Summary of the Cauchy Functional Equation

The equation:
\[ f(x + y) = f(x) + f(y), \ x, y \in \mathbb{R}. \] (1.1)

The objective is to identify all functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying (1.1).

The theorem.

1. All solutions of (1.1) are obvious, that is, linear functions of the form
   \[ f(x) = cx, \ c = f(1), \] (1.2)
   under additional assumptions on \( f \), such as one of the following:
   - Continuity everywhere (Cauchy, 1821);
   - Continuity at one point (Darboux, 1875);
   - monotonicity on an interval;
   - boundedness on an interval;
   - Lebesgue measurability (Banach, Sierpiński, 1920).

2. In general, there are many solutions to (1.1) that are not of the form (1.2) (Hamel, 1905).

Ramifications. The following equations are reduced to (1.1), leading to the corresponding obvious solutions under reasonable conditions:

1. \( g(x + y) = g(x)g(y) \) becomes (1.1) after setting \( f(x) = \ln g(x) \), with solutions \( g(x) = a^x \);
2. \( g(xy) = g(x) + g(y) \) becomes (1.1) after setting \( f(x) = e^x \), with solutions \( g(x) = c\ln x \);
3. \( g(xy) = g(x)g(y) \), also known as the Hamel equation, becomes (1.1) after setting \( f(x) = \ln g(a^x) \), with solutions \( g(x) = x^r \).

Details of the proofs. After noticing that (1.1) implies \( f(0) = f(0 + 0) = 2f(0) \), that is \( f(0) = 0 \), and also \( 0 = f(x - x) = f(x) + f(-x) \), that is \( f(x) = -f(-x) \), we take a positive integer \( n \) and use (1.1) \( n \) times to conclude that \( f(n) = nf(1) \) and \( f(1/n) = (1/n)f(1) \), that is, \( f(m/n) = (m/n)f(1) \), that is, (1.2) holds for all rational. If \( f \) is continuous, then, after passing to the limit, (1.2) holds for all real \( x \); if \( f \) is continuous at only one point, then (1.1) implies that \( f \) is continuous everywhere. Then Banach used a theorem of Lusin to argue that Lebesgue measurability of \( f \), together with (1.1), leads to continuity of \( f \) at least at one point; Sierpiński used a different property of the Lebesgue measure, in the spirit of geometric measure theory.

Construction of non-linear functions satisfying (1.1) relies on existence of the Hamel basis in \( \mathbb{R} \) considered as a linear space over the rationals \( \mathbb{Q} \). In other words, there is a collection \( \mathcal{H} \) of real numbers such that every real \( x \) can be written as a finite sum \( x = \sum_{i=1}^{k} r_i x_i \), with \( x_i \in \mathcal{H} \) and \( r_i \in \mathbb{Q} \). Existence of such a basis is essentially equivalent to the axiom of choice; no explicit constructions are known. Then a function \( f \) can be defined in any way on the elements of \( \mathcal{H} \), and then extended by linearity to all of \( \mathbb{R} \). If the result is not (1.2), then the graph of \( f \) is dense in \( \mathbb{R}^2 \).

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2Two different proofs, by S. Banach and W. Sierpiński, were published in the first issue of the journal Fundamenta Mathematicae
3Georg Karl Wilhelm Hamel (1877–1954) was a student of D. Hilbert.