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A Closer Look at the Compensating Polar Planimeter

John Eggers

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John Eggers (jeggers@ucsd.edu, MR ID 1228813, ORCID ID 000-0001-8064-7004) received his Ph.D. in Mathematics from the University of California at San Diego in 1995 where he is now a faculty member of the Mathematics Department. His primary work is in undergraduate education. This may have led to his interest in juggling, making and throwing boomerangs, and collecting and playing with planimeters.

Although much has been written on the subject of polar planimeters, they still remain relatively obscure instruments. This is unfortunate: not only do they provide a remarkably quick and precise method for measuring areas, they have a fascinating history and are a delightful example of Green's theorem to show calculus students (or colleagues, for that matter). Perhaps part of the reason polar planimeters are not discussed in calculus courses is that an explicit computation with the relevant vector field using Green's theorem is somewhat complicated. (There are geometric explanations of planimeter operation that avoid Green's theorem, but it is debatable whether they are simpler. See [**6**, **9**].) In this paper, we avoid tedious computation by applying Green's theorem to an *implicit* representation of the planimeter vector field. The resulting simplification should make the explanation of the polar planimeter accessible to any calculus student with a knowledge of partial derivatives and a modest background of linear algebra. We will also discuss two other important features specific to polar planimeters: (1) the neutral circle, and (2) how compensating polar planimeters compensate.

A brief description of planimeter operation

A polar planimeter is a mechanical device used to measure the area of a region by tracing the boundary of the region. Figure 1 shows a compensating polar planimeter from the author's collection that indicates how it is used.

The instrument consists of three major components: a (1) pole arm, (2) tracer arm, and (3) measuring wheel. The pole arm merely rotates about the pole, the tracer arm is connected to the free end of the pole arm by a pivot joint (a ball and socket joint in the case of the compensating polar planimeter pictured in Figure 1), and the measuring wheel is attached to the tracer arm with its axis parallel to the tracer arm. The area enclosed by a simple closed curved is measured by moving the tracer along the curve clockwise and recording the amount the measuring wheel moves which, as we will see, is proportional to the area enclosed by the curve. The dial keeps track of

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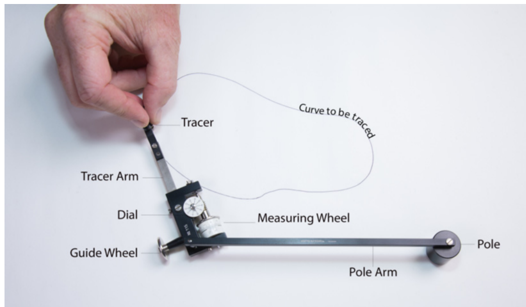

Figure 1. A compensating polar planimeter.

how many complete rotations are made by the measuring wheel, and the guide wheel merely balances the instrument so that it does not tip over while tracing. See [10] for a more complete description of the technical aspects of using a polar planimeter, including how to take into account the area of the neutral circle and how to make use of a compensating polar planimeter.

The polar planimeter was invented by Jakob Amsler in 1854 (see [9]) and Amsler published a paper describing it in 1856 (see [1], and [4] for a link to [1]). The polar planimeter is not the only type of planimeter; there are other types of planimeters. See [11] for a brief mathematical discussion of several types of planimeters; see [9, 12] for a more complete historical discussion of planimeters.

Analysis of planimeter operation

The basic design of a polar planimeter is remarkable in its simplicity: as indicated above, it consists of two rigid arms connected by a pivot joint. An idealized version is depicted in Figure 2. One arm, called the pole arm with length P, has an end fixed at a point (0, 0) called the pole, about which the pole arm is free to rotate. The other arm, called the tracer arm of length T, is connected at one end to the free end of the pole arm by a pivot joint (a, b) that is free to rotate. At the other end of the tracer arm is the tracer point (x, y) which is used to trace the boundary curve C (assumed to be a simple closed curve) of the region D. A measuring wheel W is attached to the tracer arm with its axis parallel to the tracer arm. In order to simplify the ensuing computations, the measuring wheel W of our idealized planimeter is drawn coincident with the tracer point. The vector $\boldsymbol{\tau}$ in Figure 2 represents the unit vector perpendicular to the tracer arm in the direction of positive measuring wheel motion.

It is clear that placing the measuring wheel coincident with the tracer point is an impractical design for a usable planimeter. However, the computations resulting from placing it there in our idealized planimeter will be correct since, as we will see, the displacement M of the measuring wheel during a normal tracing operation is independent of the placement of the measuring wheel W along the tracer arm. We define *normal tracing operation* to mean that the curve is traced with the pole of the planimeter outside the region enclosed by the curve. We will consider what happens when the pole of the planimeter is placed inside the region enclosed by the curve in the "The neutral circle" section of this paper.

To see that the total wheel displacement M during a normal tracing operation is independent of the placement of the measuring wheel, we first observe that the displacement of a tracer arm can be decomposed into a component perpendicular to the tracer arm and a component arising from rotation about the pivot, as indicated in Figure 3. The parameter w represents the displacement of the measuring wheel W

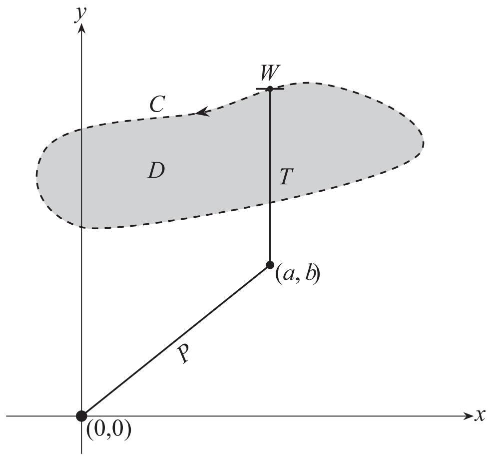

Figure 2. An idealized polar planimeter.

from the pivot (a, b) along the tracer arm. Note that on some planimeters, the pivot is between the measuring wheel and the tracer point, in which case w would be negative.

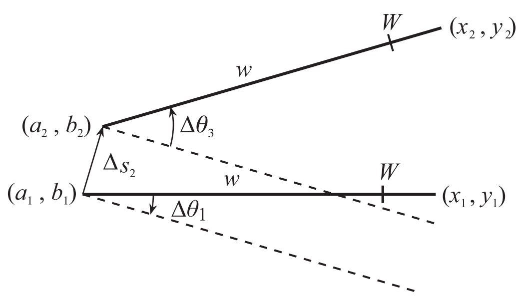

Figure 3. Decomposition of wheel motion.

As indicated in Figure 3, a wheel displacement ΔM resulting from a tracer displacement from $(x_1, y_1) - (a_1, b_1)$ to $(x_2, y_2) - (a_2, b_2)$ can be expressed in the form $\Delta M = w\Delta\theta_1 + \Delta s_2 + w\Delta\theta_3$, where the displacements are performed in the order indicated by the numeric subscripts. $\Delta\theta_1$ is the angle of rotation that places the tracer arm perpendicular to the line joining the initial pivot position (a_1, b_1) to the final pivot position (a_2, b_2) so that Δs_2 is a displacement perpendicular to the tracer arm. $\Delta\theta_3$ is the angle of rotation placing the tracer point at its final position (x_2, y_2). We see that $\Delta M = \Delta s + w\Delta\theta$.

We conclude that the total wheel displacement can be expressed as $M = \int_C ds + w\int_C d\theta$, where $\int_C ds$ is the contribution from motion perpendicular to the tracer arm and $w\int_C d\theta$ is the contribution from rotational motion about the pivot. We now observe that the geometric constraint on the tracer arm during a normal tracer operation implies that $\int_C d\theta = 0$ since the tracer arm must return to its original position without making a complete revolution (since the pole is not inside the region enclosed by the curve). Thus, during a normal tracing operation, $M = \int_C ds$, which is independent of the placement of the measuring wheel along the tracer arm, as claimed.

In Amsler's original design for the polar planimeter (see Figure 4), the pivot joint (a, b) was constrained to lie on one side of the line joining the pole $(0, 0)$ and the tracer point (x, y). With this constraint, the coordinates (a, b) of the pivot joint are uniquely determined by the coordinates (x, y) of the tracer point.

We will denote by Ω the open disk of radius $P + T$ centered at the pole $(0, 0)$ and write $\partial\Omega$ for its bounding circle. Note that $\partial\Omega$ is the circle the planimeter traces with its pole and tracer arms fully extended and that all points accessible to the planimeter lie in Ω.

However, not all points of Ω are accessible to the planimeter. Thus, we define the set $\mathcal{A} \subset \Omega$ of accessible points to be the open annular region $\mathcal{A} = \left\{(x, y) \mid (P - T)^2 < x^2 + y^2 < (P + T)^2\right\}$. This definition precludes the possibility that the pole and tracer arms would be parallel with the tracer point at an accessible point (x, y). That is, (x, y) an accessible point implies $(x, y) \neq \lambda(a, b)$ for any λ. In particular, the pole $(0, 0)$ is not an accessible point.

We define a set S to be accessible if the closure of S is contained in $\mathcal{A}$. Thus, a simply connected domain D is accessible precisely when all points of both D and its boundary curve are accessible. Since a simple closed curve is traceable precisely when it is accessible, we will use the terminology "traceable curve" and "accessible curve" interchangeably.

Accessible curves fall into one of two categories: those that do not enclose the pole $(0, 0)$ and those that do enclose $(0, 0)$. Tracing an accessible curve C that does not enclose the pole $(0, 0)$ results in a normal tracing operation, in which case the enclosed domain D is also accessible, as depicted in Figure 2. Curves that enclose the pole will be considered in the "The neutral circle" section.

Since Green's theorem will be used to prove that a polar planimeter actually measures the area enclosed by a simple closed curve by tracing the curve, we will state Green's theorem here for reference. Although there are stronger versions of Green's theorem (see, e.g., [**2**, **3**, **5**]), we will use the form of Green's theorem stated in most textbooks:

Theorem (Green's theorem). *Let D be a simply connected domain bounded by a piecewise continuously differentiable simple closed curve C. Let $P(x, y)$ and $Q(x, y)$ be continuously differentiable on a neighborhood of $D \cup C$. Then,*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Figure 4. An Amsler polar planimeter.

There are many explanations for why a planimeter works, but they are all based on the fact that the measuring wheel is constrained to roll in a direction perpendicular to the tracer arm. This means the displacement M of the measuring wheel may be expressed as the line integral around the simple closed curve C of the unit vector $\boldsymbol{\tau}$ (see Figure 2) perpendicular to the tracer arm T and oriented to point in the direction of positive wheel motion; that is, $M = \int_C \boldsymbol{\tau} \cdot d\mathbf{r}$. Thus, adopting the standard convention that *positive orientation* corresponds to a counter-clockwise traversal around C, we have $\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y-b\right), x-a\right)$, where T is the length of the tracer arm, and

$$M = \int_C \boldsymbol{\tau} \cdot d\mathbf{r} = \frac{1}{T}\int_C -(y-b)\, dx + (x-a)\, dy,$$

where (x, y) are the coordinates of the tracer point moving along the curve C and (a, b) are the coordinates of the pivot joint connecting the pole arm and tracer arm. In practice, planimeters are manufactured so that positive wheel displacement corresponds to a *clockwise* traversal of C, but we will stay with the standard mathematical convention to avoid confusion.

Since the pivot coordinates (a, b) are functions of the coordinates (x, y), applying Green's theorem to the line integral representing the measuring wheel displacement M yields

$$M = \frac{1}{T}\iint_D \left[2 - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right)\right] dx\, dy,$$

where D is the simply connected domain bounded by C. Thus, if we demonstrate that $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 1$ on D, it will follow immediately that the displacement M of the measuring wheel is proportional to the area $A(D)$ of the region D. While it is possible to obtain an explicit expression for the partial derivatives in terms of (x, y) (see, e.g., [7]), it is easier to obtain an implicit expression for $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}$. To do this, observe that (a, b) satisfy

$$a^2 + b^2 = P^2,$$
$$(x-a)^2 + (y-b)^2 = T^2,$$

where P and T are the length of the pole arm and tracer arm, respectively. It follows that (a, b) satisfy the following system of partial differential equations:

$$(1)\quad a\frac{\partial a}{\partial x} + b\frac{\partial b}{\partial x} = 0, \qquad (2)\quad a\frac{\partial a}{\partial y} + b\frac{\partial b}{\partial y} = 0,$$

$$(3)\quad x\frac{\partial a}{\partial x} + y\frac{\partial b}{\partial x} = x - a, \text{ and} \qquad (4)\quad x\frac{\partial a}{\partial y} + y\frac{\partial b}{\partial y} = y - b.$$

Treating this as a system of four linear equations in the four unknowns $\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}, \frac{\partial b}{\partial x}$ and $\frac{\partial b}{\partial y}$, we find that

$$\frac{\partial a}{\partial x} = -\frac{b(x-a)}{ay-bx}, \qquad \frac{\partial a}{\partial y} = -\frac{b(y-b)}{ay-bx},$$

$$\frac{\partial b}{\partial x} = \frac{a(x-a)}{ay-bx}, \text{ and} \qquad \frac{\partial b}{\partial y} = \frac{a(y-b)}{ay-bx}.$$

Therefore,

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = \frac{ay - bx}{ay - bx} = 1, \quad \text{provided} \quad (x, y) \neq \lambda(a, b) \quad \text{for any constant } \lambda.$$

We saw previously that the condition $(x, y) \neq \lambda(a, b)$ for any λ holds for all accessible points (x, y) and that the domain D enclosed by an accessible simple closed curve C not encircling the pole $(0, 0)$ is also accessible. In this situation, the above argument shows that the planimeter vector field $\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y - b\right), x - a\right)$ is continuously differentiable in a neighborhood of $D \cup C$. If, in addition, C is piecewise continuously differentiable, it follows from Green's theorem that the wheel displacement $M = \frac{1}{T}\iint_D dx\, dy = \frac{1}{T} A(D)$, and we have the following:

Theorem (Planimeter theorem). *Given a polar planimeter with tracer arm length T and an accessible piecewise continuously differentiable simple closed curve C. If the domain D enclosed by C does not contain the pole of the planimeter, then*

$$T\, M = A(D)$$

where A(D) is the area of D and M is the displacement of the planimeter's measuring wheel.

A practical consequence of this result is that the tracer arm length T acts as a scale factor for the measuring wheel. In fact, this is how planimeters are calibrated and many of the early planimeter models had adjustable-length tracer arms so that they could be quickly adjusted to read the enclosed area using different scales.

The neutral circle

As we have seen, Green's theorem cannot be applied directly to the situation where the pole $(0, 0)$ is in the domain D. To see what happens in this situation, we first consider the special case of the disk Ω_β of radius $R = P\cos(\alpha) + T\cos(\beta)$ centered at the pole $(0, 0)$. The boundary circle $\partial\Omega_\beta$ can be traced with the tracer arm inclined at a small fixed angle β with the radial segment from $(0, 0)$ to the tracer point, as in Figure 5. Note that $P\sin(\alpha) = T\sin(\beta)$ by the law of sines, so that $P^2\cos^2(\alpha) = P^2 - T^2 + T^2\cos^2(\beta)$. Denoting by θ the angle the radial vector (x, y) makes with the positive x-axis, the planimeter vector field $\boldsymbol{\tau} = (-\sin(\theta - \beta), \cos(\theta - \beta))$ along $\partial\Omega_\beta$ and its component tangent to $\partial\Omega_\beta$ is $\cos(\beta)$. Thus, the measuring wheel displacement $M_{\partial\Omega_\beta}$ is $M_{\partial\Omega_\beta} = 2\pi\cos(\beta)\,[P\cos(\alpha) + T\cos(\beta)]$; whereas, the area enclosed by Ω_β is

$$A(\Omega_\beta) = \pi\,[P\cos(\alpha) + T\cos(\beta)]^2 = \pi[P^2 - T^2 + 2T\cos^2(\beta) + 2PT\cos(\alpha)\cos(\beta)].$$

We conclude that

$$A(\Omega_\beta) = T\, M_{\partial\Omega_\beta} + \pi\,\left(P^2 - T^2\right).$$

Next, we consider a simple closed curve C interior to Ω and enclosing $(0, 0)$, as in Figure 6. Since C is interior to Ω, C is interior to a disk Ω_β for a sufficiently small angle β. Since the region $\Omega_\beta \setminus D$ interior to $\partial\Omega_\beta$ and exterior to C does not contain $(0, 0)$, by an application of Green's theorem, its area can be measured dire ctly by the

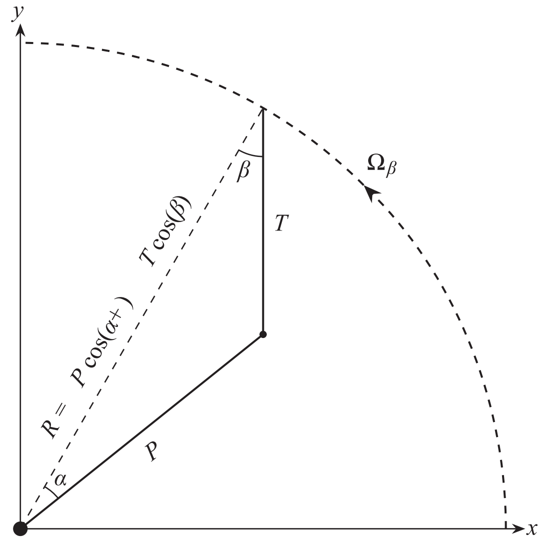

Figure 5. Circle centered at pole.

planimeter: $A(\Omega_\beta) - A(D) = T\, M_{\partial\Omega_\beta} - T\, M_C$. Thus, $A(D) = T\, M_C + A(\Omega_\beta) - T\, M_{\partial\Omega_\beta} = T\, M_C + \pi\,(P^2 - T^2)$ for every simple closed curve C enclosing an accessible domain D containing $(0, 0)$. For most polar planimeters, the pole arm length P is greater than the tracer arm length T, and under the assumption that $P > T$, the quantity $R_N = \sqrt{P^2 - T^2}$ is called the radius of the neutral circle. When our idealized planimeter is set up to trace a circle centered at $(0, 0)$ with radius $R_N = \sqrt{P^2 - T^2}$, we see that T, R_N, and P form a right triangle with hypotenuse P so that the measuring wheel's axis is tangent to the circle and its displacement after tracing the circle is zero. Hence the name "neutral circle."

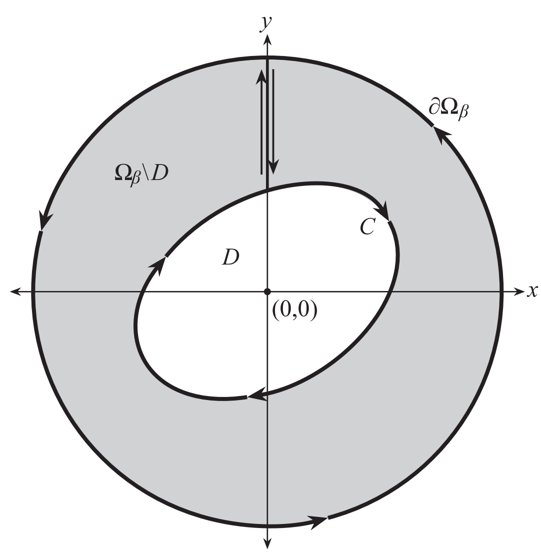

Figure 6. Pole inside.

We have seen that when the curve C enclosed the pole $(0,0)$, the area $A(D) = TM_C + \pi R_N^2$, where the radius of the neutral circle $R_N = \sqrt{P^2 - T^2}$ for our idealized polar planimeter. However, unlike the situation where the curve C does not enclose the pole $(0,0)$, the measuring wheel displacement M_C along C *does* depend on the placement of the measuring wheel along the tracer arm T when C encloses the pole $(0,0)$. This implies that the radius of the neutral circle R_N depends on the placement of the measuring wheel along the tracer arm of the particular planimeter being used.

Observe that the relationship $A(\Omega_\beta) = T\, M_{\partial\Omega_\beta} + \pi\, \left(P^2 - T^2\right)$ that we derived above remains true in the limit as β tends to 0; that is, $A(\Omega) = T\, M_{\partial\Omega} + \pi\, \left(P^2 - T^2\right)$ since $\pi\, (P+T)^2 = T \cdot 2\pi\, (P+T) + \pi\, \left(P^2 - T^2\right)$. This means we could have determined the radius of the neutral circle by doing the above computation with the planimeter fully extended, even though the hypotheses of Green's theorem are not met when the planimeter is fully extended.

To compute R_N for a planimeter with its measuring wheel displaced a distance w from the pivot (a,b) along the tracer arm T, we fully extend the planimeter, as indicated in Figure 7. (Note: Some planimeters situate the measuring wheel on the pole side of the pivot. The computation of R_N in that case is similar and left to the interested reader.)

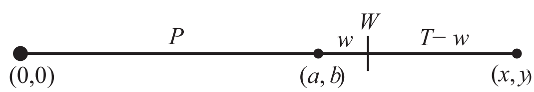

Figure 7. Extended planimeter.

Tracing the circle $\partial\Omega$ of radius $P + T$ with the planimeter fully extended, we see that the enclosed area is $A(D) = \pi\, (P+T)^2$ and the measuring wheel displacement $M_{\partial\Omega} = 2\pi\, (P + w)$. Thus, the area of the neutral circle is

$$\pi R_N^2 = A(D) - TM_{\partial\Omega} = \pi\, (P+T)^2 - 2\pi T\, (P+w) = \pi \left[\left(P^2 - w^2\right) + (T-w)^2\right].$$

It follows that the radius of the neutral circle $R_N = \sqrt{\left(P^2 - w^2\right) + (T-w)^2}$.

The geometry of the neutral circle is depicted in Figure 8. Note how the direction the measuring wheel rolls is orthogonal to the path it traverses as the neutral circle C_N is traced so that the measuring wheel displacement after tracing the neutral circle is zero. Perhaps it's worth mentioning that $w = T$ for our idealized planimeter and, in that case, the formula for the radius of the neutral circle R_N reduces to $R_N = \sqrt{P^2 - T^2}$, as it should.

In practice, most planimeter manufacturers tested each instrument they produced to determine the area of its neutral circle and included this information with the instrument, as in Figure 9.

The compensating polar planimeter

In the original design of polar planimeters pioneered by Jakob Amsler, the tracer and pole arms are permanently attached by a hinged joint. In 1894, Gottlieb Coradi introduced the compensating polar planimeter, based on a design patented by Lang (and sometimes called the Lang–Coradi planimeter) in 1893 in Switzerland [9]. This design has persisted essentially unchanged to the present day. (See Figure 10.)

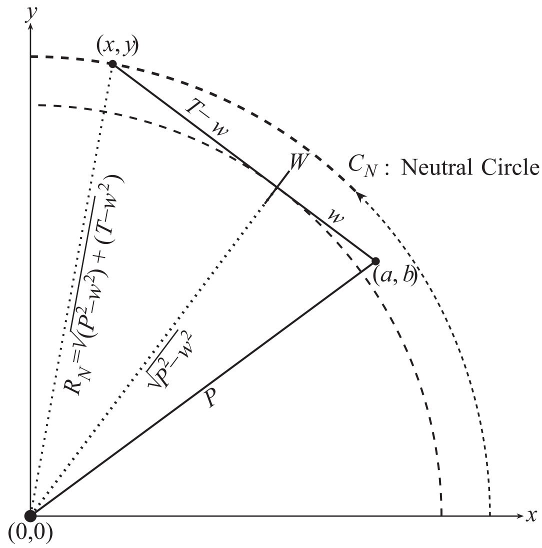

Figure 8. Geometry of neutral circle.

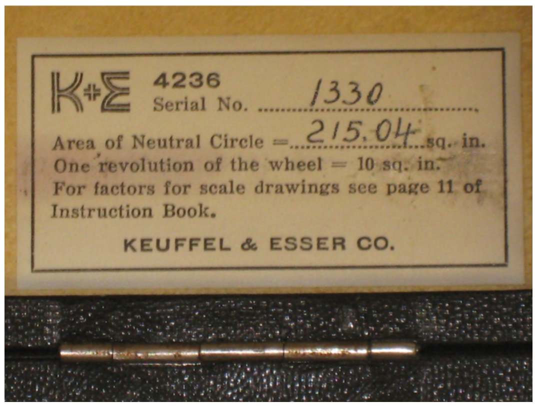

Figure 9. Area of neutral circle on data card.

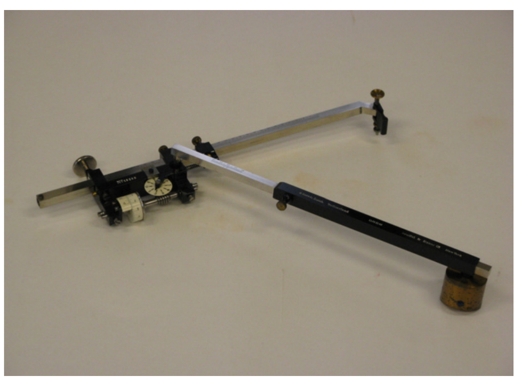

Figure 10. A Coradi compensating polar planimeter.

In a compensating polar planimeter, the pole and tracer arms are separate pieces that fit together via a ball-and-socket pivot joint; Figure 11 shows this for the planimeter used in Figure 1. This design allows the instrument to be set up in two distinct orientations with the pivot joint on either side of the line through the pole and tracer point (see Figure 13). By taking readings with each orientation of the pivot joint and averaging the results, errors caused by misalignment of the measuring wheel exactly cancel; thus, the design allows one to *compensate* for this type of error.

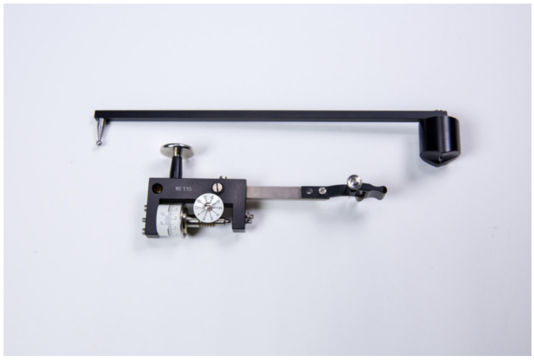

Figure 11. Parts of a compensating polar planimeter.

To see that this design actually compensates for a misaligned measuring wheel, we recall that $\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y-b\right), x-a\right)$ is the unit vector perpendicular to the tracer arm and in the direction of positive wheel motion, and we set $\boldsymbol{\rho} = \frac{1}{T}(x-a, y-b)$ be the unit vector parallel to the tracer arm in the direction from the pivot (a, b) toward the tracer point (x, y). Then, if the measuring wheel axis is misaligned by an angle ϑ (hopefully small!), the unit vector $\boldsymbol{w}$ in the direction of positive wheel displacement is no longer perpendicular to the tracer arm and can be expressed as $\boldsymbol{w} = \cos(\vartheta)\boldsymbol{\tau} - \sin(\vartheta)\boldsymbol{\rho}$. (See Figure 12.)

Thus, the displacement M of the measuring wheel after tracing a curve C enclosing a domain D is given by

$$\begin{aligned} M &= \cos(\vartheta)\int_C \boldsymbol{\tau}\cdot d\mathbf{r} - \sin(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r} \\ &= \frac{\cos(\vartheta)}{T}A(D) - \sin(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r}. \end{aligned}$$

In other words,

$$A(D) = T\ \sec(\vartheta)\, M + T\tan(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r}.$$

Assuming that ϑ is constant, the factor $\sec(\vartheta)$ can be compensated for by adjusting the tracer arm length T; however, the term involving $\int_C \boldsymbol{\rho}\cdot d\mathbf{r}$ cannot be "calibrated away" since it depends on the curve C.

Consider what happens if the compensating polar planimeter is placed in the two possible configurations. At any point $\boldsymbol{r} = (x, y)$ on the curve C, there are two possible values for the unit vector $\boldsymbol{\rho}$: $\boldsymbol{\rho}_L$ and $\boldsymbol{\rho}_R$ (see Figure 13).

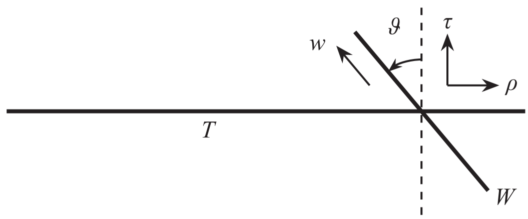

Figure 12. Measuring wheel misaligned by ϑ.

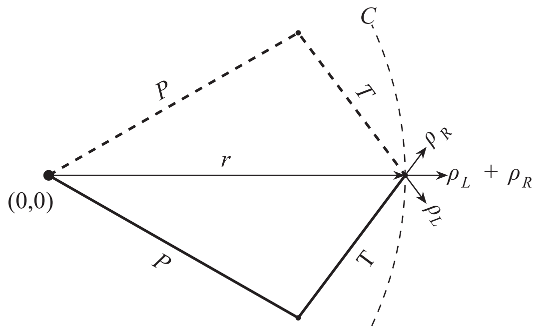

Figure 13. Two configurations of a compensating polar planimeter.

Since the two configurations are symmetric about the line connecting the pole $(0, 0)$ and the tracer point (x, y), $\boldsymbol{\rho}_L + \boldsymbol{\rho}_R$ is parallel to the vector $\boldsymbol{r} = (x, y)$ and the length of $\boldsymbol{\rho}_L + \boldsymbol{\rho}_R$ depends only on the distance $r = \sqrt{x^2 + y^2}$ between the pole $(0, 0)$ and the tracer point (x, y). In fact,

$$\boldsymbol{\rho}_L + \boldsymbol{\rho}_R = \frac{1}{T}\left[1 - \frac{P^2 - T^2}{r^2}\right]\mathbf{r} = \frac{1}{T}\nabla\left[\frac{1}{2}r^2 - \left(P^2 - T^2\right)\log(r)\right].$$

Thus, $\boldsymbol{\rho}_L + \boldsymbol{\rho}_R$ is a gradient field and it follows that $\int_C \boldsymbol{\rho}_L \cdot d\boldsymbol{r} + \int_C \boldsymbol{\rho}_R \cdot d\boldsymbol{r} = 0$ for every simple closed curve not passing through the pole $(0, 0)$. This shows that averaging the readings of the compensating polar planimeter taken with the two configurations eliminates the error due to misalignment of the measuring wheel.

The design of the compensating polar planimeter allows more precise measurement and by the 1930s the compensating polar planimeter had essentially displaced the original Amsler design.

Epilogue

Planimeters are fascinating instruments that deserve to be better known. They have an interesting history (see, e.g., [**9**, **12**]) and are a perfect example of a mechanical implementation of Green's theorem. In fact, planimeters are still manufactured today and find a variety of interesting uses (see, e.g., [**8**]).

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Summary. A polar planimeter measures the area of a region by tracing its perimeter. In this paper, we (1) show a simple approach to analyzing the operation of a polar planimeter, (2)

explain what a neutral circle is and its significance, (3) explain what it is that a compensating polar planimeter compensates for, and (4) provide a glimpse of the fascinating history of this instrument.

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