MATH 445 Mid-Term Exam 1 Answers
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Instructions:

- No notes, no books or other printed materials (including printouts from the web), no collaboration with anybody (or anything, like AI).
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- $e^z = 1+z+z^2/(2!)+z^3/(3!)+\cdots$; $\sin(z) = z-z^3/(3!)+z^5/(5!)-\cdots$; $\cos(z) = 1-z^2/(2!)+z^4/(4!)-\cdots$.

Problem 1.

(a) [5 pts] Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = 3x^4y^5z^6$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,1,1)$.

Solution: by the Fundamental Theorem of Calculus,

$$\int_C \nabla f \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 3.$$

(b) [5 pts] Compute the flux of the vector field $\mathbf{F} = (3x+2xy-z^2)\,\hat{\boldsymbol{\imath}} + (x^3+z^2-y^2)\,\hat{\boldsymbol{\jmath}} + (4+x-y)z\,\hat{\boldsymbol{\kappa}}$ out of the sphere $(x+1)^2+y^2+(z-1)^2 = 9$.

Solution: keeping in mind that the triple integral is over a ball with center at $(-1,0,1)$ and volume $(4\pi/3)3^3 = 36\pi$, use divergence theorem to conclude

$$\iint \mathbf{F}\cdot d\boldsymbol{\sigma} = \iiint \operatorname{div}\mathbf{F}dV = \iiint (3+2y-2y+4+x-y)dV = 36\pi(7-1) = 216\pi.$$

Problem 2. [10 pts] Compute the Laurent series expansion of the function $f(z) = \dfrac{z^2-6z+5}{(z-5)^3}$ around the point $z_0 = 5$.

Solution: keeping in mind that $z^2-6z+5 = (z-1)(z-5)$ and $z-1 = z-5+4$, we get

$$f(z) = \frac{4}{(z-5)^2} + \frac{1}{z-5}.$$

Problem 3. [10 pts] Compute $\oint_C \dfrac{z^2}{e^z-1}dz$, where C is the circle $|z| = 8$, oriented counterclockwise.

Solution: C encloses three isolated singular points of the function $f(z) = z^2/(e^z-1)$: $z_0 = 0, z_+ = 2\pi i$, $z_- = -2\pi i$. Because $e^z = 1+z+z^2/2+\ldots$ near zero, we conclude that z_0 is a removable singularity and therefore does not contribute to the integral. Both z_+ and z_- are simple poles, with the same residue $-4\pi^2$. The answer is therefore $2\cdot 2\pi i\cdot(-4\pi^2) = -16\pi^3 i$.

Problem 4. [10 pts] Solve the initial value problem

$$y''(x) - 2xy'(x) + 8y(x) = 0, \quad y(0) = 12, \quad y'(0) = 0.$$

Solution: plugging in $y(x) = 12 + \sum_{k=2}^{\infty} a_k x^k$, we conclude that $a_{2k-1} = 0, k \geq 2$, and $a_2 = -48, a_4 = 16$, $a_{2k} = 0, k \geq 3$, and get the answer $y(x) = 16x^4 - 48x^2 + 12$ (one of Hermite polynomials).

Problem 5. This is a multiple choice part. For each question, circle (or otherwise indicate) the answer you think is correct (there is always only one correct answer). You get two points for each correct selection, zero points for each wrong selection. No need to show your work.

(a) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$\boldsymbol{a}\times\boldsymbol{b} - \boldsymbol{b}\times\boldsymbol{a}$ $\qquad$ $\boldsymbol{a}\cdot\boldsymbol{a}$ $\qquad$ $\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})$ $\qquad$ $\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = 0$

(b) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Assuming that all the necessary partial derivatives exist and are continuous, identify the ONE expression that is always equal to zero.

$\operatorname{curl}(\boldsymbol{F}) \cdot \big(\operatorname{grad}(f)\big)$ $\quad$ $\operatorname{grad}\big(\operatorname{div}(f\boldsymbol{F})\big)$ $\quad$ $\boxed{\operatorname{div}\big(\operatorname{curl}(f\boldsymbol{F})\big) = 0}$ $\quad$ $\operatorname{grad}\big((\operatorname{grad} f) \cdot \boldsymbol{F}\big)$

(c) What is the type of singularity of the function $f(z) = z^{-2}\cos(z)$ at the point $z = 0$?

Removable $\quad$ Simple pole $\quad$ **Pole of order 2** $\quad$ Essential

(d) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z-5}{z+3}$ around the point $z_0 = -4i$?

1 $\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ $\boxed{5}$

(e) Which ONE of the following functions $u = u(x, y)$ is NOT harmonic?

$u = 2x^2 - 3y^3$ $\quad$ $u = e^x \sin(y)$ $\quad$ $u = x^2 - 10xy - y^2$ $\quad$ $u = e^x \cos(y)$

$u = 2x^2 - 3y^3$ is not harmonic: $u_{xx} + u_{yy} = 4 - 18y \not\equiv 0$.