Principles of Mathematical Analysis: Highlights of the book by W. Rudin, including exercises, together with some other random thoughts[1]

Chapter 1: Sets from algebraic point of view.

Main topics: ordered sets, field (an algebraic object), ordered field, $\mathbb{R}^n$, complex numbers, construction of real numbers using `Dedekind cuts`.

Some facts to keep in mind: extended real line $[-\infty, +\infty]$ is not a field, but still can be useful sometimes; Archimedean property of real line (there is always a rational number between any to real numbers).

More important than it looks: the (Cauchy-Bunyakovky-)Schwarz inequality and parallelogram identity both extend to an `inner product space`.

Chapter 2: Sets from topological point of view.

Main topics: cardinality (finite, countable, uncountable); diagonalization argument; metric, metric space and related topological concepts.

The key concepts: (a) neighborhood of a point (an *open* set containing the point); (b) compact set (by definition, finite cover from *every possible* open cover; this implies *sequential compactness*, that is, every sequence has a converging subsequence).

Important notations: for a set A in a metric (topological) space, A^c is the complement, $\overline{A}$ is the closure, A' is the collection of limit points, A° is the interior. These notations can be *very* different in other books/papers.

Some other important concepts: convergence; limit point of a set (every neighborhood contains at least one more point); a perfect set (closed and every point is a limit point); compact set (finite cover from *every possible* open cover); sequentially compact set (every sequence has a converging sub-sequence); "the" Cantor set (constructed on $[0, 1]$ by removing open middle thirds); condensation point (every neighborhood contains uncountably many points); algebraic numbers; separable space; base (of topology).

Some facts to keep in mind: $\overline{A} = A \bigcup A'$, but in general A is not a sub-set of A' because of possible `isolated points`; Heine-Borel Theorem (in $\mathbb{R}^n$, compact sets are exactly those that are closed and bounded); non-empty intersections of compact sets.

Related topics for independent study: axiom of choice; cardinal numbers (hierarchy of infinity) and the continuum hypothesis; topological space; infinite-dimensional spaces with the Heine-Borel property; other Cantor-type sets; transcendental numbers.

Chapter 3: Sequences and series of numbers.

Main topics: convergence in a metric space; subsequence; Cauchy sequence; Cauchy criterion for series; root and ratio tests; power series and the radius of convergence.

Also of interest: diameter of a set; conditional convergence, rearrangement, and the theorem of Riemann on the topic; summation by parts; Cauchy's condensation test for convergence; product of two series; a proof that $e \approx 2.718$ is irrational.

More important than it looks: Stirling's approximation, with upper and lower bounds; completion of a metric space; Bair's category theorem.

Related topic for independent study: p-adic numbers as a different *completion* of the rationals.

[1]Sergey Lotocky, USC; version of January 3, 2024

Chapter 4: Continuity.

Main topics: *topological* continuity (pre-image of an open set is open); uniform continuity; continuous extension (from a *closed* set); restriction.
Also of interest: coordinate function; continuity on a compact set implies uniform continuity on that set; continuity of the inverse map on compacts; continuous image of compact is compact; (an abstract) graph of a function; a function discontinuous on a given countable set ($\sum_n n^{-2} I(x > x_n)$); distance from a point to a set.
More important than it looks: fixed point theorem(s); convexity; intermediate value theorem; discontinuities of monotone functions; Minkowski sum of two sets in a linear space.

Related topics for independent study: functions of bounded variation (difference of two non-decreasing functions); possible sets of discontinuities of a function[2] (for example, a function $f : [0,1] \to \mathbb{R}$ cannot be continuous only at rational points); other types of extension (for example, analytic extension/continuation and natural boundary); how different a restriction can be from the original (for example, every continuous function on the real line can be restricted to a *perfect set* to become differentiable and, for a *typical* continuous function, such a restriction can be taken as a constant function, and there are uncountably many of them.[3])

Chapter 5: Derivatives for one real variable.

Main topics: (counter)examples based on the function $\sin(1/x)$; Theorems of Rolle, Lagrange, and Cauchy; Darboux theorem (MVT for derivatives); (basic) l'Hospital; (basic) Taylor.
More important than it looks: Newton's method; uniqueness for ODEs.

Related topics for independent study: functions with *strange* properties (for example, a function $f : [0,1] \to \mathbb{R}$ can be unbounded in every neighborhood of every point; can be strictly increasing and still have zero derivative almost everywhere; can have a dense graph but pass the vertical line test at every point; have the intermediate value property and be discontinuous everywhere [Conway's base 13 function]).

Chapter 6: Riemann-Stieltjes integral.

Main topic: construction of $\int_a^b f(x) d\alpha(x)$ for non-decreasing α using upper and lower integrals.
Also of interest: arc length.
(Much) More important than it looks: Hölder inequality; Riemann zeta function; Fresnel integrals $\int_0^\infty e^{\sqrt{-1}x^2}\, dx$.

Related topics for independent study: criteria for Riemann-Stieltjes integrability; other types of integrals (with Lebesgue coming up at the end of the book, and Henstock–Kurzweil completing the story on the interval), the case when both f and α are discontinuous at the same point.

Chapter 7: Sequences and series of functions.

The main point: in general, the order the limits are taken makes a difference, but sometimes it does not.

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[2]It is necessarily a countable union of closed sets.

[3]See Remark 3.1 in K. Ciesielski, Juan B. Seoane Sepúlveda, Differentiability versus continuity: Restriction and extension theorems and monstrous examples, Bulletin of the American Mathematical Society, Volume 56, Number 2, Pages 211–260.

Main topic: uniform convergence.

Also of interest: The Dirichlet function as a double limit; a series can converge uniformly but not absolutely; Weierstrass M test; a construction of a continuous non-differentiable function; equicontinuity and the Arzela-Ascolli theorem; the Weierstrass approximation theorem and its proof; Stone's extension of Weierstrass and numerous related ideas that can potentially lead all the way to von Neumann algebras; Helly's selection theorem; space-filling curves and a surjection from Cantor set to the unit square.

(Much) More important than it looks: polynomials are dense in $L_2((0,1))$; existence and uniqueness for systems of ODEs.

Chapter 8: Special functions AND Fourier series.

The bottom line: A LOT of VERY IMPORTANT material.

Basic results: (real) analytic functions, Abel's theorem on power series; (a particular case of) Fubini-Tonelli; uniqueness of power series; elementary transcedental functions via power series; Fundamental Theorem of Algebra; orthogonal expansions and Bessel's inequality; point-wise convergence of Fourier series; Parseval's identity; Gamma function as the unique *natural* interpolation of factorials, and the resulting Stirling asymptotic; Beta function in terms of Gamma function; Euler's product formula for the Riemann zeta function and divergence of the series $\sum_{p \text{ prime}} 1/p$; Euler–Mascheroni constant.

Related topics for independent study: various identities for the Gamma and Beta functions; the derivatives of the log of the Gamma function; power series method for ODEs, including the Fuchs-Frobenius theory, and the resulting special functions, especially the four types of the Bessel functions; different types of convergence for the Fourier series; Gibbs phenomenon; Fourier transform and the Plancherel identity; applications to PDEs.

Chapter 9: Multivariable[4] differential calculus.

Main topic: the *intrinsic* definition of the (Frechet) derivative as a linear operator.

Basic results: (normed) linear spaces and mappings between them; projections; contraction mapping theorem; theorems on inverse and implicit functions; a non-linear extension of the rank-nullity theorem; determinants and Jacobians; integrals with a parameter and the Feynman trick (method); multi-dimensional Taylor formula.

Related topics for independent study: *intrinsic* definitions of higher-order derivatives; multi-linear mappings; covariant tensor (a suitable interpretation of the gradient of a scalar function of several arguments) and contravariant tensor (the usual vector).

Chapter 10: Differential forms (multivariable integral calculus, with a more detailed summary).

Basic objects in $\mathbb{R}^n$

- n-cell I^n is the *(hyper) box*

$$I^n = \{x = (x_1, \dots, x_n) : a_1 \le x_1 \le b_1, \dots, a_n \le x_n \le b_n\} = \prod_{m=1}^{n} [a_m, b_m].$$

  The volume of I_n is, *by definition*, $\prod_{m=1}^{n}(b_m - a_m)$. When all $b_m - a_m$ are the same, I^n is called the hyper-cube.

[4] or multivariate

- n-simplex Q^n is the *(hyper) tetrahedron*
$$Q^n = \{x = (x_1, \dots, x_n) : x_1 \geq 0, \dots, x_n \geq 0,\ x_1 + \cdots + x_n \leq 1\}$$
The volume of Q^n is $1/n!$ (for example, by induction).
- The closed ball $B_a^n(r)$ of radium r, with center at the point a, is $\{x : |x - a| \leq r\}$. The volume of $B_a^n(r)$ is
$$|B_a^n(r)| = r^n \ \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)},$$
where $\Gamma(\cdot)$ is the Gamma function. There are numerous proofs of this result.
- The sphere $S_a^n(r)$ of radius r, with center at the point a, is the boundary of $B_a(r)$, that is, $\{x : |x - a| = r\}$. The surface area of $S_a(r)$ is
$$|S_a^n(r)| = r^{n-1} \ \frac{2\pi^{n/2}}{\Gamma(n/2)},$$
which is the derivative of $|B_a^n(r)|$ with respect to r.
- The k-surface can mean either a ("k-dimensional") subset of $\mathbb{R}^n$ or a nice *mapping* from a nice sub-set of $\mathbb{R}^k$ to $\mathbb{R}^n$.

Basic technical tools:

- Primitive maps, flips, and a decomposition of a general *invertible* map as a composition of flips and primitive maps.
- Partition of unity (localization) on a compact set: the main point is that the size of the elements of the open cover can be arbitrarily small.
- Change of variables via Jacobian.
- Wedge (exterior) product: $dx \wedge dx = 0$ and $dy \wedge dx = -dx \wedge dy$.
- Exterior derivative.
- Affine k-chain.
- Oriented affine k-simplex.
- Boundary of an oriented affine k-simplex.

The main point: a `differential form` of order k is what you integrate over a k-surface.

The ultimate prize: The theorem of Stokes in the form
$$\int_{\partial M} \omega = \int_M d\omega$$
that covers all the integral theorems of multivariable calculus, including `Green's identities` (multi-dimensional integration by parts).

Examples:

(1) $d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) = P_x dx \wedge dy \wedge dz + Q_y dy \wedge dz \wedge dx + R_z dz \wedge dx \wedge dy = (P_x + (-1)^2 Q_y + (-1)^2 R_z) dx \wedge dy \wedge dz$, corresponding to the Gauss-Ostrogradskii (divergence) theorem.

(2) $d(Pdx + Qdy + Rdz) = P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dy + R_y dy \wedge dz = (R_y - Q_z) dy \wedge dz - (R_x - P_z) dz \wedge dx + (Q_x - P_y) dx \wedge dy$, corresponding to the Stokes (curl) theorem.

More on differential forms:

(1) The `volume element`, one and only $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ in $\mathbb{R}^n$ for every n.

(2) The `surface element`, only in $\mathbb{R}^3$, and *kind of* corresponds to $dy \wedge dz + dz \wedge dx + dx \wedge dy$.

(3) An `exact` form $\omega = d\alpha$ vs a `closed` form $d\omega = 0$; existence of forms that are closed but not exact can measure various types of *holes* in the region. In particular, open convex sets do not have any holes, and all closed forms there are exact.

An alternative reference on the subject of this chapter: Ib H. Madsen, Jxrgen Tornehave. *From calculus to cohomology: De Rham cohomology and characteristic classes*, Cambridge University Press, 1997 (about 200 pages of text).

Related topics for independent study: physical meaning of divergence and curl, leading to Maxwell's equations; (convex) polytopes; push forward and pull back; tensors and tensor fields; (smooth) manifolds; Lie derivative; Riemannian manifolds; connections; the `de Rham cohomology groups` [formalizing the idea how closed but not exact forms correspond to holes in the region]; Hairy ball/"you cannot comb a sphere" theorem [a continuous vector field on $\mathbb{R}^3$ that is tangent to the sphere at every point of the sphere must be zero at some point on the sphere].

Chapter 11: Lebesgue-Stieltjes integral.

Main point: Partitioning the range rather then the domain.
Building blocks: sigma-algebra/ring/field of sets; countably additive regular set function; measurable functions; simple functions.
Main results: Fatou's lemmas (theorem), theorems on monotone and dominated convergence; a criterion for Riemann integrability.
Main new piece of terminology: almost everywhere.
Further results: spaces L^p, $p \geq 1$, especially for $p = 1$ and $p = 2$; the Riesz-Fischer theorem; non-compactness of the closed ball in L^2.
Potential source of confusion: characteristic function vs indicator function.

Related topics for independent study: the set theory and the Russell paradox; non-measurable sets and the Banach-Tarski paradox; theorems of Fubini and Tonelli.

Food for thought. Consider the following properties a real-valued measurable function $f = f(x)$, $x \in [0, 1]$, can have:

(A) f has bounded variation, that is, $\sup_{\{x_k\}} \sum_{k=1}^{N} |f(x_{k+1}) - f(x_k)| < \infty$;
(B) f is absolutely continuous, that is, $f(x) = f(0) + \int_0^x h(t)\,dt$ for some Lebsgue-integrable function h;
(C) f is differentiable almost everywhere;
(D) There exists a differentiable function $g = g(x)$ such that $f(x) = g(x)$ for almost all x.

Determine, with an explanation, the pair-wise relationships between these properties. That is, for each of the six pairs ((I), (J)), determine whether (I) implies (J) and/or (J) implies (I).