MATHEMATICAL NOTES

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A REMARK ON STIRLING'S FORMULA

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We shall prove Stirling's formula by showing that for $n=1, 2, \cdots$

$$n! = \sqrt{2\pi}n^{n+1/2}e^{-n}\cdot e^{r_n} \tag{1}$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}. \tag{2}$$

The usual textbook proofs replace the first inequality in (2) by the weaker inequality

$$0 < r_n$$

or

$$\frac{1}{12n+6} < r_n.$$

Proof. Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$\log(p+1) = A_p + b_p - \epsilon_p \tag{3}$$

where

$$A_p = \int_p^{p+1} \log x\, dx, \quad b_p = \tfrac{1}{2}[\log(p+1) - \log p],$$

$$\epsilon_p = \int_p^{p+1} \log x\, dx - \tfrac{1}{2}[\log(p+1) + \log p].$$

The partition (3) of $\log(p+1)$, regarded as the area of a rectangle with base $(p, p+1)$ and height $\log(p+1)$, into a curvilinear area, a triangle, and a small sliver* is suggested by the geometry of the curve $y=\log x$. Then

* Taken from G. Darmois, *Statistique Mathématique*, Paris, 1928, pp. 315–317. The only novelty of the present note is the inequality (7) which permits the first part of the estimate (2).

$$S_n = \sum_{p=1}^{n-1} (A_p + b_p - \epsilon_p) = \int_1^n \log x \, dx + \tfrac{1}{2} \log n - \sum_{p=1}^{n-1} \epsilon_p.$$

Since $\int \log x \, dx = x \log x - x$ we can write

$$S_n = (n + \tfrac{1}{2}) \log n - n + 1 - \sum_{p=1}^{n-1} \epsilon_p, \tag{4}$$

where

$$\epsilon_p = \frac{2p+1}{2} \log \left(\frac{p+1}{p} \right) - 1.$$

Using the well known series

$$\log \left(\frac{1+x}{1-x} \right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right)$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \frac{1}{7(2p+1)^6} + \cdots. \tag{5}$$

We can therefore bound ϵ_p above and below:

$$\begin{aligned} \epsilon_p &< \frac{1}{3(2p+1)^2} \left\{ 1 + \frac{1}{(2p+1)^2} + \frac{1}{(2p+1)^4} + \cdots \right\} \\ &= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \frac{1}{(2p+1)^2}} = \frac{1}{12} \left(\frac{1}{p} - \frac{1}{p+1} \right), \end{aligned} \tag{6}$$

$$\begin{aligned} \epsilon_p &> \frac{1}{3(2p+1)^2} \left\{ 1 + \frac{1}{3(2p+1)^2} + \frac{1}{[3(2p+1)^2]^2} + \cdots \right\} \\ &= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \frac{1}{3(2p+1)^2}} > \frac{1}{12} \left(\frac{1}{p + \frac{1}{12}} - \frac{1}{p + 1 + \frac{1}{12}} \right). \end{aligned} \tag{7}$$

Now define

$$B = \sum_{p=1}^{\infty} \epsilon_p, \qquad r_n = \sum_{p=n}^{\infty} \epsilon_p, \tag{8}$$

where from (6) and (7) we have

$$\frac{1}{13} < B < \frac{1}{12}. \tag{9}$$

Then we can write (4) in the form

$$S_n = (n + \tfrac{1}{2}) \log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = C \cdot n^{n+1/2} e^{-n} \cdot e^{r_n},$$

where r_n is defined by (8), ϵ_p by (5), and from (6) and (7) we have

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

The constant C, known from (9) to lie between $e^{11/12}$ and $e^{12/13}$, may be shown by one of the usual methods to have the value $\sqrt{2\pi}$. This completes the proof.

The preceding derivation was motivated by the geometrically suggestive partition (3). The editor has pointed out that the inequalities (6) and (7) permit the following brief proof* of (2). Let

$$u_n = n! n^{-(n+1/2)} e^n.$$

Then the series

$$\log \left(1 + \frac{1}{n}\right)^{n+1/2} = 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots$$

together with (6) and (7) yield the inequalities

$$\exp \left\{ \frac{1}{12} \left(\frac{1}{n + \frac{1}{12}} - \frac{1}{n + 1 + \frac{1}{12}} \right) \right\} < \frac{u_n}{u_{n+1}} = e^{-1} \left(1 + \frac{1}{n}\right)^{n+1/2}$$

$$< \exp \left\{ \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) \right\}.$$

Hence

$$v_n = u_n e^{-1/12n}$$

increases and

$$w_n = u_n e^{-1/(12n+1)}$$

decreases, while

$$v_n < w_n = v_n e^{1/12n(12n+1)}.$$

Since

$$v_1 = e^{11/12}, \qquad w_1 = e^{12/13}$$

* A modification of that attributed to Cesàro by A. Fisher, *Mathematical theory of probabilities*, New York, 1936, pp. 93–95.

it follows that

$$v_n \to C, \qquad w_n \to C, \qquad v_n < C < w_n, \qquad e^{11/12} < C < e^{12/13}.$$

Thus

$$u_n = Ce^{r_n}$$

where r_n satisfies (2).

ON RESTRICTED FUNCTIONS

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Let $f(x)$ be a real function of a real variable defined on a set E. Let P be a point property of f. The function f is said to be *peculiar* with respect to the property P if there exists a partition of E into two subsets, E_1 and E_2, each everywhere dense in E, such that the property P holds at every point of E_1 and fails to hold at every point of E_2. Functions peculiar with respect to continuity and differentiability are well known.

In a recent paper [2], H. P. Thielman investigated two generalizations of continuity—neighborliness, a concept defined by Bledsoe [1], and cliquishness. He showed that, although there are functions peculiar with respect to continuity, neighborliness, and differentiability, there exists no function which is peculiar with respect to cliquishness. It is the purpose of this note to define another point property which is similar to cliquishness in this respect.

Let $f(x)$ be defined on a set E, and let a be a limit point of E. The function $f(x)$ is said to be *restricted* at the point a if $\limsup_{x\to a} f(x)$ and $\liminf_{x\to a} f(x)$ are both finite. Otherwise, $f(x)$ is said to be *unrestricted* at a.

If a function is restricted (unrestricted) at each point of a set E, it is said to be restricted (unrestricted) on E.

A function may be restricted on a set but not bounded on that set; the function $f(x)=1/x$, defined on the open interval $(0, 1)$, is such a function.

THEOREM 1. *If* $\limsup_{x\to b} f(x)=+\infty$ *for all* $b\in E$ *and if* a *is a limit point of* E, *then* $\limsup_{x\to a} f(x)=+\infty$.

Proof: Let M and δ be arbitrary positive numbers. Let $b\in E$ be such that $0<|b-a|<\delta/2$. Choose δ_1 so that $0<\delta_1<|b-a|$. Then, since $\limsup_{x\to b} f(x) =+\infty$, there exists $c\in E$ such that $f(c)>M$ and $|c-b|<\delta_1$. But

$$|c-a| \leqq |c-b| + |b-a| < \delta.$$

Since M and δ are arbitrary, $\limsup_{x\to a} f(x)=+\infty$.

COROLLARY. *If* $f(x)$ *is unrestricted on* E *and if* a *is a limit point of* E, *then* $f(x)$ *is unrestricted at* a.