

MATH 445 Mid-Term Exam 1 with solutions
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Instructions:

- No notes, no books, or other printed materials (including printouts from the web), no collaboration with anybody (or anything, like AI), no calculators.
- Answer all questions, show your work, and clearly indicate your answers; upload the solutions to GradeScope.
- There are five problems, 10 points each.
- $e^z = 1 + z + z^2/(2!) + z^3/(3!) + \dots$; $\sin(z) = z - z^3/(3!) + z^5/(5!) - \dots$; $\cos(z) = 1 - z^2/(2!) + z^4/(4!) - \dots$.

Problem 1.

(a) [5 pts] Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = x^3 y^2 z^4$, ∇f is the gradient of f , and C is a straight line segment from the point $(0, 0, 0)$ to the point $(1, 2, 1)$.

Solution: by the Fundamental Theorem of Calculus,

$$\int_C \nabla f \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 0, 0) = 4.$$

(b) [5 pts] Compute the flux of the vector field

$$\mathbf{F} = (3x + 2xy - z^2) \hat{\mathbf{i}} + (x^3 + z^2 - y^2) \hat{\mathbf{j}} + (xz - yz - 2z^2) \hat{\mathbf{k}}$$

out of the sphere $(x - 1)^2 + (y - 2)^2 + (z + 1)^2 = 9$.

Solution: keeping in mind that the triple integral is over a ball with center at $(1, 2, -1)$ and volume $(4\pi/3)3^3 = 36\pi$, use divergence theorem and the “centroid trick” to conclude

$$\iint \mathbf{F} \cdot d\boldsymbol{\sigma} = \iiint \operatorname{div} \mathbf{F} dV = \iiint (3 + 2y - 2y + x - y - 4z) dV = 36\pi(3 + 1 - 2 + 4) = 216\pi.$$

Problem 2. Consider the function

$$f(z) = \frac{z^2 - 6z + 5}{(z - 1)^5}.$$

- (a) [6pt] Compute the Laurent series expansion of the function $f(z)$ around the point $z_0 = 1$.
(b) [4pt] What type of singularity does the function f have at the point $z_0 = 1$?

Solution: keeping in mind that $z^2 - 6z + 5 = (z - 1)(z - 5)$ and $z - 5 = z - 1 - 4$, we get

$$f(z) = \frac{z - 1 - 4}{(z - 1)^4} = \frac{-4}{(z - 1)^4} + \frac{1}{(z - 1)^3}.$$

so f has pole of order 4 at $z_0 = 1$.

Problem 3. [10 pts] Compute

$$\oint_C \frac{z}{e^{z^2} - 1} dz$$

where C is the circle $|z| = 3$, oriented counterclockwise.

Solution: C encloses five isolated singular points of the function $f(z) = z/(e^{z^2} - 1)$: $z_0 = 0, z_+^2 = 2\pi i, z_-^2 = -2\pi i$. All are simple poles. In particular, because $e^{z^2} = 1 + z^2 + z^4/2 + \dots$ near zero,

we conclude that z_0 is a simple pole with residue 1. The residue at each of the other four poles is $1/(2e^{z^2})$ evaluated at $e^{z^2} = 1$, which is $1/2$. This leads to the final answer $6\pi i$.

Problem 4. [10 pts] Solve the initial value problem

$$y''(x) - xy'(x) + 3y(x) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: plugging in $y(x) = \sum_{k=1}^{\infty} a_k x^k$, we conclude that $a_{2k} = 0, k \geq 2$, and $a_3 = -1/3$, $a_{2k+1} = 0, k \geq 2$, and get the answer $y(x) = a_1 x + a_3 x^3 = (3x - x^3)/3$.

Problem 5. Let $f(x) = \frac{x^5}{8}$, $|x| < 2$, and let $S_f = S_f(x)$, $x \in (-\infty, +\infty)$ be the sum of the Fourier series of f .

- (a) [3pt] Draw the graph of S_f for $x \in [-8, 8]$;
- (b) [2pt] Compute $S_f(6)$
- (c) [2pt] Compute $S_f(-9)$
- (d) [3pt] True or false: the infinite sum in S_f converges uniformly on $[-5, 5]$? Explain your conclusion.

Solution. (a) The graph of $f(x) = x^5/8$ for $|x| < 2$ “looks like” the graph of $f(x) = x^3$ in the sense that it is an odd increasing function with a horizontal tangent at zero. For S_f you take the periodic extension with period 4 and set $S_f(2+4k) = 0$. The points $x_k = 2+4k$, $k = 0, \pm 1, \pm 2, \dots$ are the points of discontinuity of S_f . Note that the function f is not defined there. (b) By above, $S_f(6) = 0$ (c) $S_f(-9) = S_f(-9+8) = S_f(-1) = f(-1) = -1/8$. (d) The partial sums of S_f are continuous everywhere on the real line; if the convergence were uniform, the limit S_f would also be continuous. Because S_f is not continuous on $[-5, 5]$, the convergence is not uniform.