You are encouraged to disagree with everything that follows.

Homework 1.

PROBLEM 1.

(1) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle -1, 0, 2\rangle - \langle 1, 1, 1\rangle = \langle -2, -1, 1\rangle$, $\overrightarrow{PR} = \langle 0, -2, -2\rangle$, $\overrightarrow{PS} = \langle a, -1, -2a\rangle$.

(2) The vertex of the angle is P, so you need $\overrightarrow{PR}\cdot\overrightarrow{PS} = 0$, or $2+4a = 0$. Therefore, $\boxed{a = -1/2}$.

(3) The area is $(1/2)|\overrightarrow{PQ} \times \overrightarrow{PR}|$ and

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 1 \\ 0 & -2 & -2 \end{vmatrix} = \langle 4, -4, 4\rangle = 4\langle 1, -1, 1\rangle.$$

Consequently, the area is $\boxed{2\sqrt{3}}$.

(4) The normal vector to the plane is any vector parallel to $\overrightarrow{PQ} \times \overrightarrow{PR}$, for example, $\langle 1, -1, 1\rangle$. Taking P as the point on the plane, we get the equation $(x-1) - (y-1) + (z-1) = 0$ or $\boxed{x - y + z = 1}$.

(5) You are welcome to compute the scale triple product using the determinant, but, given the work you already did, you do not have to compute another determinant. The answer is $4|1-a|$ (it has to be non-negative).

(6) You want the coordinates of S to satisfy the equation of the plane through P, Q, R, that is, $1 + a - 0 + 1 - 2a = 1$ or $\boxed{a = 1}$. You can also see it immediately from the volume formula.

(7) The direction vector for the line is the normal vector to the plane, that is, $\langle 1, -1, 1\rangle$. Then the equation of the line is

$$\mathbf{r}(t) = \langle 1 + t, -t, 1 + t\rangle.$$

At the point of intersection, $(1 + t) - (-t) + (1 + t) = 1$ or $t = -1/3$, so the point is $(2/3, 1/3, 2/3)$.

PROBLEM 2.

(1) $\mathbf{r}(1) = \langle 0, 1, 2\rangle$, so the coordinates of the point are $\boxed{(0, 1, 2)}$.

(2) $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \langle -2t, 3t^2, 2t\rangle$.

(3) $|\mathbf{v}(t)| = \sqrt{4t^2 + 9t^4 + 4t^2} = t\sqrt{8 + 9t^2}$.

(4) $\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \langle -2, 6t, 2\rangle$.

(5) The particle is at $(0, 1, 2)$ when $1 - t^2 = 0$ or $t = 1$ (by assumption, $t \geq 0$). Therefore, the equation of the tangent line is $\mathbf{R}(u) = \langle 0, 1, 2\rangle + u\dot{\mathbf{r}}(1)$. Next, $\dot{\mathbf{r}}(1) = \langle -2, 3, 2\rangle$, and so the equation of the line is $\boxed{\mathbf{R}(u) = \langle -2u, 1 + 3u, 2 + 2u\rangle}$.

(6) You want the coordinates of the particle to satisfy the equation of the plane. Then $1 + t^2 - (1 - t^2) = 2$ or $t = 1$ (remember, $t \geq 0$) So the point of intersection is $\boxed{(0, 1, 2)}$.

(7) According to the formula, the distance is

$$\int_0^1 |\mathbf{v}(t)|dt = \int_0^1 \sqrt{8 + 9t^2}\, t\, dt = \text{(simply guess antiderivative)}\ \frac{1}{27}(8 + 9t^2)^{3/2}\Big|_{t=0}^{t=1} = \boxed{\frac{17^{3/2} - 8^{3/2}}{27}}.$$

Homework 2.

PROBLEM 1.

(1) $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \langle 4x - y - 1, 2y - x + 1\rangle$.

(2) The direction vector is $\mathbf{a} = \langle -1, -1\rangle$. Therefore, the rate is

$$\frac{\nabla f(1, 1) \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{\langle 2, 2\rangle \cdot \langle -1, -1\rangle}{\sqrt{2}} = \boxed{-4/\sqrt{2}}.$$

The rate is negative, so the function is *decreasing* in that direction.

(3) The direction is given by $-\nabla f(1, 1) = \langle -2, -2\rangle$, which is toward the origin. The rate of change is $-|\nabla f(1, 1)| = -2\sqrt{2}$, which is, not surprisingly, the same as the rate of change toward the origin from the previous question.

(4) The path is $\mathbf{r}(t) = (x(t), y(t), z(t))$, where $\dot{x}(t) = 4x - y - 1$, $\dot{y}(t) = -x + 2y + 1$, $x(0) = y(0) = 0$, and $z(t) = 2x^2(t) - x(t)y(t) + y^2(t) - x(t) + y(t) - 1$. This is an inhomogeneous linear system of two ODEs and can be solve using standard methods (first, get the general solution for homogeneous equation via eigenvalues/eigenvectors [this will have two "arbitrary constants"], then "guess" a particular solution of the (full) inhomogeneous equation, and then identify the "arbitrary constants" by plugging in the initial conditions).

On the topographic map (that is, the set of points $(x(t), y(t))$), the path is a parabola of the type $y = x^\alpha$, although twisted and turned. The reason is that the level sets of the function are ellipses, also twisted and turned.

PROBLEM 2.

(1) 2π (the integrand is a potential field)
(2) 8 (Green's theorem)
(3) 5π (Stokes)

PROBLEM 3.

(1) $\sqrt{2}(2\pi + (8\pi^3/3))$ (direct integration)
(2) 3π (Green's theorem is a better choice)
(3) $1/2$ (direct integration in spherical coordinates)
(4) 4π (a better approach is to close up the surface and use the divergence theorem, then subtract the extra flux through the top; the flux through the bottom is zero).

PROBLEM 4. Draw the picture. For some orders of integration, you will have to break the region into several pieces.

Homework 3.

PROBLEM 1.

$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial f}{\partial\varphi}\right).$$

PROBLEM 2.

(1) $\frac{-11-2i}{25}$
(2) Note that $(1+i)/(1-i) = i$, so n must be a multiple of 4.
(3) $\sqrt{2}\exp(-3i\pi/4 + 2i\pi n)$
(4) The four solutions are $z_1 = \sqrt[4]{2}\exp(i\pi/8)$, $z_2 = \sqrt[4]{2}\exp(i\pi/8 + i\pi)$ $z_3 = \sqrt[4]{2}\exp(-i\pi/8)$, $z_4 = \sqrt[4]{2}\exp(-i\pi/8 + i\pi)$
(5) $\int e^{-2x}\sin(3x)dx = Im\left(\int e^{(-2+3i)x}dx\right) = Im\left(\frac{e^{(-2+3i)x}}{-2+3i}\right) = (e^{-2x}/13)Im((\cos(3x) + i\sin(3x))(-2-3i)) = \boxed{(-e^{-2x}/13)(2\sin(3x) + 3\cos(3x))}$.
(6) $\sqrt[6]{-i} = \exp(-i\pi/12 + i\pi n/3$, $n = 0, 1, 2, 3, 4, 5$.
(7) One of them is $\sqrt{6}$.
(8) $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

PROBLEM 3. (a)

$$\begin{aligned} u^2 + u &= \alpha^2 + 2\alpha^5 + \alpha^8 + \alpha + \alpha^4 = \alpha^2 + 2 + \alpha^3 + \alpha + \alpha^4; \\ u^2 + u - 2 &= \alpha + \alpha^2 + \alpha^3 + \alpha^4; \\ \alpha(u^2 + u - 2) &= \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5; \\ (1-\alpha)(u^2 + u - 2) &= \alpha - \alpha^5 = \alpha - 1. \end{aligned}$$

$$\alpha^4 = e^{8\pi i/5} = \cos(8\pi/5) + i\sin(8\pi/5) = \cos(2\pi - (8\pi/5)) - i\sin(2\pi - (8\pi/5)) = \bar{\alpha}.$$

(b) $Re\alpha = 2u$.

(c) Have fun! Gauss did, with the regular 17-gon when he was 17 years old, and the rest is history.

Homework 4.

PROBLEM 1.

(1) $z^3 - 2z + 1 = (x+iy)^3 - 2(x+iy) + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 - 2x + 2iy + 1$, so $\boxed{Re(f) = x^3 - 3xy^2 - 2x + 1,\ Im(f) = 3x^2y - y^3 + 2}$. It is analytic everywhere, because it is a polynomial (or you can verify the Cauchy-Riemann equations).

(2) $f(z) = (x+iy)(\cos y + i\sin y)e^x$; $Re(f) = e^x(x\cos y - y\sin y)$. It is analytic.

(3) $f(z) = (e^{iz} + e^{-iz})/2$; $Re(f) = \cos x \cosh y$ (hyperbolic functions appear here).

(4) If f is analytic everywhere and $Re(f) = 0$, then Cauchy-Riemann equations imply that $v = Im(f)$ satisfies $v_x = v_y = 0$ everywhere, or $\boxed{v = Im(f) = const}$.

(5) If $u(x,y) = ax^3 + bxy$, then $u_{xx} + u_{yy} = 6ax$. Therefore, $a = 0$ and b can be any real number $u(x,y) = bxy$. To find conjugate harmonic v we write $u_x = by = v_y$, $u_y = bx = -v_x$; one of the solutions is $\boxed{v(x,y) = b(y^2 - x^2)/2}$. Note that the resulting $f(z) = u(x,y) + iv(x,y)$ is $f(z) = -ibz^2/2$.

PROBLEM 2. Under the map $f(z) = 1/z$:

(1) $\{z : |z| < 1\}$ becomes $\{z : |z| > 1\}$ (kind of obvious)

(2) $\{z : Re(z) > 1\}$ becomes $\{z : |z - 1/2| < 1/2$: the circle $(x - 1/2)^2 + y^2 = 1/4$ is the image of the line $x = 1$.

(3) $\{z : 0 < Im(z) < 1\}$ becomes $\{z : Im(z) < 0$ and $|z + 1/2i| > 1/2\}$: again, the circle $x^2 + (y + 1/2)^2 = 1/4$ is the image of the line $y = 1$.

Homework 5.

PROBLEM 1.

(1) $R = (18/10)^{1/4}$

(2) $R = 37^{-1/4}$

(3) $R = 2^{2/3}$

(4) $R = \sqrt{e}$

(5) $R = 9e^2/4$.

PROBLEM 2.

(1) $\sum_{n\geq 0} \frac{(-2)^n}{3^{n+1}} z^n$, $R = 3/2$

(2) $1 + (z-1) + \sum_{n\geq 2} \frac{(-1)^n}{2^n}(z-1)^n,\ R = 2$

(3) $\sum_{n\geq 0}(n+1)2^n z^n,\ R = 1/2$ (differentiate a suitable function)

(4) $f(z) = \dfrac{1}{2i}\left(\dfrac{1}{z+1-i} - \dfrac{1}{z+1+i}\right)$ (partial fractions). You can take it from here. $R = \sqrt{2}$.

PROBLEM 3.

(1) $\dfrac{1}{2z} + \sum_{n\geq 0} \dfrac{z^n}{2^{n+2}}$

(2) $\sum_{n\geq 1} \dfrac{(-1)^n 2^{n-1}}{(z-2)^{n+1}}$

(3) $-\dfrac{1}{2(z-2)} + \sum_{n\geq 0}(-1)^n \dfrac{(z-2)^n}{2^{n+2}}$

(4) $\dfrac{1}{2}\sum_{n\geq 0} \dfrac{1}{(z+1)^{n+1}} + \dfrac{1}{6}\sum_{n\geq 0} \dfrac{(z+1)^n}{3^n}$

PROBLEM 4.

(1) removable

(2) removable

(3) second-order pole

(4) essential

(5) not an isolated singularity

A good variation would be to find all other singular points of these functions and determine their type.

Problem 5. Note that the equations of Laguerre and Bessel have regular singular points at zero and therefore might not have all solution represented as a power series. In that case, the Fuchs-Frobenius theory is the way to go.

Homework 6.

Problem 1.

(1) 3
(2) 33
(3) $-\pi i$
(4) -3π
(5) $-1/6$.

Problem 2.

(1) $8\pi^2$
(2) $2\pi/3$
(3) $\pi\sqrt{3}/72$

Problem 3.

(1) $w(z) = z + \displaystyle\sum_{k\ge 1} \frac{z^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3k)\cdot(3k+1)}$.
(2) $w(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2n}$.
(3) $w(z) = z^2 - 1$.

Problem 4.

(1) $\lambda = 2n$ (Hermite polynomials)
(2) $\lambda = n^2$ (Chebyshev polynomials of the first kind)
(3) $\lambda = n(n+1)$ (Legendre polynomials)
(4) $\lambda = n$ (Laguerre polynomials)

The main thing to keep in mind is that if $w(z) = \sum_{k\ge 0} a_k z^k$, then

$$w''(z) = \sum_{k\ge 2} k(k-1)a_k z^{k-2} = \sum_{k\ge 1}(k+1)k a_{k+1} z^{k-1} = \sum_{k\ge 0}(k+1)(k+2)a_{k+2} z^k.$$

Homework 7.

Problem 1. $\sum_{n\ge 1} z^n/n$. On the boundary you have $|z - z_0| = R$, so absolute convergence of $\sum_n a_n(z-z_0)^n$ even at one point of the boundary implies convergence of $\sum_n |a_n|R^n$.

Problem 2.

(1) 0, not uniform: $(1-(1/n))^n \to 1/e \ne 0$;
(2) 0, uniform: $|\sin(x/n)| \le |x/n| \le 4/n$;
(3) 1, not uniform: if $x = 1/n$, you get $1/2$;
(4) x^2, uniform: $|nx^2/(n+x) - x^2| \le 1/n$.

Problem 3.

(1) absolutely but not uniformly
(2) absolutely and uniformly
(3) absolutely and uniformly
(4) absolutely but not uniformly

Problem 4. Draw the pictures. Then everything is clear:

(1) $g(x) = (2/\pi)f(\pi(x+1/2)) - 1$,
(2) $g(x) = 2f(2\pi x) - 1$,
(3) $g(x) = (1/2\pi)\Big(f(2\pi x - \pi) + \pi\Big)$.

Problem 5.

(1) $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k\ge 0}\frac{\cos((2k+1)x)}{(2k+1)^2}$, $g(x) = \frac{8}{\pi^2}\sum_{k\ge 0}\frac{(-1)^k\sin(\pi(2k+1)x)}{(2k+1)^2}$.

(2) $f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k\ge 0}\frac{\sin((2k+1)x)}{2k+1}$, $g(x) \sim \frac{4}{\pi}\sum_{k\ge 0}\frac{\sin(2\pi(2k+1)x)}{2k+1}$.

(3) $f(x) \sim 2\sum_{k\ge 1}\frac{(-1)^{n+1}}{n}\sin(nx)$, $g(x) \sim \frac{1}{2} - \frac{1}{\pi}\sum_{k\ge 1}\frac{1}{n}\sin(nx)$.

(A discontinuous function is not equal to its Fourier series. This is why sometimes I write $=$ and sometimes $\sim$.)

Problem 6. Only option (b) results in the *continuous* periodic function. For continuous periodic functions, the Fourier series converges better than for discontinuous; therefore, I would go with option (b).

Problem 7.

(1) $\pi/4$
(2) $\pi^4/96$
(3) $\pi^2/8$
(4) $\pi^2/6$

Problem 8. The graph is the periodic (with period 2) extension of f, except that $S_f(k) = 0$ for $k = 0, \pm 1, \pm 2, \ldots$. Therefore, $S_f(3) = 0$ and $S_f(5/2) = S_(1/2) = 1$.

Homework 8.

Problem 1.

(1) $\hat{f}(w) = \frac{1}{\sqrt{2\pi}(2+iw)}$ (direct integration).

(2) $\sqrt{2\pi}\hat{f}(w) = \frac{i}{w}(be^{-iwb} - ae^{-iaw}) + \frac{1}{w^2}(e^{-iwb} - e^{-iaw})$ (direct integration by parts).

(3) $\hat{f}(w) = \sqrt{2/\pi}\frac{\sin w}{w(1+w^2)}$.

Problem 2. $\hat{f}(w) = \sqrt{2/\pi}\frac{1}{1+w^2}$. Then, using the formula for the inverse Fourier transform, and keeping in mind that $e^{iwx} = \cos(wx) + i\sin(wx)$, where cos is even and sin, odd, we get $\int_0^\infty \frac{\cos(wx)}{1+w^2}dw = \frac{\pi}{2}e^{-|x|}$. Also, the result means that the Fourier transform of $g(x) = 1/(1+x^2)$ is $\hat{g}(w) = \sqrt{\pi/2}e^{-|w|}$. Therefore, by Parseval's identity,

$$\int_{-\infty}^{+\infty}\frac{dx}{(1+x^2)^2} = \frac{\pi}{2}\int_{-\infty}^{+\infty}e^{-2|w|}dw = \pi\int_0^{+\infty}e^{-2w}dw = \frac{\pi}{2}.$$

Problem 3. (a) $g(x) = f(\sqrt{2a}x)$, so $\hat{g}(w) = \frac{1}{\sqrt{2a}}\hat{f}(w/\sqrt{2a}) = (\sqrt{2a})^{-1}e^{-x^2/(4a)}$. (b) The Fourier transform of $f(t) = 1/(1+t^2)$ is, not surprisingly, $\widehat{f}(\omega) = \sqrt{2/\pi}e^{-|\omega|}$. Then note that $a/(b+ct^2) = (a/b)/(1+(\sqrt{c/b}t)^2)$ and use linearity and scaling.

Problem 4. Just move the integrals around.

Problem 5. Interpret u times something as a Fourier transform of f times something; then use Parseval. Keep in mind that $|e^{it}| = 1$ for all real t.

Homework 9.

Problem 1. Transport equation: $F(x+\cos t)$, where F is an arbitrary continuously differentiable function.

Problem 2. Method of characteristic: $F(x^{-1}+y^{-1})$, where F is an arbitrary continuously differentiable functions.

Homework 10.

Problem 1.

(1) $F''/F + G''/G = 0$, $F'' = cF$, $G'' = -cG$. If $c = a^2$, we get $F(x) = C_1\sinh(ax) + C_2\cosh(ax)$, $G(y) = C_3\sin(ay) + C_4\cos(ay)$ and $u(x,y) = (C_1\sinh(ax) + C_2\cosh(ax))(C_3\sin(ay) + C_4\cos(ay))$, C_i are arbitrary constants. If $c = -a^2$, then just switch x and y in the above expression for u. If $c = 0$, then $u = (C_1 + C_2x)(C_3 + C_4y)$.

(2) $F'/(x^2F) = G'/(y^2G) = 3c$, $u(x,y) = Ae^{c(x^3+y^3)}$. In fact, the general solution is $u(x,y) = h(x^3+y^3)$, where $h = h(s)$ is a continuously-differentiable function.

(3) $(F'/F) - x = -(G'/G) + y = c$, $u = Ae^{c(x-y)+(x^2+y^2)/2}$.

(4) $xF'/F = -2yG'/G = c$, $u = Ax^c e^{-y^2/c}$, $c \neq 0$; $u = 0$ is also a solution; it is included in the family.

(5) (i) If $a = r = 1$, then $u = x/t$ is one possible solution. (b) If $a = 1$, then one possible solution is $u = x/(1-e^{-t})$.

(6) Let us say you try $u(x,t) = f(t)g(x)$. Then you get

$$\frac{f'}{f}(t) = \frac{(g^\gamma)''}{g}(x) = a$$

with an arbitrary constant a.

A special case is $a = 0$, leading to

$$u(t,x) = (bx+c)^{1/\gamma},$$

with arbitrary constants b, c, which works as long as $bx + c > 0$. If $a \neq 0$, then the equation for f is still easy, and you get

$$f(t) = \Big(a(1-\gamma)t + c\Big)^{1/(1-\gamma)}$$

with an arbitrary constant c, as long as $\gamma \neq 1$ and $a(1-\gamma)t + c > 0$. To solve the equation for g, which is

$$\gamma g^{\gamma-1}g'' + \gamma(\gamma-1)g^{\gamma-2}(g')^2 = ag,$$

proceed as follows. Introduce a new function h by $g'(x) = h(g(x))$ [then, once you know h, you find g by solving this equation]. Then $g'' = hh'$, and you will get a first-order equation for h

$$\gamma g^{\gamma-1}hh' + \gamma(\gamma-1)g^{\gamma-2}h^2 = ag,$$

in which you now treat g as an independent variable. Next, introduce yet another function $w = h^2$. Then you will get a linear first-order equation for w

$$w' + \frac{2(\gamma-1)}{g}w = \frac{2a}{\gamma}g^{2-\gamma}.$$

Solve it using integrating factor:

$$w(g) = \frac{2a}{\gamma(\gamma+1)}g^{3-\gamma} + bg^{2(1-\gamma)}$$

with an arbitrary constant b. To get back to g, we take $b = 0$, to simplify subsequent computations, and then have to assume $a > 0$ so that

$$h(g) = \sqrt{\frac{2a}{\gamma(\gamma+1)}}\, g^{(3-\gamma)/2},$$

and

$$g(x) = \left(\sqrt{\frac{2a}{\gamma(\gamma+1)}}\cdot\frac{\gamma-1}{2}x + c\right)^{2/(\gamma-1)}$$

with an arbitrary constant c and the usual disclaimer that the expression inside the parentheses must be positive. When $c = 0$, it is actually not hard to check that $(g^\gamma)'' = ag$.

If $\gamma > 1$, you can try a more sophisticated approach and look for a `self-similar solution` in the form

$$u(t,x) = t^{-\alpha}v(x/t^\beta),$$

which is also sort of separation of variables.

The result is highly non-trivial [known as the `Barenblatt` solution, after Grigory Isaakovich Barenblatt (1927–2018)]:

$$u(t,x) = \frac{1}{t^\alpha}\left(b - \frac{\gamma-1}{2\gamma(\gamma+1)}\frac{x^2}{t^{2\beta}}\right)_+^{1/(\gamma-1)}, \quad \alpha = \beta = \frac{1}{\gamma+1},$$

where $b > 0$ is an arbitrary constant and the notation $(y)_+$ stands for $y1_{y>0}$.

Let us see if we can make any sense out of this result.

First of all, a solution $u = u(t, x)$ of any equation is called self-similar if there exist real numbers p, q such that, for every $c > 0$, the function

$$U(t, x, c) = c^p u(ct, c^q x)$$

is also a solution of the same equation. Now, if $u = u(t, x)$ satisfies

$$u_t = (u^\gamma)_{xx}$$

AND has the form $u(t, x) = t^{-\alpha} v(xt^{-\beta})$, then, on the one hand,

$$c^p u(ct, c^q x) = c^{p-\alpha} t^{-\alpha} v(c^{q-\beta} xt^{-\beta}),$$

which will equal $u(t, x)$ if $p = \alpha$ and $q = \beta$. On the other hand,

$$U_t = c^{1+p} u_t, \ (U^\gamma)_{xx} = c^{p\gamma+2q}(u^\gamma)_{xx}$$

so that we need

$$1 + p = p\gamma + 2q.$$

With one equation and two unknowns (p and q), we can further assume that $p = q$, and then we get

$$p = \alpha = \beta = \frac{1}{\gamma + 1}.$$

Now that we know α and β, we need the function $v = v(y)$ such that

$$u(t, x) = t^{-\alpha} v(xt^{-\beta})$$

satisfies $u_t = (u^\gamma)_{xx}$.

Direct substitution, with $y = xt^{-\beta}$, results in the following equation for v:

$$-\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}} yv'(y) = \frac{1}{t^{\alpha\gamma+2\beta}} (v^\gamma(y))''$$

This is not too bad because, with $\alpha = \beta = 1/(\gamma + 1)$, we have

$$\alpha + 1 = \alpha\gamma + 2\beta = \frac{\gamma + 2}{\gamma + 1},$$

so the powers of t go away and we get

(1) $$-\frac{v(y) + yv'(y)}{\gamma + 1} = (v^\gamma(y))''.$$

Now we can confirm that, if we look for a solution in the form

$$v(y) = (b - ay^2)^{1/(\gamma-1)},$$

then

$$v'(y) = -\frac{2ay}{\gamma - 1} v^{-\gamma}(y), \quad (v^\gamma(y))' = -\frac{2a\gamma y}{\gamma - 1} v(y),$$
$$(v^\gamma(y))'' = -\frac{2a\gamma}{\gamma - 1} v(y) + \frac{4a^2\gamma y^2}{(\gamma - 1)^2} v^{-\gamma}(y)$$

and (1) will indeed hold if

$$a = \frac{\gamma - 1}{2\gamma(\gamma + 1)}.$$

Problem 2.

(1)

$$u(x, t) = (1/4) + \frac{1}{4\pi} \sin(4\pi t) \cos(2\pi x) - \sum_{n\geq 0} \frac{2\cos(2(2n + 1)\pi x)}{\pi^2(2n + 1)^2} \cos(4(2n + 1)\pi t).$$

(2) $u(x, t) = \cos(2t) \sin(2x) + \sin(t) \sin(x) + \sin(x) - \cos(t) \sin(x)$.

(3) No solution exists: the initial speed $u_t(x, 0)$ cannot be written in the required form $\sum_{k\geq 1} 2\pi k b_k \cos(\pi kx)$ because $\int_0^1 f(x)dx \neq 0$.

Problem 3. The answer is 0.

Problem 4. $u(x,t) = \big(\sin(x+ct) + \sin(x-ct)\big)/2$; use a suitable trig identity.

Homework 11.

Problem 1.

(1) After separation of variable, you find that $u(x,t) = \sum_{n\geq 1} A_n e^{-a_n^2 t}\sin(a_n x)$, and the second boundary condition implies $a_n = (n-0.5)\pi$.
(2) After separation of variable, you find that $u(x,t) = \sum_{n\geq 1} A_n e^{-a_n^2 t}\sin(a_n x)$, and the second boundary condition implies $a_n = \tan(a_n)$.
(3) Note that $u_0(x,t) = 1-x$ is a solution of the equation, and it satisfies the boundary conditions. Since $u_0|_{t=0} = 1-x$, we find that $u(x,t) = u_0(x,t)+v(x,t)$, where v is the solution of $v_t = v_{xx}$, $t>0$, $0<x<1$, $v|_{t=0} = x-1+u(0,x)$, $v|_{x=0} = 0$, $v|_{x=1} = 0$, which you know how to solve.
(4) Note that $v(t,x) = u(t,x)\exp(-\int_0^t s^2 ds) = u(t,x)e^{-t^3/3}$ satisfies $v_t = v_{xx}$ with the same initial and boundary conditions.

Problem 2.

(1) $u(x,y) = \frac{\sin(x)\sinh(y/2)}{\sinh(\pi/2)}$.
(2) $u(x,y) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{(1-(-1)^n)(1-(-1)^m)}{(m^2+n^2)mn}\sin(mx)\sin(ny)$
(3) $u(r) = \sum_{n\geq 1} c_n J_0(\alpha_n r)$, $r = \sqrt{x^2+y^2}$,
$$c_n = \frac{\int_0^1 J_0(\alpha_n r)rdr}{\alpha_n^2\int_0^1 J_0^2(\alpha_n r)rdr},$$
where $0<\alpha_1<\alpha_2<\alpha_3<\ldots$ are the zeros of Bessel's function J_0.
(4) No solution exists.

Problem 4.

(1) Use the formula for the solution and the results about the Fourier transform to get
$$u(x,t) = \frac{1}{\sqrt{2\pi(1+t)}}e^{-\frac{x^2}{2(1+t)}}.$$
(2) This follows from the formula for the solution because the heat kernel is positive and integrates to one.

Homework 12.

Problem 1.

(1) With a radially-symmetric initial condition, the solution should be radially symmetric as well (no dependence on θ). Therefore the basis functions come from J_0. Writing $\lambda_k = -\alpha_k^2/4$, $\varphi_k(r) = J_0(\alpha_k r/2)$, where $0<\alpha_1<\alpha_2<\alpha_3<\ldots$ are the zeros of Bessel's function J_0, we get
$$u(t,r,\theta) = \sum_{k\geq 1} f_k e^{\lambda_k t}\varphi_k(r),$$
where
$$c_k = \frac{\int_0^2 \varphi_k(r)f(r)rdr}{\int_0^2\varphi_k^2(r)rdr}$$
(2) This time, the initial condition suggests that the basis functions come from J_1 (recall that the general basis function is $J_N(\cdot)\psi(N\theta)$, where ψ is either sin or cos). Writing $\varphi_k(r) = J_1(\beta_k r)$, where $0<\beta_1<\beta_2<\beta_3<\ldots$ are the zeros of Bessel's function J_1, we get
$$u(t,r,\theta) = \frac{\cos(\theta)}{\pi}\sum_{k\geq 1} f_k\cos(\beta_k t)\varphi_k(r),$$
where
$$c_k = \frac{\int_0^1 \varphi_k(r)f(r)rdr}{\int_0^1\varphi_k^2(r)rdr}.$$
The factor $1/\pi$ comes from $\int_0^{2\pi}\cos^2\theta\,d\theta = \pi$.