Calculus I: Summary of Theory[1]

Elementary functions are polynomials, trig, exponential, and a result of any *finite* number of the following operations on them: algebraic operations, compositions, and taking the inverse.

Definitions of Limits:

Sequences[2]

$$\lim_{n\to\infty} x_n = L \quad \Leftrightarrow \ \forall \varepsilon > 0 \ \exists \ N = N(\varepsilon) \ \forall n > N : |x_n - L| < \varepsilon;$$
$$\lim_{n\to\infty} x_n = +\infty \quad \Leftrightarrow \ \forall L > 0 \ \exists \ N = N(L) \ \forall n > N : x_N > L;$$
$$\lim_{n\to\infty} x_n = -\infty \quad \Leftrightarrow \ \forall L > 0 \ \exists \ N = N(L) \ \forall n > N : x_N < -L;$$
$$\lim_{n\to\infty} x_n \text{ DNE} \quad \Leftrightarrow \ \forall L \in \mathbb{R} \ \exists \ \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists \ n > N : |x_n - L| \geq \varepsilon.$$

Functions

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \ \forall \varepsilon > 0 \ \exists \ \delta = \delta(\varepsilon) \ \forall \ 0 < |x-a| < \delta : |f(x) - L| < \varepsilon;$$
$$\lim_{x\to a} f(x) = +\infty \quad \Leftrightarrow \ \forall L > 0 \ \exists \ \delta = \delta(L) \ \forall \ 0 < |x-a| < \delta : f(x) > L;$$
$$\lim_{x\to a} f(x) = -\infty \quad \Leftrightarrow \ \forall L > 0 \ \exists \ \delta = \delta(L) \ \forall \ 0 < |x-a| < \delta : f(x) < -L;$$
$$\lim_{x\to a} f(x) \text{ DNE} \quad \Leftrightarrow \ \forall L \in \mathbb{R} \ \exists \ \varepsilon > 0 \ \forall \ \delta > 0 \ \exists \ 0 < |x-a| < \delta : |f(x) - L| \geq \varepsilon;$$
$$\lim_{x\to +\infty} f(x) = L \quad \Leftrightarrow \ \forall \varepsilon > 0 \ \exists \ M = M(\varepsilon) > 0 \ \forall \ x > M : |f(x) - L| < \varepsilon;$$

and so on.

Extreme value/intermediate value theorem. If f is continuous on $[a, b]$, that is, $f(x)$ is continuous on (a, b), and $f(a) = \lim_{x\to a+} f(x)$, $f(b) = \lim_{x\to b-} f(x)$, then we have EVT by Weierstrass and IVT by Cauchy.

Weierstrass: $\exists \ x_1, x_2 \in [a, b] : \max_{x\in[a,b]} f(x) = f(x_1), \ \min_{x\in[a,b]} f(x) = f(x_2).$

Cauchy: For every $a \leq u < v \leq b$ such that $f(u) \neq f(v)$ and for every number c between $f(u)$ and $f(v)$, there exists a $w \in (u, v)$ such that $f(w) = c$.

Neither theorem is true if f is continuous on (a, b) and not on of $[a, b]$.

The theorem of Weierstrass is a particular case of the more general result: a continuous function on a compact set achieves the minimal and maximal values. The fact that the *closed and bounded* interval $[a, b]$ is compact sounds obvious but, under closer inspection, becomes a consequence of the Heine-Borel theorem. The fact that the image of $[a, b]$ is a compact set also requires a proof.

The theorem of Cauchy can be deduced from a more general statement, that the image of a connected set under a continuous mapping is also connected, but can also be proved directly, and in a more constructive manner, using a *bisection argument*.

"Mean-value" Theorems: Derivatives.

Fermat: If $x_0 \in (a, b)$ is a *local extreme point of* $f = f(x)$ [that is, f is defined in some neighborhood of x_0 and the values of $f(x)$ in that neighborhood are either bigger than or less then $f(x_0)$] and if $f'(x_0)$ exists, then $f'(x_0) = 0$. This follows directly from the definition of the derivative: the differential quotient is non-positive on one sided of x_0 and non-negative on the other side, so the limit, if exists, must be zero.

Rolle: if $f = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$. This follows by combining the theorems of Weierstrass (to identify the point c as the *global* extreme of f on $[a, b]$) and Fermat (to get the zero derivative).

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[2]$\forall$ means *for every*; $\exists$ means *there exists*.

Lagrange/"The" mean value theorem: If $f = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then $f(b) - f(a) = f'(c)(b - a)$ for some $c \in (a, b)$. This follows from Rolle applied to

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

An application: $f = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f'(x) > 0$, $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

A counterexample: the function $f(x) = 1/x$, $x \neq 0$, satisfies $f'(x) = 1/x^2 > 0$, $x \neq 0$, but is not monotone on $[-1, 1]$. There are no contradictions because f is not continuous on $[-1, 1]$.

Cauchy: If $f = f(x)$ and $g = g(x)$ are two functions such that each is continuous on $[a, b]$ and differentiable on (a, b), and moreover $g'(x) \neq 0$ on (a, b), then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (a, b)$. This follows from Rolle applied to

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}\big(g(x) - g(a)\big).$$

An application: derivation of the `Schlömilch-Roche` version of the remainder in the Taylor formula.

Darboux: A derivative f' satisfies the conclusion of the intermediate values theorem [even if f' is not continuous.] More precisely, if f is differentiable on (a, b), $a < u < v < b$, and $f'(u) \neq f'(v)$, then, for every c between $f'(u)$ and $f'(v)$, there is a $w \in (u, v)$ such that $f'(w) = c$. For the proof, consider the function $F(x) = f(x) - cx$. By assumption, $F'(u) = f'(u) - c$ and $F'(v) = f'(v) - c$ have opposite signs. Continuity of F and the theorem of Weierstrass then imply existence of a *global*, on $[u, v]$, extremum $F(w)$ of F, for some $w \in (u, v)$. Then the theorem of Fermat implies $F'(w) = 0$, that is, $f'(w) = c$, as desired.

Takeaway: a derivative does not have to be continuous, but not all discontinuous functions can be derivatives. In particular, derivatives cannot jump.

The table below summarizes the connections among all the theorems from above.

To prove	Apply	to
Weierstrass	Heien-Borel	$[a, b]$ and $f([a, b])$
Cauchy (IVT)	bisection to locate zero	$F(x) = f(x) - c$
Fermat	definition of derivative	$f(x)$ at x_0
Rolle	Weierstrass and Fermat	f
Lagrange	Rolle	$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$
Cauchy (MVT)	Rolle	$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x) - g(a)\big)$
Darboux	Weierstrass and Fermat	$F(x) = f(x) - cx$

"Mean-value" Theorems: Integrals. In what follows, we assume that all function are Riemann-integrable on $[a, b]$.

(1) The function

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$; it is differentiable on (a, b), with $F'(x) = f(x)$, if f is continuous on $[a, b]$.

(2) If $f(x) \geq 0$ on $[a, b]$, then
$$\int_a^b f(x)\, dx \geq 0.$$
(3) If $f(x) \leq g(x)$, then
$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$
In particular,
$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx,$$
and if $m \leq f(x) \leq M$, then
$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$
(4) If $m \leq f(x) \leq M$, then
$$\int_a^b f(x) = K(b-a)$$
for some $K \in [m, M]$. In particular, if f is continuous on $[a, b]$, then
$$\int_a^b f(x)\, dx = f(c)(b-a)$$
for some $c \in [a, b]$.

(5) If $m \leq f(x) \leq M$ and $g(x) \geq 0$ on $[a, b]$, then
$$\int_a^b f(x)g(x)\, dx = K \int_a^b g(x)\, dx$$
for some $K \in [m, M]$. In particular, if f is continuous on $[a, b]$, then
$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx$$
for some $c \in [a, b]$.

(6) If $g = g(x)$ is non-negative and decreasing, then
$$\int_a^b f(x)g(x)\, dx = g(a) \int_a^c f(x)\, dx$$
for some $c \in [a, b]$.