

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

- (1) Choose X uniformly on $[0, 2]$, and then Y uniformly on $[0, X]$. Put $Z = XY$.
- (a) Find the joint density $f(x, y)$ of (X, Y) . (b) Find $\mathbb{E}[Z]$.
- (c) Find $\text{Var}(Z)$. (d) Find the density $f_Z(z)$ of Z .
- (2) Let $X_n \in \mathbb{R}^n$ be a sequence of random vectors, whose entries are i.i.d. normal $N(0, 1)$ random variables. For $\vec{x} \in \mathbb{R}^n$, let $\|\vec{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ denote the usual Euclidean norm.
- (a) If X is a normal $N(0, 1)$ variable find $\mathbb{E}[X^4]$. Hint: You may use that $\mathbb{E}[e^{tX}] = e^{t^2/2}$.
- (b) Find $\text{Var}(\|X_n\|_2^2)$. (c) Show that $n^{-1/2}\|X_n\|_2 \rightarrow 1$ in probability.
- (d) Show that $\sqrt{n} \cdot \left(\frac{\|X_n\|_2^2}{n} - 1 \right)$ converges weakly and identify the limit.
- (3) Eight tennis players enter a single-elimination tournament arranged in a fixed 8-player bracket (a “ladder”), see Figure 1. The starting places of the players on the ladder are chosen uniformly at random among all $8!$ possible permutations.
- (a) Assume that the players can be ranked $1, 2, \dots, 8$ in order of strength and that in every match the stronger player always wins. The loser of the final receives the runner-up prize. What is the probability that the second strongest player (player 2) is the runner-up?
- (b) Assume now that all players are evenly matched, and each player wins each game independently with probability $1/2$. Suppose that two of the players are Alice and Bob. What is the probability that Alice and Bob play with each other?
- (c) Generalize your answers to parts (a) and (b) to a tournament with 2^n players for all $n \geq 1$. Simplify your answers.

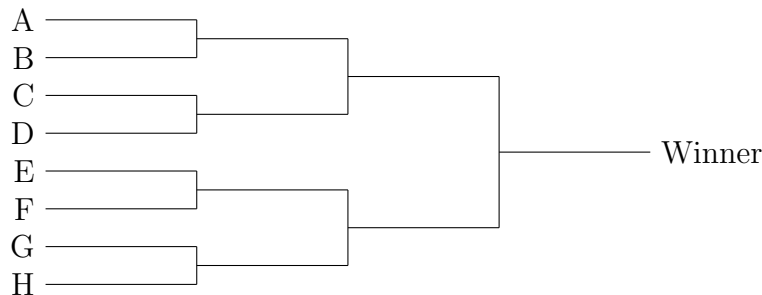


Figure 1: Eight-player tournament ladder