# Extreme Values: In distribution vs Path-wise[1]

**Background**. Given a sequence of random variables

$$X_1, X_2, X_3, \dots,$$

the objective is to understand the behavior of $X_n^* = \max(X_1, \dots, X_n)$ and $X_{*,n} = \min(X_1, \dots, X_n)$. Because of the equality

$$\min(X_1, \dots, X_n) = -\max(-X_1, \dots, -X_n),$$

it is enough to understand one of the two.

**The iid case.** If $X_k$ are iid as $X$, then, depending on the upper tail of the distribution of $X$, the *suitably normalized* distribution of $X_n^*$ converges to Weibull, Fréchet, or Gumbel distribution. By Borell-Cantelli, the normalization leads to the *almost sure* behavior of $X_n^*$.

*Example 1.* $X$ is uniform on $[0,1]$. Then

$$P(1 - X_n^* > x) = P(X_n^* < 1 - x) = (1 - x)^n,$$

so that

$$\lim_{n\to\infty} P\big(n(1 - X_n^*) > x\big) = e^{-x}$$

(a particular case of Weibull), and then, by Borell-Cantelli, we get the following limit with probability one:

$$\limsup_{n\to\infty} \frac{-\ln(1 - X_n)}{\ln n} = 1.$$

*Example 2.* $X$ is exponential with mean 1:

$$P(X > x) = e^{-x},\ x > 0.$$

Then

$$P(X_n^* < x) = (1 - e^{-x})^n,$$

so that

$$\lim_{n\to\infty} P\big(X_n^* < x + \ln n\big) = e^{-e^{-x}}$$

(Gumbel), and then, by Borell-Cantelli, we get the following limits with probability one:

$$\lim_{n\to\infty} \frac{X_n^*}{\ln n} = 1, \quad \limsup_{n\to\infty} \frac{X_n}{\ln n} = 1.$$

*Example 3.* $X$ is normal, mean zero, variance one. Now the exact normalization leading to Gumbel involves the `Lambert function` $W = W(z)$, the inverse of $f(z) = ze^z$. The corresponding almost sure limit is rather explicit:

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} = 1. \tag{1.1}$$

**Non-iid case** is obviously much harder. The two examples where something can be done are

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- Gaussian case: if $X_n$ is a stationary Gaussian sequence with mean zero and *reasonably weak* dependence, then (1.1) still holds.
- The CLT setting: if $S_n$ is a sum of iid random variables with mean zero and variance one, so that $S_n/\sqrt{n}$ converges in distribution to standard Gaussian, then we have the *law of iterated logarithm*
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\ln(\ln n)}} = 1;$$
in fact, the limit points of the sequence $\left\{ \frac{S_n}{\sqrt{2n\ln(\ln n)}},\ n \geq 5 \right\}$ are dense in $[-1, 1]$. Note also that, even though $S_n/\sqrt{n}$ is (almost) standard Gaussian for large $n$, the sup grows much slower than (1.1): effects of dependence.