Taylor expansion, one variable[1]

Taylor polynomial for a function[2] f at a point a:

$$T_{N;f,a}(x) = f(a) + \sum_{k=1}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This polynomial is defined if f is N time differentiable at the point a; the polynomial might not have degree N because it can happen that $f^{(N)}(a) = 0$.

Taylor series for a function f at a point a:

$$T_{f,a}(x) = f(a) + \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The series is defined if f has derivatives of every order at the point a. For $x \neq a$, this series does not have to converge, and, if converges, the sum does not have to equal $f(x)$. The two standard examples:

$$f(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \cos(xy)\, e^{-(\ln y)^2} dy,$$

when

$$T_{f,0}(x) = e^{1/4} + \sum_{k=1}^{+\infty} \frac{(-1)^k e^{(2k+1)^2/4}}{(2k)!}\, x^{2k}$$

diverges for all $x \neq 0$, and

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

when $T_{f,0}(x) = 0$ for all x.

The remainder in the Taylor formula is

$$R_{N;f,a}(x) = f(x) - T_{N;f,a}(x).$$

Here are the main representations for R.

Peano : If $f^{(N)}(a)$ exists (minimal assumption), then $R_{N;f,a}(x) = o\big((x-a)^N\big),\ x \to a;$

Schlömilch-Roche: If $f^{(N+1)}(x)$ exists near $x = a,$ then

$$R_{N;f,a}(x) = \frac{f^{(N+1)}\big(a + \theta(x-a)\big)}{N!K}(x-a)^{N+1}(1-\theta)^{N+1-K} \text{ for some } 0 \le \theta \le 1,\ 1 \le K \le N+1;$$

Lagrange is Schlömilch-Roche with $K = N+1,$ that is

$$R_{N;f,a}(x) = \frac{f^{(N+1)}\big(a + \theta(x-a)\big)}{(N+1)!}(x-a)^{N+1}, 0 \le \theta \le 1;$$

Cauchy is Schlömilch-Roche with $K = 1,$ that is

$$R_{N;f,a}(x) = \frac{f^{(N+1)}\big(a + \theta(x-a)\big)}{N!}(x-a)^{N+1}(1-\theta)^N,\ 0 \le \theta \le 1;$$

The intergral form: If $f^{(N+1)}$ is integrable near $x = a,$ then

$$R_{N;f,a}(x) = \frac{1}{N!} \int_a^x (x-t)^N\, f^{(N+1)}(t)\, dt.$$

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[2]In what follows, $F^{(n)}$ denotes n-th derivative of the function F

An outline of the proofs. If $f'(a)$ exists, then

$$\lim_{x\to a}\frac{|R_{1;f,a}(x)|}{|x-a|}=\lim_{x\to a}\left|\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}\right|=\lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|=0,$$

which is Peano for $N=1$. For general N, note that $R^{(n)}_{N;f,a}(a)=0$, $n=1,2,\ldots,N$, and then use induction to conclude that

$$\lim_{x\to a}\frac{|R_{N;f,a}(x)|}{|x-a|^N}=0.$$

The Schlömilch-Roche version, for $x>a$, follows after *fixing* x and applying the Cauchy mean value theorem to the functions

$$F(t)=f(x)-f(t)+\sum_{k=1}^{N}\frac{f^{(k)}(t)}{k!}(x-t)^k,\ G(t)=(x-t)^K,\quad a\le t\le x,$$

on the interval $[a,x]$. Note that $F(a)=R_{N;f,a}(x)$ and $F(x)=0$.

The basic bound on the reminder is

$$|R_{N;f,a}(x)|\le\frac{M_N|x-a|^{N+1}}{(N+1)!},\quad M_N=\max_{\theta\in[0,1]}\left|f^{(N+1)}\big(a+\theta(x-a)\big)\right|;$$

this is best seeing from the Lagrange form of the remainder.

If the Taylor series is alternating and converging, then the alternating series bound is better.

Some applications of the Peano reminder.

(1) If $f''(a)>0$, then the graph of f is strictly above the tangent line at $x=a$ in some neighborhood of a. In particular, if $f'(a)=0$ (horizontal tangent) then $f(a)$ is a strict local minimum of f.

Indeed, by Peano, $f(x)-f(a)-f'(a)(x-a)=\frac{f''(a)}{2}(x-a)^2\big(1+o(1)\big)$, so that, for x close to a, $f(x)-f(a)-f'(a)(x-a)>\frac{f''(a)}{4}(x-a)^2>0$.

(2) If $f(x)=\sum_{k=0}^{N}A_n(x-a)^k+o((x-a)^N)$, $x\to a$, then $A_0=f(a)$ and $A_k=f^{(k)}(a)/k!$, $k=1,\ldots,N$ (uniqueness of the Taylor polynomial).

Indeed, setting $x=a$ gives $f(a)=A_0$; taking the limit, as $x\to a$, of $(f(x)-f(a))/(x-a)$ gives $f'(a)=A_1$, etc.