W_{n,\alpha}$ and its zero-biased $W^*_{n,\alpha}$ are coupled as follows:

$$W_{n,\alpha} = V_{n,\alpha} + \eta_{n,\alpha} \quad \text{and} \quad W^*_{n,\alpha} = V_{n,\alpha} + \eta^*_{n,\alpha}$$

where $\eta^*_{n,\alpha}$ is $\eta_{n,\alpha}$-zero-biased, $V_{n,\alpha}, \eta_{n,\alpha}$ and $\eta^*_{n,\alpha}$ are defined on the same probability space,

$$V_{n,\alpha} = \sum_{x \in \nu} c_\alpha(x) \Big/ \sqrt{\alpha \binom{n}{2}} = \sqrt{\frac{n-2}{n}} W_{n-1,\alpha},$$

$$\eta_{n,\alpha} = c_\alpha(\lambda/\nu) \Big/ \sqrt{\alpha \binom{n}{2}},$$

ν is a partition of $n-1$ chosen from the Jack_α measure, and $c_\alpha(\lambda/\nu)$ denotes the α-content of the box added to ν to obtain λ.

Some moment bounds on $\eta_{n,\alpha}$ are also obtained.

Sketch of Proof of Main Theorems

Chen and Thanh (2014), *Preprint*

Lemma 5

For $p \geq 1$ *and* $\alpha \geq 1$,

$$P\left(|\eta_{n,\alpha}| \geq \frac{p\sqrt{2e^3}}{\sqrt{n-1}}\right) \leq \frac{n}{2\pi(p^2 e)^p \sqrt{e^3 n/\alpha}},$$

$$P\left(|\eta^*_{n,\alpha}| \geq \frac{p\sqrt{2e^3}}{\sqrt{n-1}}\right) \leq \frac{\alpha n^2}{2\pi(p^2 e)^p \sqrt{e^3 n/\alpha}}.$$

Recall that $W^*_{n,\alpha} - W_{n,\alpha} = \eta^*_{n,\alpha} - \eta_{n,\alpha}$.

Theorem 6

Let $EW = 0$ *and* $\mathrm{Var}(W) = 1$. *Suppose the zero-biased* W^* *is defined on the same probability space as* W. *Let* $T = W^* - W$ *and let* $\epsilon \geq 0$.

1. Then

$$\sup_{x\in\mathbb{R}} |P(W \leq x) - \Phi(x)| \leq \sqrt{ET^2} + \frac{\sqrt{2\pi}}{4} E|T| + \frac{\epsilon}{\sqrt{2\pi}} + P(|T| > \epsilon).$$

2. Assume $E|T|^{2p} \leq 1$ *for some* $p \geq 2$. *Then for all* $x \in \mathbb{R}$,

$$|P(W \leq x) - \Phi(x)| \leq \frac{C_p(\sqrt{ET^2} + \sqrt{ET^4} + E|T|^{p+1} + E|T|^{p+2})}{1 + |x|^p}.$$

$$+\frac{\epsilon + \sqrt{P(|T| > \epsilon)}}{1 + |x|^p}.$$

Sketch of Proof of Main Theorems

- Theorem 6 is deduced from Theorem 4.
- Combine Lemma 5 and Theorem 6 to prove Theroem 1 and Theorem 2 for $\alpha \geq 1$.
- For $0 < \alpha < 1$, note that from the definition of the Jack_α measure, $P_\alpha(\lambda) = P_{1/\alpha}(\lambda^t)$, where λ^t is the transpose partition of λ.
- Also from its defintion, $W_{n,\alpha}(\lambda) = -W_{n,1/\alpha}(\lambda^t)$.

Summary

- The Jack_α measure on partitions of a positive integer is a discrete analog of Dyson's β ensembles in random matrix theory.
- For $\alpha = 1$, the Jack_α measure agrees with the Plancherel measure on the irreducible representations of the symmetric group.
- We obtained both uniform and non-uniform Berry-Esseen bounds for $W_{n,\alpha} = \frac{\sum_i \left(\alpha\binom{\lambda_i}{2} - \binom{\lambda_i'}{2}\right)}{\sqrt{\alpha\binom{n}{2}}}$, where $\alpha > 0$ and the partition λ is chosen from the Jack_α measure. If $\alpha = 1$, $W_{n,\alpha}$ coincides with $\sqrt{\binom{n}{2}}\frac{\chi^\lambda(12)}{d_\lambda}$.
- For $\alpha \geq 1$, we came close to solving a conjecture of Fulman (2004).

Thank You