MATH 507a: The Final Project

Instructions:

- Do at least three out of the following five problems, and try your best with the last one.
- You are welcome to use any software package and any help, including any on-line resource you can find.
- If you are not comfortable with computers, start early and ask for help often.
- Do not miss classes: throughout the semester, we will discuss various questions related to the project.

Problem 1. Find a procedure for sampling uniformly on the surface of the sphere.

(a) Use computer to generate a thousand points that are random, independent, and uniform on the unit sphere, and print the resulting picture.

(b) By putting sufficiently many independent uniform points on the surface of the Earth (not literally but using a computer model, of course), estimate the areas of Antarctica and Africa, compare your results with the actual values [about 5.5 million square miles/15 million square kilometers for Antarctica and about 12 million square miles/30 million square kilometers for Africa], and make a few comments (e.g. are the relative errors similar? would you expect them to be similar? if not, which one should be bigger? how does accuracy improve if you use more points? etc.) *You can also skip this part if you find programming too challenging.*

Problem 2. Get a computer program for distinguishing a randomly generated sequence of zeroes and ones from a cooked-up one. You are welcome to write the program yourself or use what can you find on the web or in some book. Test your program on the following two sequences: the sequence consisting of the concatenation of all numbers in binary form[1]

$$0\,1\,10\,11\,100\,101\,110\,111\,1000\ldots$$

and a similar sequence consisting of the concatenation of all prime numbers in binary form

$$0\,1\,10\,11\,101\,111\,1011\,\ldots$$

The first sequence (the fractional part of the Champernowne number) is known to be random when considered in base 10; the second sequence (the fractional part of the Copeland-Erdös constant in binary form) is known to be random. In both cases, randomness is understood in a very specific way,[2] and you are welcome to discuss this point too.

Problem 3. Generate a sample path of the Poisson process. Try the following two ways: (a) Set up an "exponential clock" and jump every time the clock "rings" (b) Given the time interval, generate the number of events as a Poisson random variable and then generate the times of events using the corresponding number of iid uniform random variables. Try to include the intensity of the Poisson process as a parameter in your procedure. Can you think of any other ways of generating the process? (c) Generate a two-dimensional picture: a realization of the Poisson point process in a unit square (or any other region of your choice).

Problem 4. Generate a random variable having a symmetric α-stable distribution for a given $\alpha \in (0, 2)$, $\alpha \neq 1$.

Problem 5. Generate a random orthogonal matrix. For the "minimal effort solution" you can only consider the 2×2 case.

Problem 6. Create and solve your own comprehensive, Ph.D.-level exam in measure-theoretic probability.

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[1] spacing is introduced only for convenience: to indicate how the numbers are appearing

[2] The official name is *normal*, as in *normal number*; while it is known that most numbers are normal in every base, it is very hard to show that a particular number is normal, even in one particular base.