

The Relation Between the Root and Ratio Tests

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Two well-known tests for the convergence of a series $\sum_{n=0}^{\infty} a_n$, $a_n \neq 0$, are the ratio test and the root test:

Ratio Test: *If*

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \quad (1)$$

then the series $\sum a_n$ converges.

We call the values a_{n+1}/a_n , $n \geq 0$, the *consecutive ratios* of the series.

Root Test: *If*

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} < 1 \quad (2)$$

then the series $\sum a_n$ converges.

(For proofs, see Krantz [1] or Rudin [2].)

Since the limit in (1) is always greater than or equal to the limit in (2), the root test is stronger than the ratio test: there are cases in which the root test shows convergence but the ratio test does not. (In fact, the ratio test is a corollary of the root test: see Krantz [1].)

We can illuminate the relationship between these two tests with a simple calculation:

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \cdot a_0 \right|^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \cdots \frac{a_1}{a_0} \right|^{1/n}.$$

(The last equality holds since $a_0^{1/n} \rightarrow 1$ as $n \rightarrow \infty$.) The right-hand side is the limit of the geometric means of the first n consecutive ratios of the series. In other words, while the ratio test depends on the behavior (in the limit) of each consecutive ratio, the root test only considers the average behavior of these ratios. Clearly, if all the consecutive ratios get small then their average value will get small as well. The converse is false, which is why the root test is stronger. Thus, for example, the ratio test fails on the rearranged geometric series

$$1/2 + 1 + 1/8 + 1/4 + 1/32 + 1/16 + \cdots, \quad (3)$$

since the consecutive ratios alternate in value between 2 and 1/8. However, the geometric mean of the first $2n$ consecutive ratios is

$$\left| 2^n \cdot \frac{1}{8^n} \right|^{1/2n} = 1/2,$$

so the root test shows that the series converges.

Interpreting the root test in terms of averages suggests substituting another mean for the geometric mean. By the arithmetic-geometric mean inequality, the arithmetic mean of n consecutive ratios of a series is always larger than their geometric mean. This yields a new convergence test:

Arithmetic Mean Test: *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{a_n}{a_{n-1}} + \frac{a_{n-1}}{a_{n-2}} + \dots + \frac{a_1}{a_0} \right) < 1$$

then the series $\sum a_n$ converges.

This test is stronger than the ratio test but weaker than the root test. However, in some cases it may be easier to compute the arithmetic mean of the consecutive ratios than it is to compute their geometric mean. For example, consider the series $\sum a_n$, where the a_n 's are defined inductively by

$$a_0 = 1, \quad a_n = a_{n-1} \cdot \frac{\log(1 + 1/(n + 1))}{\log(n + 1)\log(n + 2)}, \quad n \geq 1.$$

By l'Hôpital's rule,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$

so the ratio test fails. To apply the root test we would have to evaluate the limit

$$\lim_{n \rightarrow \infty} \left| \prod_{k=1}^n \frac{\log(1 + 1/(k + 1))}{\log(k + 1)\log(k + 2)} \right|^{1/n}.$$

The arithmetic mean test, however, requires only the following calculation:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left| \frac{\log(1 + 1/(k + 1))}{\log(k + 1)\log(k + 2)} \right| &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{\log(k + 1)} - \frac{1}{\log(k + 2)} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{\log 2} - \frac{1}{\log(n + 2)} \right) \\ &= 0. \end{aligned}$$

Hence the series converges.

An example of a series for which the arithmetic mean test fails is series (3) above: In this case the arithmetic mean of the consecutive ratios converges to $17/16$. As a exercise for the reader we leave the problem of determining the values of α , $0 < \alpha < 1$, for which the convergence of a similar rearrangement of $\sum \alpha^n$ can be shown using the arithmetic mean test.

REFERENCES

1. S. Krantz, *Real Analysis and Foundations*, CRC Press, Boca Raton, FL, 1991.
2. W. Rudin, *Principles of Mathematical Analysis*, 3rd Ed., McGraw Hill, New York, NY, 1976.