# ASSOCIATION OF RANDOM VARIABLES, WITH APPLICATIONS

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**1. Introduction and summary.** It is customary to consider that two random variables $S$ and $T$ are associated if $\mathrm{Cov}[S, T] = EST - ES \cdot ET$ is nonnegative. If $\mathrm{Cov}[f(S), g(T)] \geqq 0$ for all pairs of nondecreasing functions $f$, $g$, then $S$ and $T$ may be considered more strongly associated. Finally, if $\mathrm{Cov}[f(S, T), g(S, T)] \geqq 0$ for all pairs of functions $f$, $g$ which are nondecreasing in each argument, then $S$ and $T$ may be considered still more strongly associated.

The strongest of these three criteria has a natural multivariate generalization which serves as a useful definition of association:

Definition 1.1. We say random variables $T_1, \cdots, T_n$ are *associated* if

$$\mathrm{Cov}[f(\mathbf{T}), g(\mathbf{T})] \geqq 0 \tag{1.1}$$

for all nondecreasing functions $f$ and $g$ for which $Ef(\mathbf{T})$, $Eg(\mathbf{T})$, $Ef(\mathbf{T})g(\mathbf{T})$ exist.

(Throughout, we use $\mathbf{T}$ for $(T_1, \cdots, T_n)$; also, without further explicit mention we consider only test functions $f$, $g$ for which $\mathrm{Cov}[f(\mathbf{T}), g(\mathbf{T})]$ exists.)

In Section 2 we develop the fundamental properties of association: Association of random variables is preserved under (a) taking subsets, (b) forming unions of independent sets, (c) forming sets of nondecreasing functions, (d) taking limits in distribution.

In Section 3 we develop some simpler criteria for association. We show that to establish association it suffices to take in (1.1) nondecreasing test functions $f$ and $g$ which are either (a) binary or (b) bounded and continuous.

In Section 4 we develop the special properties of association that hold in the case of binary random variables, i.e., random variables that take only the values 0 or 1. These properties turn out to be quite useful in applications. We also discuss association in the bivariate case. We relate our concept of association in this case to several discussed by Lehmann (1966).

Finally, in Section 5 applications in probability and statistics are presented yielding results by Robbins (1954), Marshall-Olkin (1966), and Kimball (1951). An application in reliability which motivated our original interest in association will be presented in a forthcoming paper.

**2. Properties of association.** Association has the following two properties desirable in any classification of multivariate distributions:

($P_1$) *Any subset of associated random variables are associated.*

Proof. ($P_1$) follows immediately from the definition by choosing nondecreasing functions $f$ and $g$ that depend only on the variables in the subset. ▯

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($P_2$) *If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.*

PROOF. Let $\mathbf{S} = (S_1, S_2, \cdots, S_n)$ be associated, $\mathbf{T} = (T_1, \cdots, T_m)$ be associated, and $\mathbf{S}$ and $\mathbf{T}$ be independent of each other. Let $f$, $g$ be nondecreasing functions. Writing $f$ for $f(\mathbf{S}, \mathbf{T})$, and $g$ for $g(\mathbf{S}, \mathbf{T})$, we have

$$\begin{aligned}\text{Cov}[f, g] &= E_{\mathbf{S},\mathbf{T}}fg - E_{\mathbf{S},\mathbf{T}}f \cdot E_{\mathbf{S},\mathbf{T}}g \\ &= E_{\mathbf{S}}E_{\mathbf{T}}fg - E_{\mathbf{S}}\{E_{\mathbf{T}}f \cdot E_{\mathbf{T}}g\} + E_{\mathbf{S}}\{E_{\mathbf{T}}f \cdot E_{\mathbf{T}}g\} \\ &\qquad - E_{\mathbf{S}}E_{\mathbf{T}}f \cdot E_{\mathbf{S}}E_{\mathbf{T}}g = E_{\mathbf{S}}\,\text{Cov}_{\mathbf{T}}[f, g] + \text{Cov}_{\mathbf{S}}[E_{\mathbf{T}}f, E_{\mathbf{T}}g],\end{aligned}$$

where $E_{\mathbf{S}}$ denotes expectation over the distribution of $\mathbf{S}$, $E_{\mathbf{T}}$ expectation over the distribution of $\mathbf{T}$, and $E_{\mathbf{S},\mathbf{T}}$ expectation over the joint distribution of $\mathbf{S}$ and $\mathbf{T}$. $E_{\mathbf{S},\mathbf{T}} = E_{\mathbf{S}}E_{\mathbf{T}}$ from the independence of $\mathbf{S}$ and $\mathbf{T}$. Since $\text{Cov}_{\mathbf{T}}[f(\mathbf{s}, \mathbf{T}), g(\mathbf{s}, \mathbf{T})] \geqq 0$ for each fixed $\mathbf{s}$, then $E_{\mathbf{S}}\,\text{Cov}[f, g] \geqq 0$. Since $E_{\mathbf{T}}f(\mathbf{s}, \mathbf{T})$, $E_{\mathbf{T}}g(\mathbf{s}, \mathbf{T})$ are nondecreasing functions in $\mathbf{s}$, $\text{Cov}_{\mathbf{S}}[E_{\mathbf{T}}f, E_{\mathbf{T}}g] \geqq 0$. □

Another standard multivariate property, valid for association, is

($P_3$) *The set consisting of a single random variable is associated.*

PROOF. The result is a consequence of a classical inequality for similarly ordered functions due to Chebyshev (Hardy, Littlewood, and Pólya, (1934) Section 2.17). A simple direct proof is given in Section 3. □

Properties $P_1$, $P_2$, and $P_3$ permit some standard multivariate manipulations with associated random variables. Additional manipulations having useful applications become possible using

($P_4$) *Nondecreasing functions of associated random variables are associated.*

PROOF. Let $T_1, \cdots, T_n$ be associated, $f_i$ be nondecreasing, and $S_i \equiv f_i(\mathbf{T})$, $i = 1, \cdots, m$. If $f$ and $g$ are nondecreasing, then $f(f_1, \cdots, f_m)$ and $g(f_1, \cdots, f_m)$ are nondecreasing. Thus by Definition 1.1, $\text{Cov}_{\mathbf{S}}[f(\mathbf{S}), g(\mathbf{S})] = \text{Cov}_{\mathbf{T}}[f(\mathbf{f}(\mathbf{T})), g(\mathbf{f}(\mathbf{T}))] \geqq 0$. □

At this point we state an additional natural property of associated random variables:

($P_5$) *If $T_1^{(k)}, \cdots, T_n^{(k)}$ are associated for each $k$, and $\mathbf{T}^{(k)} \to \mathbf{T}$ in distribution, then $T_1, \cdots, T_n$ are associated.*

We defer the proof until Section 3.

$P_2$ and $P_3$ imply

THEOREM 2.1. *Independent random variables are associated.*

The case of independent random variables represents one extreme of association. An opposite extreme is represented by the case of random variables $\mathbf{T}$ taking values only along a nondecreasing curve $\mathbf{t}(\theta)$; i.e., $t_i(\theta)$ is nondecreasing in $\theta$ ($i = 1, \cdots, n$). The set containing only the random variable $\Theta$ defined by $T_i = t_i(\Theta)$, $i = 1, \cdots, n$, is associated by $P_3$. It follows that $T_1, T_2, \cdots, T_n$ are associated by $P_4$.

**3. Equivalent criteria for association.** In this section we show that association of random variables may be established by taking in (1.1) nondecreasing test functions $f$ and $g$ which are (a) binary, or (b) bounded and continuous. We also develop an alternate criterion for association under which it suffices to show association of indicator functions of the random variables.

We will need the following identity: Define $X_f(t) = 1$ if $f(\mathbf{T}) > t$, $X_f(t) = 0$ if $f(\mathbf{T}) \leqq t$. Then

$$\text{Cov}[f(\mathbf{T}), g(\mathbf{T})] \doteq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Cov}[X_f(s)X_g(t)]\, ds\, dt. \tag{3.1}$$

The identity is presented in Lehmann (1966) along with a proof attributed to Hoeffding. See also Marshall-Olkin (1966) for a comparable result on moment generating functions.

Using (3.1) we may now prove

THEOREM 3.1. *Let*

$$\text{Cov}[\gamma(\mathbf{T}), \delta(\mathbf{T})] \geqq 0 \textit{ for all binary nondecreasing functions } \gamma, \delta. \tag{3.2}$$

*Then* $T_1, \cdots, T_n$ *are associated.*

PROOF. Let $f$, $g$ be nondecreasing test functions. By (3.1)

$$\text{Cov}[f(\mathbf{T}), g(\mathbf{T})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Cov}[X_f(s), X_g(t)]\, ds\, dt.$$

But $X_f(s)$, $X_g(t)$ are nondecreasing functions of $\mathbf{T}$, and so by hypothesis (3.2), $\text{Cov}[X_f(s), X_g(t)] \geqq 0$. It follows by Definition (1.1) that $T_1, \cdots, T_n$ are associated. ▯

REMARK. One proof of property $P_3$ is by the observation that for any pair of binary, nondecreasing functions $\gamma$, $\delta$ of a single argument, either $\gamma(t) \leqq \delta(t)$ for all $t$ or $\delta(t) \leqq \gamma(t)$ for all $t$. If for example $\gamma \leqq \delta$, then $\text{Cov}[\gamma(T), \delta(T)] = E\gamma\delta - E\gamma \cdot E\delta = E\gamma - E\gamma \cdot E\delta = E\gamma(1 - E\delta) \geqq 0$. Property $P_3$ then follows from Theorem 3.1. The previously cited inequality of Chebyshev can be proved by this remark and Theorem 3.1. A multivariable version of the Chebyshev inequality may be obtained readily using property $P_2$.

To show that it suffices to take bounded continuous functions in Definition (1.1), we must first prove

LEMMA 3.2. *If* $\text{Cov}[u(\mathbf{T}), v(\mathbf{T})] \geqq 0$ *for all bounded continuous nondecreasing* $u$, $v$, *then* $\text{Cov}[\phi(\mathbf{T}), \psi(\mathbf{T})] \geqq 0$ *for all binary, right continuous nondecreasing* $\phi$, $\psi$.

PROOF. Let $A = \{\mathbf{t} \mid \phi(\mathbf{t}) = 1\}$, and $d(\mathbf{t}, A)$ be the Euclidean distance from a point $\mathbf{t}$ to the set $A$. Define $u^{(k)}(\mathbf{t}) = 0$ if $d(\mathbf{t}, A) \geqq k^{-1}$, $u^{(k)}(\mathbf{t}) = 1 - k\, d(\mathbf{t}, A)$ if $d(\mathbf{t}, A) < k^{-1}$. Each function $u^{(k)}$ is nonnegative, *bounded* above by 1, *continuous*, and *nondecreasing*. In a similar way we may define $v^{(k)}$ in terms of $\psi$.

By hypothesis, $\text{Cov}[u^{(k)}(\mathbf{T}), v^{(t)}(\mathbf{T})] \geqq 0$. Since $\phi$ is right continuous, $A$ is closed, and so $u^{(k)} \downarrow \phi$. Similarly $v^{(k)} \downarrow \psi$. We conclude by monotone convergence that $\text{Cov}[\phi(\mathbf{T}), \psi(\mathbf{T})] \geqq 0$. ▯

THEOREM 3.3. *Let*

$$\text{Cov}[u(\mathbf{T}), v(\mathbf{T})] \geqq 0 \tag{3.3}$$

*for all bounded, continuous, nondecreasing functions* $u$, $v$.

*Then* $T_1, \cdots, T_n$ *are associated.*

PROOF. For a binary nondecreasing function $\gamma_i$, let $A_i = \{\mathbf{t} \mid \gamma_i(\mathbf{t}) = 1\}$, $i = 1, 2$. We can find a compact set $C_i \subset A_i$ such that $P[C_i] + \epsilon \geqq P[A_i]$. Let

$C_i^+ = \{\mathbf{c} + \mathbf{t} \mid \mathbf{c} \varepsilon C_i, t_1 \geqq 0, \cdots, t_n \geqq 0\}$, a closed set. Then $C_i \subset C_i^+ \subset A_i$. Let $\phi_i(\mathbf{t}) = 1$ if $\mathbf{t} \varepsilon C_i^+$, $\phi_i(\mathbf{t}) = 0$ otherwise. The function $\phi_i$ is *binary, right continuous, and nondecreasing*. Thus by Lemma 3.2,

$$\text{Cov}[\phi_1(\mathbf{T}), \phi_2(\mathbf{T})] \geqq 0. \tag{3.4}$$

Since $\gamma_i \geqq \phi_i$, then

$$E\gamma_1(\mathbf{T})\gamma_2(\mathbf{T}) \geqq E\phi_1(\mathbf{T})\phi_2(\mathbf{T}). \tag{3.5}$$

At the same time

$$E\phi_i(\mathbf{T}) + \epsilon \geqq E\gamma_i(\mathbf{T}). \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we obtain

$$\begin{aligned}\text{Cov}[\gamma_1(\mathbf{T}), \gamma_2(\mathbf{T})] &\geqq E\phi_1(\mathbf{T})\phi_2(\mathbf{T}) - [E\phi_1(\mathbf{T}) + \epsilon][E\phi_2(\mathbf{T}) + \epsilon] \\ &\geqq \text{Cov}[\phi_1(\mathbf{T}), \phi_2(\mathbf{T})] - 2\epsilon - \epsilon^2 \geqq -2\epsilon - \epsilon^2.\end{aligned}$$

Letting $\epsilon \to 0$, we conclude $\text{Cov}[\gamma_1(\mathbf{T}), \gamma_2(\mathbf{T})] \geqq 0$. By Theorem 3.1, $T_1, \cdots, T_n$ are associated. □

REMARK. The proof of property $P_5$ is an immediate consequence of Theorem 3.3 and the Helly-Bray theorem.

Next we establish a criterion for association of $T_1, \cdots, T_n$ in terms of indicator functions $X_i(t)$, defined to be 1 for $t < T_i$, 0 for $t \geqq T_i$.

THEOREM 3.4. *Let the array of random variables*

$$\begin{matrix} X_1(t_1) & \cdots & X_1(t_k) \\ \vdots & & \vdots \\ X_n(t_1) & \cdots & X_n(t_k) \end{matrix}$$

*be associated for every choice of $k$ and $t_1, \cdots, t_k$. Then $T_1, \cdots, T_n$ are associated.*

PROOF. Let $u$, $v$ be nonnegative, bounded, continuous, nondecreasing functions. For given $t_1 < \cdots < t_k$, define $u^{(k)}(\mathbf{T}) = 0$ if any $X_i(t_1) = 0$, $i = 1, \cdots, n$; $u^{(k)}(\mathbf{T}) = u(\mathbf{S})$, where $S_i = \max_j \{t_j \mid X_i(t_j) = 1\}$, if all $X_i(t_1) = 1$. The functions $u^{(k)}$ are nonnegative and *nondecreasing*, viewed either as functions of $\mathbf{T}$ or as functions of $\mathbf{X}(\mathbf{t})$. Define $v^{(k)}$ in a similar way, corresponding to $v$. Now let $\{t_1, \cdots, t_k\}$ increase with $k$ to a countable dense set in $(-\infty, \infty)$. Since $u$, $v$, and $uv$ are continuous, then $0 \leqq u^{(k)} \uparrow u$, $0 \leqq v^{(k)} \uparrow v$, and $0 \leqq u^{(k)}v^{(k)} \uparrow uv$ at each fixed value of $\mathbf{T}$. Since $u$, $v$, and $uv$ are bounded, then by monotone convergence $Eu^{(k)}(\mathbf{T}) \uparrow Eu(\mathbf{T})$, $Ev^{(k)}(\mathbf{T}) \uparrow Ev(\mathbf{T})$, and $Eu^{(k)}(\mathbf{T})v^{(k)}(\mathbf{T}) \uparrow Eu(\mathbf{T})v(\mathbf{T})$. Since by hypothesis, $0 \leqq \text{Cov}[u^{(k)}(\mathbf{T}), v^{(k)}(\mathbf{T})]$, it follows that $\text{Cov}[u(\mathbf{T}), v(\mathbf{T})] = \lim_{k\to\infty} \text{Cov}[u^{(k)}(\mathbf{T}), v^{(k)}(\mathbf{T})] \geqq 0$.

Next, given bounded, continuous, nondecreasing functions $u$, $v$, by adding a sufficiently large constant to each, we can make them nonnegative. By the result obtained just above, we conclude $\text{Cov}[u(\mathbf{T}), v(\mathbf{T})] \geqq 0$. By Theorem 3.3, $T_1, \cdots, T_n$ are associated. □

**4. Special cases of interest.** In this section we consider two special cases:

(1) association of binary random variables,

(2) various concepts of bivariate dependence.

4.1. *Association of binary random variables.* In the special case of binary random variables, association leads to some interesting applications. We first obtain the intuitively reasonable property:

($BP_1$) *If $X_1, \cdots, X_n$ are associated binary random variables, then $1 - X_1, \cdots, 1 - X_n$ are associated binary random variables.*

Proof. If $\gamma$ is a binary, nondecreasing function, then the dual function $\gamma^D(\mathbf{x}) = 1 - \gamma(\mathbf{1} - \mathbf{x})$, where $\mathbf{1} - \mathbf{x} = (1 - x_1, \cdots, 1 - x_n)$, is also binary and nondecreasing. Let $\mathbf{Y} = \mathbf{1} - \mathbf{X}$. Then

$$\text{Cov}_{\mathbf{Y}}[\gamma(\mathbf{Y}), \delta(\mathbf{Y})] = \text{Cov}_{\mathbf{X}}[\gamma^D(\mathbf{X}), \delta^D(\mathbf{X})] \geqq 0,$$

for binary nondecreasing $\gamma$, $\delta$. □

Next we obtain

Theorem 4.1. *Let $X_1, \cdots, X_n$ be associated binary random variables. Then*

$$P[X_1 = 1, \cdots, X_n = 1] \geqq P[X_1 = 1] \cdots P[X_n = 1], \tag{4.1}$$

*and*

$$P[X_1 = 0, \cdots, X_n = 0] \geqq P[X_1 = 0] \cdots P[X_n = 0]. \tag{4.2}$$

Proof. Choose $\gamma(\mathbf{X}) = X_1$, $\delta(\mathbf{X}) = X_2 \cdots X_n$, both nondecreasing functions of $\mathbf{X}$. Since $X_1, \cdots, X_n$ are associated, we have $EX_1 \cdots X_n \geqq EX_1 \cdot EX_2 \cdots X_n$. Repeated use of this argument yields $EX_1 \cdots X_n \geqq EX_1 \cdots EX_n$. Since for a binary random variable $X$, $EX = P[X = 1]$, we obtain (4.1).

Using $BP_1$ and (4.1), we obtain (4.2). □

Theorem 4.2. *For binary random variables $X$, $Y$, association is equivalent to* $\text{Cov}[X, Y] \geqq 0$.

Proof. If $X$, $Y$ are associated, then $\text{Cov}[X, Y] \geqq 0$ by definition.

Now suppose $\text{Cov}[X, Y] \geqq 0$. We list all possible binary nondecreasing functions $\gamma(X, Y)$:

$$(\gamma \equiv 0) \leqq (\gamma = XY) \leqq \begin{matrix}(\gamma = X)\\(\gamma = Y)\end{matrix} \leqq (\gamma = X + Y - XY) \leqq (\gamma \equiv 1).$$

The covariance between any pair of binary functions $\gamma$, $\delta$ such that $\gamma \leqq \delta$ is automatically nonnegative. The remaining pair $\gamma = X$, $\delta = Y$ has nonnegative covariance by hypothesis. □

4.2. *Various concepts of bivariate dependence.* As observed in the introduction, the conditions

$$\text{Cov}[S, T] \geqq 0, \tag{4.3}$$

$$\text{Cov}[f(S), g(T)] \geqq 0 \quad \text{for all nondecreasing } f, g, \tag{4.4}$$

$$S, T \text{ associated}, \tag{4.5}$$

are successively stronger, i.e., (4.5) ⇒ (4.4) ⇒ (4.3). We can add to the list the condition described by Lehmann (1966) as *positive regression dependence* of

$T$ on $S$,

$$P[T > t \mid S = s] \text{ is nondecreasing in } s. \tag{4.6}$$

We prove

THEOREM 4.3. *If $T$ is positively regression dependent on $S$, then $S$ and $T$ are associated, i.e.,* (4.6) *implies* (4.5).

PROOF. Write $\mathrm{Cov}[f(S, T), g(S, T)] = Efg - EfEg = E_S E_{T|S} fg - E_S E_{T|S} f E_S E_{T|S} g$, where we have omitted the arguments of $f$ and $g$ and where $E_{T|S}$ denotes expectation over the conditional distribution of $T$ given $S$. Thus

$$\begin{aligned}\mathrm{Cov}[f, g] &= E_S E_{T|S} fg - E_S\{E_{T|S} f \cdot E_{T|S} g\} \\ &\quad + E_S\{E_{T|S} f \cdot E_{T|S} g\} - E_S E_{T|S} f \cdot E_S E_{T|S} g \\ &= E_S \,\mathrm{Cov}_{T|S}[f, g] + \mathrm{Cov}_S[E_{T|S} f, E_{T|S} g].\end{aligned}$$

Now assume $f$, $g$ nondecreasing. Then $\mathrm{Cov}_{T|S}[f, g] \geqq 0$ by $P_3$, and so $E_S \,\mathrm{Cov}_{T|S}[f, g] \geqq 0$. To show $\mathrm{Cov}_S[E_{T|S} f, E_{T|S} g] \geqq 0$, note that $P[T > t \mid S = s]$ nondecreasing in $s$ implies that $P[f(T, s') > t \mid S = s]$ is nondescreasing in $s$, which in turn implies $P[f(T, s) > t \mid S = s]$ is nondecreasing in $s$, which finally implies $E_{T|S=s} f(T, s)$ is nondecreasing in $s$. Thus $\mathrm{Cov}_S[E_{T|S} f, E_{T|S} g] \geqq 0$ by $P_3$. □

Lehmann (1966) calls $S$, $T$ *positively quadrant dependent* if

$$P[S \leqq s, T \leqq t] \geqq P[S \leqq s]P[T \leqq t] \quad \text{for all } s, t. \tag{4.7}$$

We prove

THEOREM 4.4. *$S$, $T$ are positively quadrant dependent if and only if* $\mathrm{Cov}[f(S), g(T)] \geqq 0$ *for all nondecreasing $f$, $g$; i.e.,* (4.4) *is equivalent to* (4.7).

PROOF. To show (4.4) implies (4.7), choose, in particular, $f(x) = 0$ for $x \leqq s$, 1 for $x > s$, and $g(x) = 0$ for $x \leqq t$, 1 for $x > t$. Then $f$, $g$ are nondecreasing, and so by (4.4)

$$\begin{aligned}0 \leqq \mathrm{Cov}[f(S), g(T)] &= \mathrm{Cov}[1 - f(S), 1 - g(T)] \\ &= P[S \leqq s, T \leqq t] - P[S \leqq s]P[T \leqq t].\end{aligned}$$

Next assume $S$, $T$ satisfy (4.7). Then for $f$, $g$ nondecreasing, $f(S)$, $g(T)$ also satisfy (4.7), as shown in Theorem 1 of Lehmann (1966). Using Lemma 3 of Lehmann (1966), (4.4) follows. □

Next we point out that *for general random variables no two of the conditions* (4.3), (4.4), (4.5), (4.6) *are equivalent*. It is easy to find $S$, $T$ satisfying (4.3) but not (4.4). To show $S$, $T$ may satisfy (4.4) but not (4.5), let $S$, $T$ take on values $a_1 < a_2 < a_3$ with probabilities

| | $S = a_1$ | $S = a_2$ | $S = a_3$ |
|---|---|---|---|
| $T = a_3$ | 8/64 | 0 | 15/64 |
| $T = a_2$ | 0 | 18/64 | 0 |
| $T = a_1$ | 15/64 | 0 | 8/64 |

Finally, to show $S$, $T$ may satisfy (4.5) but not (4.6), let $S$, $T$ take on values $a_1 < a_2 < a_3$ with probabilities:

| | $S = a_1$ | $S = a_2$ | $S = a_3$ |
|---|---|---|---|
| $T = a_3$ | 1/8 | 0 | 1/4 |
| $T = a_2$ | 0 | 1/4 | 0 |
| $T = a_1$ | 1/4 | 0 | 1/8 |

However, *for binary random variables* $X$, $Y$, *conditions* (4.3), (4.4), (4.5), (4.6) *are equivalent*. This is a consequence of the following two facts:

(1) As pointed out in Example 11 of Lehmann (1966), binary $X$, $Y$ satisfy (4.6) if and only if

$$(4.8)\quad P[X = 0, Y = 0]P[X = 1, Y = 1] \geqq P[X = 0, Y = 1]P[X = 1, Y = 0].$$

(2) Binary $X$, $Y$ satisfy (4.3) if and only if (4.8) holds, as may be verified directly.

**5. Applications to probability and statistics.** Several interesting applications may be obtained as a consequence of

THEOREM 5.1. *Let* $T_1, \cdots, T_n$ *be associated,* $S_i \equiv f_i(\mathbf{T})$, *and* $f_i$ *be nondecreasing,* $i = 1, \cdots, k$. *Then*

$$(5.1)\qquad P[S_1 \leqq s_1, \cdots, S_k \leqq s_k] \geqq \textstyle\prod_{i=1}^k P[S_i \leqq s_i]$$

*and*

$$(5.2)\qquad P[S_1 > s_1, \cdots, S_k > s_k] \geqq \textstyle\prod_{i=1}^k P[S_i > s_i]$$

*for all* $s_1, \cdots, s_k$.

PROOF. By $P_4$, $S_1, \cdots, S_k$ are associated. Let $X_i(s) = 1$ if $S_i > s$, $X_i(s) = 0$ if $S_1 \leqq s$. Then $X_i(s)$ is nondecreasing in $S_i$, and so by $P_4$, $X_1(s_1), \cdots, X_k(s_k)$ are associated. Using (4.1) and (4.2), we obtain (5.1) and (5.2). $\square$

*Partial sums* (Robbins, 1954) Let $T_1, \cdots, T_n$ be independent random variables, and $S_i \equiv \sum_{j=1}^i T_j$, $i = 1, \cdots, n$. Then

$$P[S_1 \leqq s_1, \cdots, S_n \leqq s_n] \geqq \textstyle\prod_{i=1}^n P[S_i \leqq s_i]$$

for all $s_1, \cdots, s_k$.

The inequality follows immediately from Theorem 5.1 by noting that independent random variables are associated, and that each $S_i$ is nondecreasing in $T_1, \cdots, T_n$.

*Order statistics*. Let $S_1 \leqq \cdots \leqq S_n$ be the order statistics in a sample $T_1, \cdots, T_n$. Then

$$P[S_{i_1} \leqq s_{i_1}, \cdots, S_{i_k} \leqq s_{i_k}] \geqq \textstyle\prod_{j=1}^k P[S_{i_j} \leqq s_{i_j}],$$

$$P[S_{i_1} > s_{i_1}, \cdots, S_{i_k} > s_{i_k}] \geqq \textstyle\prod_{j=1}^k P[S_{i_j} > s_{i_j}]$$

for every choice of $1 \leqq i_1 < \cdots < i_k \leqq n$ and $s_{i_1} < \cdots < s_{i_k}$.

This is obtained by noting that each $S_i$ is a nondecreasing function of the independent random variables $T_1, \cdots, T_n$.

*Multivariate exponential.* Marshall and Olkin (1966) consider the multivariate exponential distribution

$$(5.3)\quad F(s_1, \cdots, s_m) = 1 - \exp\left[-\sum_1^m \lambda_i s_i - \sum_{i<j} \lambda_{ij} \max(s_i, s_j) - \sum_{i<j<k} \lambda_{ijk} \max(s_i, s_j, s_k) - \cdots - \lambda_{12\cdots m} \max(s_1, s_2, \cdots, s_m)\right].$$

As pointed out by Marshall and Olkin, if random variables $S_1, \cdots, S_m$ are distributed according to (5.3), then there exist independent exponential random variables $T_1, \cdots, T_n$ such that $S_j = \min(T_i; i \varepsilon A_j)$, where $A_j \subset \{1, 2, \cdots, n\}$ Since $T_1, \cdots, T_n$ are independent and each $S_j$ is a nondecreasing function of $T_1, \cdots, T_n$, we obtain from Theorem 5.1 that

$$(5.4)\quad F(s_1, \cdots, s_m) \geqq \prod_{i=1}^m F_i(s_i)$$

and

$$(5.5)\quad 1 - F(s_1, \cdots, s_m) \geqq \prod_{i=1}^m [1 - F_i(s_i)],$$

where $F_i$ is the marginal distribution of $S_i$. Marshall and Olkin obtain inequalities (5.4) and (5.5) for the bivariate exponential, and give a further, quantitative analysis of that case.

*Analysis of variance.* A. W. Kimball (1951) considers the case of analysis of variance in which two hypotheses are tested using the same error variance for each test. As an example of particular importance, he cites the case in which the effects of both rows and columns are to be tested. As usually formulated, three quadratic forms, $q_1, q_2, q_3$, are computed, independently distributed as $\chi^2$ with $n_1, n_2, n_3$ degrees of freedom respectively, $q_1$ representing the sum of squares between rows, $q_2$ the sum of squares between columns, and $q_3$ the sum of squares due to error. The likelihood ratio test statistics for testing the two hypotheses are

$$F_1 = (q_1/n_1)/(q_3/n_3) \quad \text{and} \quad F_2 = (q_2/n_2)/(q_3/n_3).$$

The probability of making no errors of the first kind is $P[F_1 \leqq F_{1\alpha}, F_2 \leqq F_{2\alpha}]$, where $F_{1\alpha}(F_{2\alpha})$ is the $100\alpha$ per cent point of the distribution of $F_1(F_2)$. Kimball proves

$$(5.6)\quad P[F_1 \leqq F_{1\alpha}, F_2 \leqq F_{2\alpha}] > P[F_1 \leqq F_{1\alpha}]P[F_2 \leqq F_{2\alpha}].$$

In other words, the assurance of no errors of the first kind is greater following the standard experimental procedure than if separate experiments had been performed.

Kimball's result is an immediate consequence of Theorem 5.1 if we note that $q_1, q_2, q_3^{-1}$ are associated (since independent), and $F_1, F_2$ are nondecreasing functions of $q_1, q_2, q_3^{-1}$.

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