MATH 445 Mid-Term Exam 1
Wednesday, October 12, 2022

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Instructions:

- No notes, no books or other printed materials (including printouts from the web), and no calculators.
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- $e^z = 1 + z + z^2/(2!) + z^3/(3!) + \cdots; \sin(z) = z - z^3/(3!) + z^5/(5!) - \cdots.$

Problem 1.

(a) [5 pts] Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = 2x^3y^5z^7$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,1,1)$.

(b) [5 pts] Compute the flux of the vector field $\mathbf{F} = (3x + 2xy)\,\hat{\boldsymbol{\imath}} + (z^2 - y^2)\,\hat{\boldsymbol{\jmath}} + (4 + x)z\,\hat{\boldsymbol{\kappa}}$ out of the sphere $(x-1)^2 + y^2 + (z+1)^2 = 1$.

Problem 2.

(a) [5 pts] Compute $\oint_C \frac{e^z - 1}{z^2}dz$, where C is the circle $|z| = 4$, oriented counterclockwise.

(b) [5 pts] Compute the Laurent series expansion of the function $f(z) = \dfrac{z+1}{z-5}$ around the point $z_0 = 5$.

Problem 3. Solve the initial value problem

$$y''(x) - xy'(x) + 4y(x) = 0, \quad y(0) = 3, \quad y'(0) = 0.$$

Problem 4. Let $f(x) = \frac{x^3}{2}$, $|x| < 2$, and let $S_f = S_f(x),\ x \in (-\infty, +\infty)$ be the sum of the Fourier series of f.

(a) [3pt] Draw the graph of S_f for $x \in [-10, 10]$;
(b) [2pt] Compute $S_f(6)$
(c) [2pt] Compute $S_f(5)$
(d) [3pt] True or false: the infinite sum in S_f converges uniformly on $[-5, 5]$? Explain your conclusion.

Problem 5. This is a multiple choice part. For each question, circle (or otherwise indicate) the answer you think is correct (there is always only one correct answer). You get two points for each correct selection, zero points for each wrong selection. No need to show your work.

(a) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{a}$ $\qquad$ $\boldsymbol{a} \cdot \boldsymbol{a}$ $\qquad$ $(\boldsymbol{b} \times \boldsymbol{a}) \times \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$

(b) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Assuming that all the necessary partial derivatives exist and are continuous, identify the ONE expression that is always equal to zero.

$\boldsymbol{F} \cdot \operatorname{curl}\big(\operatorname{grad}(f)\big)$ $\qquad$ $\operatorname{grad}\big(\operatorname{div}(f\boldsymbol{F})\big)$ $\qquad$ $\operatorname{curl}\big(\operatorname{curl}(f\boldsymbol{F})\big)$ $\qquad$ $\operatorname{grad}\big((\operatorname{grad} f) \cdot \boldsymbol{F}\big)$

(c) What is the type of singularity of the function $f(z) = z^{-2}\sin(z)$ at the point $z = 0$?

Removable $\qquad$ Simple pole $\qquad$ Pole of order 2 $\qquad$ Essential

(d) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z+5}{z-3}$ around the point $z_0 = 4i$?

1 $\qquad$ 2 $\qquad$ 3 $\qquad$ 4 $\qquad$ 5

(e) Which ONE of the following functions $u = u(x,y)$ is NOT harmonic?

$u = x^2 - y^2$ $\qquad$ $u = e^x \sin(y)$ $\qquad$ $u = x^2 + y^2$ $\qquad$ $u = e^x \cos(y)$