MATH 445 Final Exam, Fall 2024

Instructor — S. Lot otsky (KAP 248D; x0–2389; `lototsky@usc.edu`)

Instructions:

- No notes, no books or other printed materials (including printouts from the web), no calculators, no collaboration with anybody (or anything, like AI).
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- **There are nine problems. Each of the problems 1–9 is worth 10 points. Problem 10 is a collection of 5 multiple choice questions**
- $e^z = 1 + z + z^2/(2!) + z^3/(3!) + \cdots$; $\sin(z) = z - z^3/(3!) + z^5/(5!) - \cdots$; $\cos(z) = 1 - (z^2/2!) + (z^4/4!) + \cdots$.

Problem 1.

(a) [5 pts] Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = 2x^3 + y^5 z^7$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,1,1)$.

Solution. By the Fundamental Theorem of Calculus,

$$\int_C \nabla f \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 3.$$

(b) [5 pts] Compute the flux of the vector field $\mathbf{F} = (3x + 2xz)\,\hat{\boldsymbol{\imath}} + (z^2 - y^2)\,\hat{\boldsymbol{\jmath}} + (4z + xz)\,\hat{\boldsymbol{\kappa}}$ out of the sphere $(x+1)^2 + (y-3)^2 + (z-1)^2 = 9$.

Solution. Using the divergence theorem and the fact that the centroid of a ball is its center, we get, with G denoting the ball centered at $(-1,3,1)$,

$$\iint_{\partial G} \mathbf{F} \cdot d\vec{\sigma} = \iiint_G \operatorname{div} \mathbf{F}\, dV = \iiint_G (3 + 2z - 2y + 4 + x) dV$$
$$= \operatorname{vol}(G)(7 + \bar{x} - 2\bar{y} + 2\bar{z}) = (4\pi 3^3/3)(7 - 1 - 6 + 2) = 72\pi.$$

Problem 2.

(a) [5 pts] Compute $\oint_C \frac{z^2}{e^{2z} - 1} dz$, where C is the circle $|z| = 5$, oriented counterclockwise.

Solution. With $f(z) = z^2/(e^{2z} - 1)$, the singular points of f are $z_k = \mathrm{i}\pi k$, $k = 0, \pm 1, \pm 2, \ldots$: those are the points where $e^{2z} = 1$. Of those points, only three (z_0, z_1, z_{-1}) are inside C. Moreover, z_0 is a removable singularity (either by Taylor or L'Hospital, $\lim_{z\to 0} f(z) = 0$), and the other two are simple poles; by the standard formula, the residue at $z_{\pm 1}$ is $(z_{\pm 1})^2/(2e^{z_{\pm 1}}) = -\pi^2/2$. As a result, by the residue theorem,

$$\oint_C \frac{z^2}{e^{2z} - 1} dz = -2\pi^3 \mathrm{i}.$$

(b) [5 pts] Compute the Laurent series expansion of the function $f(z) = \frac{z^2 - 25}{(z-5)^3}$ around the point $z_0 = 5$.

Solution. With $z - 5$ already at the bottom, we need to re-write the function in the form of the Laurent series:

$$f(z) = \frac{(z-5)(z+5)}{(z-5)^3} = \frac{z+5}{(z-5)^2} = \frac{z-5+10}{(z-5)^2} = \frac{10}{(z-5)^2} + \frac{1}{z-5}.$$

Problem 3. Solve the initial value problem

$$y''(x) - xy'(x) + 3y(x) = 0, \quad y(0) = 0, \quad y'(0) = -3.$$

Solution. With $y(x) = \sum_{k\geq 0} a_k x^k$, the equation suggests using $y'(x) = \sum_{k\geq 0} k a_k x^{k-1}$ and $y''(x) = \sum_{k\geq 0} a_{k+2}(k+1)(k+2)x^k$, Then

$$\sum_{k\geq 0} \big((k+1)(k+2)a_{k+2} - (k-3)a_k\big)x^k = 0, \quad a_{k+2} = \frac{k-3}{(k+1)(k+2)}\, a_k.$$

With $a_0 = y(0) = 0$ and $a_1 = y'(0) = -3$, we get $a_2 = a_4 = \ldots = 0$, and $a_3 = \frac{1-3}{(1+1)(1+2)} \cdot (-3) = 1$, $a_5 = a_7 = \ldots = 0$, for the final answer

$$y(x) = x^3 - 3x.$$

Problem 4. Let $f(x) = 1 + \cos(\pi x)$, $0 < x < 1$, and let $S_f = S_f(x)$, $x \in (-\infty, +\infty)$ be the sum of the Fourier series of the periodic extension of f with period 1.
(a) [4pt] Draw the graph of S_f for $x \in [-3, 3]$;
(b) [3pt] Compute $S_f(1)$;
(c) [3pt] Compute $S_f(5/2)$.

Solution. (a) We have the graph of cosine on $(0, \pi)$ (half-period, going from 1 to -1) that is compressed in the horizontal direction by the factor of π and shifted up by 1.
(b) $S_f(1) = (f(0+) + f(1-))/2 = (2 + 0)/2 = 1$, (c) $S_f(5/2) = S_f(1/2) = f(1/2) = 1$.

Problem 5. The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = e^{-(x+5)^2}$.

Solution. We have $g(x) = h(x + 5)$ and $h(x) = f(\sqrt{2}x)$. As a result,

$$\widehat{g}(\omega) = e^{5\mathrm{i}\omega}\, \widehat{h}(\omega) = e^{5\mathrm{i}\omega}\, 2^{-1/2}\, \widehat{f}(\omega/\sqrt{2}) = 2^{-1/2} e^{5\mathrm{i}\omega} e^{-\omega^2/4}.$$

Problem 6. The Fourier transform of the function $f(x) = e^{-|x|}$ is

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

Use the result to compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega.$$

Solution. The properties of the Fourier integral imply that, in this case,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{f}(\omega) e^{\mathrm{i}\omega x}\, d\omega$$

for all x, that is,

$$e^{-|x|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega x) + \mathrm{i}\sin(\omega x)}{1+\omega^2}\, d\omega.$$

Putting $x = 1$ and using that $\omega \mapsto \sin\omega$ is an odd function, we get the answer:

$$\int_{-\infty}^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega = \frac{\pi}{e}.$$

Problem 7. Use separation of variables to find a non-constant solution $u = u(t, x)$ of the equation $u_t = u^5 u_x$.

Solution. Writing $u(t, x) = F(t)G(x)$, we get $F'G = F^5 G^5 F G'$, which, after separation of t and x, results in $(F^{-5})'(t) = (G^5)'(x)$, leading to the answer

$$u(t, x) = \left(\frac{cx + a}{b - ct}\right)^{1/5}, \quad a, b, c \in \mathbb{R},\ c \neq 0.$$

Problem 8. Solve the following initial-boundary value problem:

$$\begin{aligned} u_t &= 5u_{xx},\ u = u(t,x),\ t > 0,\ x \in (0,\pi), \\ u(0,x) &= \sin(x) - \sin(3x). \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

Solution. The boundary conditions suggest the general form of the solution $u(t,x) = \sum_{k\geq 1} u_k e^{-5k^2 t}\sin(kx)$. The initial condition implies $u_1 = 1, u_3 = -1,\ u_k = 0$ otherwise, leading to the final answer

$$u(t,x) = e^{-5t}\sin(x) - e^{-45t}\sin(3x).$$

Problem 9. Solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx},\ u = u(t,x),\ t > 0,\ x \in (0,\pi), \\ u(0,x) &= 0, \\ u_t(0,x) &= \sin(2x) + 3\sin(5x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

Solution. The boundary conditions and zero initial displacement suggest the general form of the solution $u(t,x) = \sum_{k\geq 1} u_k \sin(2kt)\sin(kx)$. Initial speed gives $4u_2 = 1,\ 10u_5 = 3,\ u_k = 0$ otherwise, leading to the final answer

$$u(t,x) = \frac{\sin(4t)\sin(2x)}{4} + \frac{3\sin(10t)\sin(5x)}{10}.$$

Problem 10. This is a multiple choice part. For each of the five questions, circle (or otherwise indicate) the answer you think is correct (there is always only one correct answer). You get two points for each correct selection, zero points for each wrong selection. No need to show your work.

(i) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z + 5\sin z}{z - 3i}$ around the point $z_0 = 0$?

1   2 $\boxed{3}$ 4   5

The singular point of the function is $3\mathbf{i}$, which is 3 units away from the origin where we do the expansion.

(ii) True or false: the Fourier series of the 2π periodic extension of the function $f(x) = x^5$, $|x| < \pi$, converges uniformly on the interval $[-10, 10]$?

True $\boxed{\text{False}}$ Need more information

The sum of the Fourier series is not a continuous function, but all the partial sums are, so convergence cannot be uniform.

(iii) Identify the sequence that converges uniformly to zero on the interval $x \in [0,1]$, as $n \to \infty$:

x^n $nx/(1+nx)$ $n\ln\left(1+\frac{x}{n}\right)$ $\boxed{n\sin\left(\frac{x^4}{n^2}\right)}$

The first does not converge to 0 when $x = 1$; the second converges to 0 only for $x = 0$; the third converges to x; the last one works because $\sin(x) \leq x$.

(iv) Identify the function $f = f(x)$ for which the Fourier transform $\hat{f}$ is real [that is, the imaginary part of $\hat{f}$ is identically equal to zero.]

$e^{-x^2+\sin(2x)}$ $\boxed{e^{-|x|}\cos(2x)}$ $e^{-x^4+x^3}$ $e^{-x^2}\big(\cos(2x)+\sin(x)\big)$

Look for an even function (the one that does not change if you replace x with $-x$).

(v) Let f be a scalar field for which all partial derivatives exist and are continuous. Which of the following expressions defines the Laplacian $\boldsymbol{\Delta} f$ of f?

$\boxed{\mathrm{div}\,\big(\mathrm{grad}(f)\big)}$ $\mathrm{grad}\,\big(\mathrm{div}(f)\big)$ $\mathrm{curl}\,\big(\mathrm{grad}(f)\big)$ $\mathrm{grad}\,\big(\mathrm{curl}\,f\big)$

Note that the second and forth make no sense, whereas the third is 0.

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \sum_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$
$\sum_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \frac{\tilde{f}(0+)+\tilde{f}(0-)}{2}$	Obvious	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0)$ $= \frac{f(0+)+f(0-)}{2}$
$\lim_{\lvert k\rvert\to\infty} \lvert c_k(f)\rvert = 0$	Riemann-Lebesgue: $f \in L_1$	$\lim_{\lvert\omega\rvert\to\infty} \lvert\hat{f}(\omega)\rvert = 0,\ \hat{f}$ continuous
$\sum_{k=-\infty}^{+\infty} \lvert c_k(f)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert f(x)\rvert^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\int_{-\infty}^{+\infty} \lvert\hat{f}(\omega)\rvert^2\,d\omega = \int_{-\infty}^{+\infty} \lvert f(x)\rvert^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(x)$
$\int f(x)\,dx$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\frac{f(x)}{x}$	$\frac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega\rvert}$
$1(\lvert x\rvert \le 1)$	$\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$	$\frac{\sin\omega}{\omega}$	$\sqrt{\frac{\pi}{2}}1(\lvert x\rvert \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$