Spring 2024, MATH 408, Exam 2

Monday, April 15; 11–11:50am

Instructor — S. Lototsky (KAP 248D; x0–2389; lototsky@usc.edu)

Instructions:

- No notes, no books or other printed materials (including printouts from the web), no collaboration with anybody (or anything, like AI).
- You should have access to a calculator or some other computing device, and to the χ^2 and F distribution tables. Instead of the tables, you are welcome to use the statistical functions available on your computing device.
- Answer all questions, show your work, and clearly indicate your answers; upload the solutions to GradeScope.
- **Each problem is worth 10 points.**

Problem 1.
Below is part of a two-way ANOVA table for $b = 5$ blocks and $k = 6$ treatments. Fill out the rest of the table.

Source	SS	df	MS	F	Prob $> F$
Blocks	80	4	20	2.105	0.118045835...
Treatments	210	5	42	4.42	0.007124324...
Error	190	20	9.5		
Total	480	29			

`Answers are in smaller font. Note that, using the table, you can only conclude that` $\mathbb{P}(F_{4,20} > 2.105) > 0.1$ `and` $\mathbb{P}(F_{5,20} > 4.42) \in (0.005, 0.01)$

Problem 2.
To test whether a die is fair, 64 rolls were made, and the corresponding outcomes were as follows:

Face value	Observed frequency
1	8
2	9
3	15
4	15
5	9
6	8

Estimate the P-value if the χ^2 test is used.

`Solution: for the value` φ^* `of the test statistic, which is the sum of observed-minus-` `with expected equal to` $64/6 = 32/3$, `we get`

$$\varphi^* = 2((24-32)^2 + (27-32)^2 + (45-32)^2)/(32 \cdot 3) = (64+25+169)/48 = 5.375$$

and the P-value is

$$\mathbb{P}(\chi_5^2 > \varphi^*) = 0.37184731...$$

Using the table, you can only conclude that the P-value is bigger than 0.1 (and less than 0.9).

Problem 3. Assume that the following is an independent random sample from population X with a continuous cdf $F_X(x) = F(x)$:

$$14.4 \quad 15.5 \quad 13.3 \quad 11.1 \quad 12.2,$$

and assume that the following is an independent random sample from population Y with cdf $F_Y(x) = F(x+\theta)$:

$$8.8 \quad 10.0 \quad 7.7 \quad 4.4 \quad 0.6.$$

Compute the P-value of the sign test for the null hypothesis $\theta = 0$ against the alternative $\theta > 0$. Note that the alternative means that the random variable X is more likely to be large, that is, $\mathbb{P}(X > Y) > 1/2$.

Solution: with the test statistic $M = \sum_{k=1}^{5} I(X_k > Y_k)$ we get $M^* = 5$ (all X samples are bigger than the corresponding Y samples), and therefore
P-value$= \mathbb{P}(\mathcal{B}(5, 1/2) \geq 5) = \mathbb{P}(\mathcal{B}(5, 1/2) = 5) = 2^{-5} = 1/32$.

Problem 4. For the two samples in Problem 3, compute the Spearman rank correlation coefficient.

Solution. For the ranks of X, that is, the positions of X_k in the sample arranged in increasing order, we get $4, 5, 3, 1, 2$; the corresponding ranks of Y_k are $4, 5, 3, 2, 1$ and the sum of the squares of the differences of the ranks is 2.
Using the ''no-tie formula'' for r_s, with $n = 5$, we get $r_s = 1-(6\cdot 2)/(5\cdot 24) = 1-0.1 = 0.9$

Problems 5. A coin-making machine produces pennies with unknown probability p to turn up heads; this probability is equally likely to be any number between 0 and 1.

A coin pops out of the machine, flipped 22 times and lands heads 5 times. Compute the Bayesian estimate $\hat{p}$ of p.

Solution. Using the idea of conjugate priors (Beta/Binomial), we conclude that, given the prior $\mathrm{Beta}(1, 1)$ (uniform), the posterior is $\mathrm{Beta}(5+1, 17+1)$, and then $\hat{p}$, being the posterior mean, is $(5+1)/(22+2) = 1/4$.