Summary of the Cauchy Functional Equation[1]

The equation:

$$f(x+y) = f(x) + f(y),\ \ x, y \in \mathbb{R}. \tag{1.1}$$

The objective is to identify all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).

The theorem.

(1) All solutions of (1.1) are *obvious*, that is, linear functions of the form

$$f(x) = cx,\ \ c = f(1), \tag{1.2}$$

under **additional assumptions** on f, such as one of the following:
- Continuity everywhere (Cauchy, 1821);
- Continuity at one point (Darboux, 1875);
- monotonicity on an interval;
- boundedness on an interval;
- Lebesgue measurability (Banach, Sierpiński, 1920).[2]

(2) In general, there are many solutions to (1.1) that are not of the form (1.2) (Hamel, 1905).

Ramifications. The following equations are reduced to (1.1), leading to the corresponding *obvious* solutions under *reasonable* conditions:

(1) $g(x+y) = g(x)g(y)$ becomes (1.1) after setting $f(x) = \ln g(x)$, with solutions $g(x) = a^x$;
(2) $g(xy) = g(x) + g(y)$ becomes (1.1) after setting $f(x) = g(e^x)$, with solutions $g(x) = c \ln x$;
(3) $g(xy) = g(x)g(y)$, also known as the `Hamel equation`,[3] becomes (1.1) after setting $f(x) = \ln g(a^x)$, with solutions $g(x) = x^r$.

Details of the proofs. After noticing that (1.1) implies $f(0) = f(0+0) = 2f(0)$, that is $f(0) = 0$, and also $0 = f(x-x) = f(x) + f(-x)$, that is $f(x) = -f(-x)$, we take a positive integer n and use (1.1) n times to conclude that $f(n) = nf(1)$ and $f(1/n) = (1/n)f(1)$, that is, $f(m/n) = (m/n)f(1)$, that is, (1.2) holds for all *rational*. If f is continuous, then, after passing to the limit, (1.2) holds for all real x; if f is continuous at only one point, then (1.1) implies that f is continuous everywhere. Then Banach used a theorem of Lusin to argue that Lebesgue measurability of f, together with (1.1), leads to continuity of f at least at one point; Sierpiński used a different property of the Lebesgue measure, in the spirit of *geometric measure theory*.

Construction of non-linear functions satisfying (1.1) relies on existence of the `Hamel basis` in $\mathbb{R}$ considered as a linear space over the rationals $\mathbb{Q}$. In other words, there is a collection $\mathcal{H}$ of real numbers such that every real x can be written as a *finite* sum $x = \sum_{i=1}^{k} r_i x_i$, with $x_i \in \mathcal{H}$ and $r_i \in \mathbb{Q}$. Existence of such a basis is *essentially equivalent* to the `axiom of choice`; no explicit constructions are known. Then a function f can be defined in any way on the elements of $\mathcal{H}$, and then extended by linearity to all of $\mathbb{R}$. If the result is not (1.2), then the graph of f is dense in $\mathbb{R}^2$.

[1] Sergey Lototsky, USC; version of May 13, 2024

[2] Two different proofs, by S. Banach and W. Sierpiński, were published in the first issue of the journal Fundamenta Mathematicae

[3] Georg Karl Wilhelm Hamel (1877–1954) was a student of D. Hilbert.