A Summary of Some Connections Among Some Probability Distributions.[1]

Notations

- $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt,\ x>0$, the Gamma function, $\Gamma(n+1)=n!$ for positive integer n;
- $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \ x,y>0$, the Beta function, $B(m+1,n+1) = m!n!/(m+n+1)!$ for positive integer m,n;
- For random variables X,Y,V, expressions of the form $V=X+Y$, $V=X/Y$, etc. assume that X and Y are independent and equalities between random variables are understood in distribution;
- $\mathcal{B}(n,p)$, Binomial random variable,

$$\mathbb{P}\big(\mathcal{B}(n,p)=k\big) = \binom{n}{k} p^k(1-p)^{n-k}; \tag{1}$$

- $\mathcal{P}(\lambda)$, Poisson random variable,

$$\mathbb{P}\big(\mathcal{P}(\lambda)=k\big) = e^{-\lambda}\frac{\lambda^k}{k!}; \tag{2}$$

- $\mathcal{N}(0,1)$, standard normal (Gaussian) random variable,

$$\mathbb{P}\big(\mathcal{N}(0,1)\le x\big) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt; \tag{3}$$

- $\mathcal{C}(0,1)$, standard Cauchy random variable,

$$\mathbb{P}\big(\mathcal{C}(0,1)\le x\big) = \int_{-\infty}^{x} \frac{dt}{\pi(1+t^2)}; \tag{4}$$

- $\mathfrak{G}(a,\nu)$, Gamma random variable *with rate* ν,

$$\mathbb{P}\big(\mathfrak{G}(a,\nu)\le x\big) = \int_0^x \frac{t^{a-1}\nu^a}{\Gamma(a)}\,e^{-\nu t}\,dt; \tag{5}$$

- $\mathfrak{B}(r,s)$, Beta random variable,

$$\mathbb{P}\big(\mathfrak{B}(r,s)\le x\big) = \int_0^x \frac{t^{r-1}(1-t)^{s-1}}{B(r,s)}\,dt; \tag{6}$$

- $\chi^2_n = \mathfrak{G}(n/2,1/2) =$ sum of n iid $\big(\mathcal{N}(0,1)\big)^2$, Chi-squared distribution, or random variable, with n degrees of freedom;
- $t_n = \frac{\sqrt{n}\mathcal{N}(0,1)}{\sqrt{\chi^2_n}}$, t distribution, or random variable, with n degrees of freedom;
- $F_{m,n} = \frac{n\chi^2_m}{m\chi^2_n}$, F distribution, or random variable, with m and n degrees of freedom.

The main notational challenge is to distinguish between $\mathcal{B}$ and $\mathfrak{B}$.

[1]Sergey Lototsky, USC. Most recent update on March 10, 2024.

Connections

$$\mathcal{C}(0,1)=\frac{\mathcal{N}(0,1)}{\mathcal{N}(0,1)}=t_1,\quad F_{1,n}=t_1^2,\quad F_{m,n}=\frac{1}{F_{n,m}};\tag{7}$$

$$\mathfrak{B}(a,b)=\frac{\mathfrak{G}(a,\nu)}{\mathfrak{G}(a,\nu)+\mathfrak{G}(b,\nu)};\tag{8}$$

$$\mathbb{P}\left(\mathfrak{B}(m,n)<\frac{m}{m+nx}\right)=\mathbb{P}\big(F_{2n,2m}>x\big);\tag{9}$$

$$\mathbb{P}\big(\mathcal{B}(n,p)\le N\big)=\mathbb{P}\big(\mathfrak{B}(N+1,n-N)>p\big);\tag{10}$$

$$\mathbb{P}\big(\mathcal{B}(n,p)\ge N\big)=\mathbb{P}\big(\mathfrak{B}(N,n-N+1)\le p\big);\tag{11}$$

$$\mathbb{P}\big(\mathcal{P}(\nu T)\le N\big)=\mathbb{P}\big(\mathfrak{G}(N+1,\nu)>T\big);\tag{12}$$

$$\mathbb{P}\big(\mathcal{P}(\nu T)>N\big)=\mathbb{P}\big(\mathfrak{G}(N+1,\nu)\le T\big).\tag{13}$$

Outlines of the proofs

- (8): use the two-dimensional change of variables with $X=\mathfrak{G}(a,\nu)$, $Y=\mathfrak{G}(b,\nu)$, $U=X+Y$, $V=X/(X+Y)$.
- (9): on the right, using characterizations of F and χ^2, write

  $$F_{2n,2m}=\frac{m\chi^2_{2n}}{n\chi^2_{2m}}=\frac{m\mathfrak{G}(n,1/2)}{n\mathfrak{G}(m,1/2)}$$

  so that

  $$\mathbb{P}\big(F_{2n,2m}>x\big)=\mathbb{P}\left(\mathfrak{G}(n,1/2)>\frac{nx}{m}\mathfrak{G}(m,1/2)\right)=\mathbb{P}\left(\mathfrak{G}(n,1/2)+\mathfrak{G}(m,1/2)>\left(\frac{nx+m}{m}\right)\mathfrak{G}(m,1/2)\right),$$

  and then apply (8).
- (10): for fixed $n>N\ge 1$, let

  $$H(p)=\mathbb{P}\big(\mathcal{B}(n,p)\le N\big)=\sum_{k=0}^{N}\binom{n}{k}p^k(1-p)^{n-k}.$$

  Differentiation with respect to p (and using the product rule for each $p^k(1-p)^{n-k}$) leads to a *telescoping sum*

  $$H'(p)=\sum_{k=1}^{N}\frac{n!}{(k-1)!(n-k)!}\,p^{k-1}(1-p)^{n-k}-\sum_{k=0}^{N}\frac{n!}{k!(n-(k+1))!}\,p^k(1-p)^{n-(k+1)},$$

  which simplifies to

  $$H'(p)=-\frac{n!}{N!(n-(N+1))!}\,p^N(1-p)^{n-N-1};$$

  the right-hand side is the negative of the pdf of $\mathfrak{B}(N+1,n-N)$. To conclude, notice that $H(1)=0$ and so, by the Fundamental Theorem of Calculus,

  $$H(p)=-\int_p^1 H'(t)\,dt,$$

  as desired. While the final equality also holds in the special case $N=0$, this case can be considered separately:

  $$\mathbb{P}\big(\mathcal{B}(n,p)\le 0\big)=\mathbb{P}\big(\mathcal{B}(n,p)=0\big)=(1-p)^n=n\int_p^1(1-t)^{n-1}\,dt=\mathbb{P}\big(\mathfrak{B}(1,n)>p\big).$$

- (11): same computations, but now with $H(p) = \mathbb{P}\big(\mathcal{B}(n,p) \geq N\big)$ and an observation that $H(0) = 0$. This time, the special case is $N = n$.
- For (12) and (13), one approach is to replicate the derivations of (10) and (11). An alternative argument relies on analysis of events in a POISSON PROCESS $\mathbf{N}_\nu = \mathbf{N}_\nu(t)$, $t \geq 0$, with intensity ν, keeping in mind that the intervals between events are iid exponential random variables with mean $1/\nu$, that is $\mathfrak{G}(1,\nu)$, and so $\mathfrak{G}(k,\nu)$ is the time of k-th event. For example,
$$\mathbb{P}\big(\mathcal{P}(\nu T) \leq N\big) = \mathbb{P}\big(\mathfrak{G}(N+1,\nu) > T\big),$$
both representing the probability that $\mathbf{N}_\nu(T) \leq N$: the $N+1$-st event happens after time T. `Then a natural question arises: what stochastic process would lead to similar interpretations of` (10) `and` (11)`?`

A bit of history: who introduced some of the distribution and when.

- $\mathcal{B}$: around 1700 by JACOB (or James or Jacques) BERNOULLI (1655–1705, Swiss);
- $\mathcal{P}$: around 1835 by SIMÉON DENIS POISSON (1781–1840, French);
- $\mathcal{N}$: around 1800 by JOHANN CARL FRIEDRICH GAUSS (1777–1855, German mathematician, who also practiced astronomy, geodesy, and physics);
- χ^2, t: around 1875 by FRIEDRICH ROBERT HELMERT (1843–1917, German geodesist), and then re-discovered, in early 1900-s, *in English*, by WILLIAM SEALY 'STUDENT' GOSSET (1876–1937) and KARL (CARL) PEARSON (1857–1936) [who also introduced $\mathfrak{B}$ around the same time, and founded the first ever Department of Statistics (1911, UCL)];
- F: the original idea can be traced back to SIR RONALD AYLMER FISHER (1890–1962), who, in early 1920-s introduced something similar, but in a slightly different form and using a different letter (z); the current construction, including the letter F in honor of Fisher, first appeared in early 1930-s in a book by American mathematician GEORGE WADDEL SNEDECOR (1881–1974) [who also founded the first Department of Statistics in the USA, at Iowa State University, starting with a *Statistics Laboratory* in 1933, as an institute under the President's office, and eventually becoming an Academic Department in 1947.]