A Primer on the Functional Equation $f(x + y) = f(x) + f(y)$

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x + y) = f(x) + f(y), \tag{0.1}$$

for all $x, y \in \mathbb{R}$. Historical records at the École Polytechnique (see Appendix II in [2]) tell us that Cauchy's lecture on the 28th of November 1818 to the first year students was on finding all the possible continuous functions that satisfy (0.1). In addition to this, he also talked about some variations such as

$$f(x + y) = f(x)f(y),$$

for all $x, y \in \mathbb{R}$. The functional equation (0.1) is now known as Cauchy's functional equation. Cauchy showed that every continuous solution of (0.1) is linear, *i.e.*, given by $f(x) = xf(1)$, while Darboux observed that continuity at just one point is enough to get the same conclusion. Moving ahead, in 1905, Hamel constructed discontinuous solutions using the notion of what is now known as a Hamel basis. In this note, we begin by recalling these well documented contributions and then discuss two very interesting short papers written independently by Banach and Sierpiński on this topic. Both appear in the first volume of *Fundamenta Mathematicae*, bear the same title and prove the same theorem that every Lebesgue measurable solution of (0.1) is also of the form $f(x) = xf(1)$, but by different methods. In what follows, Lebesgue measure on the real line will be denoted by m.

1 Continuous Solutions

Keywords

Continuous solutions, Hamel basis, Lebesgue measure, linear functionals, Cauchy's functional equation.

To discuss continuous solutions, first restrict f to $\mathbb{Q}$, the rationals and regard $f : \mathbb{Q} \to \mathbb{R}$. By setting $x = y = 0$ in (0.1), observe that

Since f is an odd function, it follows that $f(x) = cx$ for all $x \in \mathbb{Q}$.

$f(0) = 2f(0)$ and hence $f(0) = 0$. Let $c = f(1)$ and by appealing to (0.1) once again, note that

$$f(2) = f(1) + f(1) = 2c.$$

By induction, it follows that $f(n) = cn$ for every integer $n \geq 1$. But then

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x),$$

for every $x \in \mathbb{R}$ and this shows that $f(x) = -f(-x)$, *i.e.*, f is an odd function. In particular,

$$f(-n) = -f(n) = -cn,$$

for $n \geq 1$. It follows that $f(n) = cn$ for every integer n.

For a positive rational $p/q \in \mathbb{Q}$,

$$qf(p/q) = f(p/q) + f(p/q) + \ldots + f(p/q) = f(p) = cp$$

where the second equality is a consequence of repeatedly applying (0.1). Therefore $f(p/q) = cp/q$ and hence $f(x) = cx$ for all positive rationals x. Since f is an odd function, it follows that $f(x) = cx$ for all $x \in \mathbb{Q}$. Thus, $f : \mathbb{Q} \to \mathbb{R}$ is linear and note that all of this is a consequence of just (0.1) – no further assumptions on f are needed.

Now assume that f is continuous on $\mathbb{R}$. Pick $x \in \mathbb{R}$ and choose a sequence of rationals p_i/q_i converging to x. By continuity of f at x,

$$f(x) = f\left(\lim_{i\to\infty} p_i/q_i\right) = \lim_{i\to\infty} f(p_i/q_i) = \lim_{i\to\infty} cp_i/q_i = cx,$$

and hence f is linear on $\mathbb{R}$ as claimed.

That f is linear also follows if f is assumed to be merely continuous at a single point. Indeed, let f be continuous at $x_0 \in \mathbb{R}$ and let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
\lim_{h\to x} f(h) &= \lim_{h\to x} f(h - x + x - x_0 + x_0) \\
&= \lim_{h\to x} (f(h - x + x_0) + f(x - x_0)) \\
&= \lim_{h\to x} f(h - x + x_0) + f(x - x_0) \\
&= f(x_0) + f(x - x_0) = f(x).
\end{aligned}$$

Here, the second, third and fifth equalities are due to (0.1) while the fourth one holds by the continuity of f at x_0. Hence, f is continuous everywhere and by the above discussion, it must necessarily be linear.

As f is measurable, Lusin's theorem shows that there is a closed set $F \subset [0, 1]$ with $m(F) \geq 2/3$ on which f is continuous.

2 Measurable Solutions are Linear: Banach's Proof

By the above discussion, it suffices to show that a measurable solution of (0.1) is continuous at $x = 0$. For this, it is enough to prove that for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(h) - f(0)| = |f(h)| < \epsilon,$$

for all $h \in (0, \delta)$; note that as f is odd, it is enough to consider only positive values of h. Let $\epsilon > 0$ be given. As f is measurable, Lusin's theorem shows that there is a closed set $F \subset [0, 1]$ with $m(F) \geq 2/3$ on which f is continuous. As F is compact, f is uniformly continuous on F. Therefore, there exists $\delta \in (0, 1/3)$ such that

$$|f(x) - f(y)| < \epsilon,$$

whenever $x, y \in F$ and $|x - y| < \delta$. Fix $h \in (0, \delta)$ and let

$$F - h = \{x - h : x \in F\}.$$

By translation invariance, $m(F - h) = m(F) \geq 2/3$.

Claim: The sets F and $F - h$ are not disjoint.

If F and $F - h$ were disjoint, then

$$1 + h = m([-h, 1]) \geq m(F \cup (F - h)) = m(F) + m(F - h) \geq 4/3,$$

and hence $h \geq 1/3$. But this contradicts $0 < h < \delta < 1/3$. Therefore, we may choose $a \in F \cap (F - h)$ and then both $a, a + h \in F$. Since the distance between them is h which is less than δ, the uniform continuity of f on F along with (0.1) shows that

$$|f(h)| = |f(a + h) - f(a)| < \epsilon.$$

This completes the proof.

3 Measurable Solutions are Linear: Sierpiński's Proof

Let $P, Q \subset \mathbb{R}$ be sets of positive Lebesgue measure. Then there exist $p \in P$ and $q \in Q$ such that $p - q \in \mathbb{Q}$.

Sierpiński's proof uses the following lemma that is of independent interest.

Let $P, Q \subset \mathbb{R}$ be sets of positive Lebesgue measure. Then there exist $p \in P$ and $q \in Q$ such that $p - q \in \mathbb{Q}$.

To prove this, we may assume that both P, Q are bounded. Let I be a compact interval containing Q. By the definition of Lebesgue measure, there exists, for $\epsilon = m(P)m(Q)/6m(I) > 0$, a countable collection of intervals $I_1, I_2, \ldots$ whose union covers P and

$$\sum_{i=1}^{\infty} m(I_i) < m(P) + \epsilon. \tag{3.1}$$

By subdividing the intervals I_i's, if necessary, we may suppose that the length of each I_i is less than $m(I)$. Since the sum of the lengths of the intervals I_i forms a convergent series, there exists an integer N such that

$$\sum_{i=N+1}^{\infty} m(I_i) < \epsilon. \tag{3.2}$$

Note that

$$R = (P \cap I_1) \cup (P \cap I_2) \cap \cdots \cup (P \cap I_N),$$

and $R_1 = P \setminus R$ are both measurable (since P is) and R_1 is covered by the intervals $I_{N+1}, I_{N+2}, \ldots$ the sum of whose lengths is less than ϵ by (3.2). Consequently,

$$m(R) > m(P) - \epsilon.$$

and hence

$$m(P) - \epsilon < m(R) \leq m(P \cap I_1) + m(P \cap I_2) + \ldots + m(P \cap I_N). \tag{3.3}$$

Note that $m(P) - \epsilon = m(P)(1 - m(Q)/6m(I)) > 0$. Now, if

$$m(P \cap I_i) \leq \left(\frac{m(P) - \epsilon}{m(P) + \epsilon}\right) m(I_i),$$

for all $1 \leq i \leq N$, then

$$m(P \cap I_1) + m(P \cap I_2) + \ldots + m(P \cap I_N) \leq \left(\frac{m(P) - \epsilon}{m(P) + \epsilon}\right)(m(I_1) + m(I_2) + \ldots + m(I_N)) < m(P) - \epsilon,$$

and this contradicts (3.3). Hence, there is an interval, say I_k ($1 \leq k \leq N$) with

$$m(P \cap I_k) > \left(\frac{m(P) - \epsilon}{m(P) + \epsilon}\right) m(I_k).$$

We may also suppose that both end points of I_k are rational by a suitably small adjustment. Let $I_k = [a, b]$ and consider those intervals of the form $I_k + nm(I_k) = [a+n(b-a), b+n(b-a)], n \in \mathbb{Z}$ which intersect I. Since I is compact, there are only finitely many such translated intervals with this property. Also, note that a pair of these translated intervals are either disjoint or intersect only at the end points.

Let s be the cardinality of the set

$$\mathcal{S} = \{n \in \mathbb{Z} : \text{the interval } [a + n(b - a), b + n(b - a)] \text{ intersects } I\}.$$

A moment's thought shows that $(s - 2)m(I_k) \leq m(I)$ and hence

$$sm(I_k) \leq m(I) + 2m(I_k) < 3m(I).$$

Claim: The sets Q and $\bigcup_{n\in\mathcal{S}} ((P \cap I_k) + nm(I_k))$ are not disjoint.

Note that both sets are contained in the union of $I_k + nm(I_k)$ as n varies in $\mathcal{S}$. If these sets were disjoint, then

$$\begin{aligned} sm(I_k) &= m(\bigcup_{n\in\mathcal{S}} I_k + nm(I_k)) \geq m(Q) + m(\bigcup_{n\in\mathcal{S}} ((P \cap I_k) + nm(I_k)) \\ &= m(Q) + sm(P \cap I_k) > m(Q) + s\left(\frac{m(P) - \epsilon}{m(P) + \epsilon}\right) m(I_k). \end{aligned}$$

Thus,

$$m(Q) < 2s\epsilon m(I_k)/(m(P) + \epsilon) < 6\epsilon m(I)/m(P)$$

which is false, as $\epsilon = m(P)m(Q)/6m(I)$.

The set of reals $\mathbb{R}$ is an infinite dimensional vector space over the rationals $\mathbb{Q}$.

It follows that there exists $q \in Q \cap \bigcup_{n \in S} ((P \cap I_k) + nm(I_k))$. Since $m(I_k) = b - a \in \mathbb{Q}$, there exists $p \in P$ whose distance from q is rational.

Sierpiński now considers the function $g(x) = f(x) - xf(1)$, where f is a measurable function satisfying (0.1). It can be checked that g also satisfies (0.1) and hence $g(x) = 0$ for all $x \in \mathbb{Q}$ as explained earlier. It remains to show that $g(x) = 0$ for all $x \in \mathbb{R}$. To this end, let

$$A^{\pm} = \{x \in \mathbb{R} : \pm g(x) > 0\}.$$

As g is an odd function, $x \mapsto -x$ is a bijection between A^+ and A^- and hence they have the same Lebesgue measure. If $m(A^+) = m(A^-) > 0$, the previous lemma shows that there are points $a^+ \in A^+$ and $a^- \in A^-$ such that $a^+ - a^- \in \mathbb{Q}$. Therefore $g(a^+ - a^-) = 0$. But then $g(a^+ - a^-) = g(a^+) - g(a^-)$ and this would mean that $g(a^+) = g(a^-)$ which is a contradiction. Hence, A^+ and A^- have measure zero and so does their union A. The function g vanishes precisely on $\mathbb{R} \setminus A$. If there were a point $a \in \mathbb{R}$ with $g(a) \neq 0$, then the set

$$H = \{x \in \mathbb{R} : g(x + a) = 0\},$$

which is obtained by merely translating $\mathbb{R} \setminus A$ by a units, must have positive measure since Lebesgue measure is translation invariant and $\mathbb{R} \setminus A$ has positive measure. On the other hand, for $x \in H$,

$$g(x) = -g(a) \neq 0,$$

by (0.1) and so $x \in A$. This implies that $H \subset A$ and hence $m(H) = 0$. Contradiction!

4 Discontinuous Solutions

The set of reals $\mathbb{R}$ is an infinite dimensional vector space over the rationals $\mathbb{Q}$. By using the axiom of choice, there exists a collection of reals $\{r_\alpha\}$ such that every $x \in \mathbb{R}$ can be expressed uniquely as a finite linear combination of some of the r_α's, *i.e.*,

$$x = \sum_{i=1}^{m} \lambda_i r_{\alpha_i}$$

where the λ_i's are in $\mathbb{Q}$ and m depends on x. The collection $\{r_\alpha\}$ forms a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$.

Define a function f on the set $\{r_\alpha\}$ by setting $f(r_\alpha) = s_\alpha$, where $\{s_\alpha\}$ is an arbitrary collection of reals. Extend f to all of $\mathbb{R}$ by linearity. Then f satisfies (0.1) as can be checked. This solution cannot be continuous, or for that matter measurable, since the only solutions are of the form $f(x) = cx$ for some c and this can only happen if $f(x)/x$ is constant for all x. This means that $s_\alpha/r_\alpha = c$ for all α which is certainly not always true since the choice of s_α is entirely up to us.

We conclude by observing that the graphs of such solutions to the Cauchy functional equation are dense in $\mathbb{R}^2$. Let f be a non-linear solution to (0.1). Choose non-zero reals x_1, x_2 such that $f(x_1)/x_1 \neq f(x_2)/x_2$. The vectors $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are linearly independent over $\mathbb{R}$ and so they span $\mathbb{R}^2$, *i.e.*, an arbitrary $(x, y) \in \mathbb{R}^2$ can be written as

$$(x, y) = c_1(x_1, f(x_1)) + c_2(x_2, f(x_2)),$$

for real constants c_1, c_2. Choose rational sequences a_n, b_n converging to c_1, c_2 respectively. The sequence $a_n(x_1, f(x_1)) + b_n(x_2, f(x_2))$ evidently converges to (x, y). It remains to note that each point $a_n(x_1, f(x_1)) + b_n(x_2, f(x_2))$ belongs to the graph of f since

$$f(a_n x_1 + b_n x_2) = a_n f(x_1) + b_n f(x_2),$$

by (0.1).

Suggested Reading

[1] **S Banach, Sur l'équation fonctionnelle** $f(x+y) = f(x) + f(y)$, ***Fundamenta Mathematicae*, Vol.1, 1920, available at** http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1115.pdf

[2] **B Belhoste, *Augustin–Louis Cauchy: A Biography*, Springer–Verlag, 1991.**

[3] **G Hamel, Eine Basis Zahlen und die unstetigen Lösungen der Funktionalgleichung:** $f(x) + f(y) = f(x+y)$**, *Math. Ann.*, Vol.60, pp.459–462, 1905.**

[4] **W Sierpiński, Sur l'équation fonctionnelle** $f(x+y) = f(x) + f(y)$, ***Fundamenta Mathematicae*, Vol.1, 1920, available at** http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1114.pdf

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