

Solve all the problems. Partial credit will be awarded, but in the event that you cannot fully solve a problem, state clearly what it is you have done and what you have left out. Unacknowledged omissions and guesswork can only lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, **if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.** Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1. Let U and R be independent random variables. Assume that $U \sim \text{Uniform}(0, 2)$ and that

$$\mathbb{P}\left(R = \frac{1}{2}\right) = \frac{4}{5}, \quad \mathbb{P}(R = 1) = \frac{1}{5}.$$

A sensor is placed at location U and covers every point within distance R of its location, restricted to the interval $[0, 2]$. Thus, its coverage set is

$$C = [U - R, U + R] \cap [0, 2].$$

Let L denote the length of C .

1. Express L as a function of U and R (hint: consider the cases when $R = \frac{1}{2}$ and $R = 1$ separately).
2. Compute $\mathbb{E}[L]$ and $\text{Var}(L)$.
3. Let $T \sim \text{Uniform}(0, 2)$ be independent of U and R . Compute $\mathbb{P}(T \in C)$. (hint: first, find the conditional probability given U and R)

Problem 2. Romeo has two urns, n blue balls and n red balls. He may distribute the balls into the urns any way he likes such that each urn must have at least one ball in it. Romeo chooses one urn at random and then randomly draws one ball from it. If the ball is blue, he wins.

1. Let b_k and r_k be the number of blue balls and red balls in urns $k = 1, 2$, respectively. Show that the probability that Romeo wins equals

$$P(\text{win}) = \frac{1}{2} \left(b_1 \left(\frac{1}{n_1} - \frac{1}{2n - n_1} \right) + \frac{n}{2n - n_1} \right)$$

where $n_1 = b_1 + r_1$ is the total number of balls in urn 1.

2. Recall that Romeo wants to distribute the balls in urns in order to maximize his probability of winning. What is the optimization problem that he should solve? State all the constraints carefully.

3. Find the best strategy for Romeo by solving the problem you stated in part (b)
(hint: first, consider the case when $n_1 < n$)

Problem 3. Let (X_1, X_2) have bivariate normal distribution with mean $\mu = (1, 0)^\top$ and covariance matrix $\Sigma = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$.

1. Find $a \in \mathbb{R}$ such that $\text{Cov}(X_1 - aX_2, X_2) = 0$;
2. What is the joint distribution of the random variables $X_1 - aX_2$ and X_2 ? Please justify your answer.
3. Prove that $(X_1 - aX_2)^2$ and $\sin(X_2)$ are independent random variables;
4. Find the conditional expectation $\mathbb{E}[X_1|X_2]$ (the latter is not a number but a random variable; your answer should be a function depending on X_2).