Bernoulli trials

Binomial $X \sim \mathcal{B}(n, p)$: number of successes in n trials

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, \dots, n.$$

Geometric $X \sim G_1(p)$: number of trials TO GET the first success

$$P(X = k) = (1-p)^{k-1} p, \; k \geq 1. \text{ Alternative: } G_0(p) = G_1(p) - 1.$$

Negative Binomial $X \sim \mathcal{NB}(p, m)$: number of trials TO GET m successes; $X = m, m+1, \dots$

$$P(X = k) = \binom{k-1}{m-1} p^{m-1} (1-p)^{(k-1)-(m-1)} p = \binom{k-1}{m-1} p^m (1-p)^{(k-m)}$$

($m - 1$ places to place the successes in the first $k - 1$ trials; k-th trial has to be a success; sum of m iid Geometric.)

Runs: in 64 tosses of a fair coin, 6 consecutive heads or 6 consecutive tails are likely.

Figure: Jacob (Jacques) Bernoulli (1654 – 1705), Swiss

Some contributions:

•First law of large numbers: $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = p$, if X_k are independent $\mathcal{B}(1, p)$;
•Bernoulli differential equation $y' = p(x)y + q(x)y^n$;
•polar coordinates;
•Brother of Johann Bernoulli and uncle of Daniel Bernoulli; his own two children were not mathematicians.

Poisson Distribution

$$X \sim \mathcal{P}(\lambda): \; P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \;\; k = 0, 1, 2, \ldots; E(X) = Var(X) = \lambda.$$

Poisson approximation of binomial distribution:

$$\lim_{n \to \infty, \, p \to 0, \, np \to \lambda} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Quality of Approximation:

$$\sum_{k=0}^{\infty} \left| \binom{n}{k} p^k (1-p)^{n-k} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq \frac{2\lambda}{n} \min(2, \lambda), \;\; \lambda = np,$$

a theorem of Prokhorov. Next step: the `Chen-Stein method`.

Poisson and Prokhorov

Siméon Denis Poisson (1781–1840): French (distribution: 1837).
Yuri Vasilevich Prokhorov (1929–2013): Russian.

Hyper-Geometric Distribution: Definition

The main thing to remember: 23 balls; 11 red, 12 black; take 9;

$$P(5 \text{ red}) = \frac{\binom{11}{5}\binom{12}{4}}{\binom{23}{9}}$$

General description: $X \sim \mathcal{H}(N, m; n)$
N objects of two types; m of type I; n is sample size; k is the number of Type I in the sample.

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \ k = 0, 1, \ldots, \min(m, n).$$

Key words: without replacement.

From Hyper-Geometric to Binomial

Theorem

$$\lim_{N\to\infty\, m\to\infty,\ m/N\to p} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} = \binom{n}{k} p^k (1-p)^{n-k}$$

Then can follow up with a Poisson approximation.
Example: a raffle; n^2 tickets, n winning. You buy $2n$ tickets.

$$P(3\,\mathrm{win}) = P(\mathcal{H}(n^2, n; 2n) = 3) \approx P(\mathcal{B}(2n, n^{-1}) = 3) \approx P(\mathcal{P}(2) = 3).$$

does not depend on n (exact in the limit $n \to \infty$).
Bottom line: $\mathcal{H}(N, m; n) \approx \mathcal{B}(n, m/N) \approx \mathcal{P}(nm/N)$

What else?

zeta distribution: discrete models with `heavy tails`

$P(X = k) = \frac{1}{\zeta(\alpha+1)k^{1+\alpha}}, \ \alpha > 0, \ k = 1, 2, \ldots$

Riemann's zeta function is the normalizing constant:

$$\zeta(t) = \sum_{k=1}^{\infty} \frac{1}{k^t} = \frac{1}{\prod_{p:\text{prime}}(1 - p^{-t})}, \ t > 1: \ \text{Euler's product formula}$$

Zipf distribution: Frequency of N ranked objects

$P(X = k) = C(N)k^{-1}, \ k = 1, \ldots, N.$

Erlang's distribution: truncated Poisson distribution

$$P(X = k) = C(N)\frac{\lambda^k}{k!}, \ k = 0, \ldots, N.$$

Who are those people?



Georg Friedrich Bernhard Riemann (1826–1866): German (1859)
George Kingsley Zipf (1902–1950): American (1932)
Agner Krarup Erlang (1878–1929): Danish (1906)

Construction of zeta distribution

Fix $s > 1$ and, for all prime numbers p, define independent *geometric random variables* $X_s(p)$ with

$$P(X_s(p) = k) = p^{-ks}(1 - p^{-s}), \ \ k = 0, 1, 2, \dots$$

Then the random variable

$$Z_s = \prod_{p:\text{prime}} p^{X_s(p)}$$

has zeta distribution. Indeed, by the *fundamental theorem of arithmetics*, $n = p_1^{k_1} \cdots p_n^{k_n}$, with distinct primes p_j and integer $k_j \geq 1$, and then, using `Euler's product formula`,

$$P(Z_s = n) = P\Big(X_s(p_j) = k_j, \ j = 1, \dots, n; X_s(p) = 0 \text{ otherwise}\Big)$$

$$= \prod_{j=1}^{n} p_j^{-sk_j} \prod_{p:\text{prime}} (1 - p^{-s}) = \frac{1}{n^s \zeta(s)}.$$