MATH 445 Mid-Term Exam 2 with solutions
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Instructions:

- No notes, no books or other printed materials (including printouts from the web), no collaboration with anybody (or anything, like AI).
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.

Problem 1. Let $f(x)=|x|,\ |x|<1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f for $x\in[-4,4]$ and evaluate (a) $S_f(3)$ (b) $S_f(5/2)$.

Solution: S_f has period 2 and $S_f(x)=|x|,\ |x|<1$, so S_f is continuous, $S_f(3)=S_f(1)=f(1-)=1,\ S_f(5/2)=S_f(-1/2)=1/2$.

Problem 2. The Fourier transform of the function

$$f(t)=\begin{cases}1, & \text{if } |t|\le 1,\\ 0, & \text{if } |t|>1\end{cases}$$

is $\widehat{f}(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$. Use the result to evaluate the integral $I=\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega$.

Solution: From $1=f(0)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\widehat{f}(\omega)d\omega=2(2\pi)^{-1/2}(2/\pi)^{1/2}I=(2/\pi)I$ we conclude $I=\pi/2$.

Problem 3. The Fourier transform of the function $f(x)=e^{-x^2/2}$ is $\widehat{f}(\omega)=e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x)=xe^{-(x-1)^2}$.

Solution: with $p(x)=f(2^{1/2}x)$ and $h(x)=xp(x)$, we have $g(x)=p(x-1)+h(x-1)$ so that $\widehat{g}(\omega)=e^{-\mathrm{i}\omega}(\widehat{p}(\omega)+\widehat{h}(\omega))$. Next, $\widehat{p}(\omega)=2^{-1/2}\widehat{f}(2^{-1/2}\omega)=2^{-1/2}e^{-\omega^2/4}$, , $\widehat{h}(\omega)=\mathrm{i}\widehat{p}'(\omega)=-\mathrm{i}2^{-1/2}2^{-1}\omega\, e^{-\omega^2/4}$. Putting it all together, $\widehat{g}(\omega)=e^{-\mathrm{i}\omega-(\omega^2/4)}2^{-1/2}(1-\mathrm{i}2^{-1}\omega)$. There are alternative solutions leading to the same answer.

Problem 4. Use separation of variables to find a non-constant solution $u=u(t,x)$ of the equation $u_t=u^2u_x$ such that the function $u=u(t,x)$ is defined for all $x>-1$ and $t<1$.

Solution: $u(t,x)=F(t)G(x),\ \ F'G=F^2G^2FG',\ (F'F^{-3})(t)=(GG')(x)=c,\ -(F^{-2})'=(G^2)'=2c$, $F=(a-2ct)^{-1/2},\ G=(b+2cx)^{1/2}$, taking $2c=a=b=1$ ensures that the domain of F is $t<1$ and the domain of G is $x>-1$, leading to the answer $u(t,x)=\sqrt{(1+x)/(1-t)}$.

Problem 5. Solve the following initial-boundary value problem:

$$\begin{aligned}
u_{tt} &= 25u_{xx},\ u=u(t,x),\ t>0,\ x\in(0,\pi),\\
u(0,x) &= \sin(2x)-3\sin(4x),\\
u_t(0,x) &= 0,\\
u(t,0) &= 0,\\
u(t,\pi) &= 0.
\end{aligned}$$

Solution: from the zero initial and boundary conditions, $u=\sum_{k\ge1}A_k\cos(5kt)\sin(kx)$, and then, from the non-zero initial condition, $A_2=1,\ A_4=-3,\ A_k=0$ otherwise, so $u(t,x)=\cos(10t)\sin(2x)-3\cos(20t)\sin(4x)$.

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \sum_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$
$\sum_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \frac{\tilde{f}(0+)+\tilde{f}(0-)}{2}$	Obvious	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0) = \frac{f(0+)+f(0-)}{2}$
$\lim_{\lvert k\rvert\to\infty} \lvert c_k(f)\rvert = 0$	Riemann-Lebesgue: $f \in L_1$	$\lim_{\lvert\omega\rvert\to\infty} \lvert\hat{f}(\omega)\rvert = 0,\ \hat{f}$ continuous
$\sum_{k=-\infty}^{+\infty} \lvert c_k(f)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert f(x)\rvert^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\int_{-\infty}^{+\infty} \lvert\hat{f}(\omega)\rvert^2\,d\omega = \int_{-\infty}^{+\infty} \lvert f(x)\rvert^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(x)$
$\int f(x)\,dx$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\frac{f(x)}{x}$	$\frac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega\rvert}$
$1(\lvert x\rvert \le 1)$	$\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$	$\frac{\sin\omega}{\omega}$	$\sqrt{\frac{\pi}{2}}1(\lvert x\rvert \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$