Homework 1.

PROBLEM 1.

(a) Explain why the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, with addition and multiplication mod 9, is NOT a field.

(b) For a positive integer p, explain why the set $\{0, 1, \dots, p-1\}$, with addition and multiplication mod p, is a field if and only if p is a prime number.

(c) Is there a field with exactly nine elements? [Yes: check out "finite fields" event though this is somewhat advanced.]

PROBLEM 2. Write a short and convincing explanation why the set of all real algebraic numbers is countably infinite. [An outline: the set is infinite because each $2^{1/n}$, $n = 1, 2, \dots$ is an element, being a root of $x^n = 2$; the set of all algebraic numbers is countable because there are countably many polynomials with rational coefficients, and each such polynomials has finitely many roots.]

PROBLEM 3. For real numbers x, y, define

$$\varrho(x, y) = \frac{|x-y|}{1+|x-y|}.$$

(a) Confirm that ϱ is a metric on the real line.

(b) Confirm that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \varrho(x_n, x) = 0$.

(1) Let X be the real line with metric ϱ. Is X (a) complete? (b) separable? (c) bounded? (d) compact? In each case, briefly explain your conclusion.

PROBLEM 4. Here, $\mathfrak{i} = \sqrt{-1}$ is the imaginary unit.

(a) Write the number $\frac{2-\mathfrak{i}}{3\mathfrak{i}-4}$ in the form $x + \mathfrak{i}y$.

(b) Write $-1 - \mathfrak{i}$ in the polar form.

(c) Solve the following equation: $z^4 - 2z^2 + 2 = 0$. Write the answer in the polar form. [Start by noticing that the equation is $(z^2-1)^2 = -1$.]

(d) Compute the anti-derivative $\int e^{-2x}\sin(3x)dx$ without integrating by parts. An outline: integrate $e^{(3\mathfrak{i}-2)x}$ "the Calc-I way" and then take the imaginary part of the result.

(e) Compute all the values of $\sqrt[6]{-\mathfrak{i}}$. An outline: start with $-\mathfrak{i} = \exp(-\pi\mathfrak{i}/2 + 2\pi k\mathfrak{i})$, $k \in \mathbb{Z}$, and then take the usual root on the right to recover all six distinct values.

(f) Compute all the values of $\sqrt{1+\mathfrak{i}\sqrt{3}} + \sqrt{1-\mathfrak{i}\sqrt{3}}$ [the main point: some of those values are real; the polar form/Euler formula can help].

(g) Write $\cos(5x)$ and $\cos(6x)$ as polynomials in $\cos x$. [See also: Chebyshev polynomials.] The general approach using complex numbers: start with $(\cos x + \mathfrak{i}\sin x)^n = \cos(nx) + \mathfrak{i}\sin(nx)$, $n = 1, 2, 3, \dots$, for a suitable n, expand the left-hand side using binomial formula, take the real part, and then get rid of sines using $\sin^2 x = 1 - \cos^2 x$. Chebyshev polynomials lead to an alternative procedure using a recursion. You are encouraged to learn and explore both approaches.

(h) Let $\alpha = e^{2\mathfrak{i}\pi/5}$. (i) Let $u = \alpha + \alpha^4$. Verify that $u^2 + u - 1 = 0$ and $u = \alpha + \bar{\alpha}$. (ii) Find an algebraic expression for $\cos(2\pi/5)$, the real part of α [which is also $u/2$]. (iii) As the final prize, use the results to construct a regular pentagon using compass and straight edge.

Homework 2.

PROBLEM 1. Explain why a compact metric space is separable. [For example, you can construct a countable dense set by looking at the finite sub-covers coming from an open cover with balls of radius $1/n$].

PROBLEM 2. Define the function

$$\mathfrak{S}(x) = \begin{cases} \sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases} \tag{1}$$

(a) Confirm that the function $f(x) = x + 2x^2\mathfrak{S}(x)$ is not monotone on any open interval containing $x = 0$ event though $f'(0) = 1$. Use the idea to construct a similar function for which the second derivative at a point is positive, but the function is not "convex" there: the graph is on both sides of the tangent line at that point. [Something like $x^2 + 4x^4\mathfrak{S}(x)$ could work.]

(b) Confirm that the *derivative* of the function $g(x) = e^{-3x}x^2\mathfrak{S}(x)$ *does not* achieve its minimal value on the interval $[0, 1/3]$.

PROBLEM 3. Let f be a differentiable function on $[-1, 1]$ and $|f'(x)| \le 1$. Explain why the function f' has a fixed point, that it, the equation $f'(x) = x$ has at least one solution on $[-1, 1]$. [For example, you can apply the intermediate value theorem to the function $g(x) = f'(x) - x$.]

PROBLEM 4. Confirm that differential equation $y'(x) = \sqrt{y(x)}$, $y(0) = 0$, has *uncountably many* solutions: together with $y(x) \equiv 0$, each of the following functions

$$y(x) = \begin{cases} 0, -a < x < b \\ (x-b)^2/4, x > b \\ (x+a)^2/4,\ x < -a \end{cases}$$

$a, b > 0$, is a solution.

PROBLEM 5. For $x \in [0, 1]$ define the following functions

$$\mathfrak{D}(x) = \begin{cases} 0,\ x \notin \mathbb{Q}, \\ 1,\ x \in \mathbb{Q}, \end{cases} \quad \mathfrak{D}_1(x) = \begin{cases} 0,\ x \notin \mathbb{Q}, \\ 1/q,\ x = p/q \in \mathbb{Q}, \end{cases} \quad \mathfrak{D}_2(x) = \begin{cases} 0,\ x \notin \mathbb{Q}, \\ q,\ x = p/q \in \mathbb{Q}, \end{cases} \tag{2}$$

and let $f_{r,x_0}(x) = (x - x_0)^r$ for some $r > 0$ and $x_0 \in (0, 1)$. Identify the points of continuity for each of the following functions:

(a) $\mathfrak{D}$, $\mathfrak{D}_1$, $\mathfrak{D}_2$;

(b) $f_{r,x_0}\mathfrak{D}$, $f_{r,x_0}\mathfrak{D}_1$, $f_{r,x_0}\mathfrak{D}_2$. Does it make any difference whether x_0 is rational or not? Do any of the answers depend on r?

Homework 3.

PROBLEM 1. A `norm` on $\mathbb{R}^n$ is a non-negative function $f = f(x)$ such that $f(x) = 0$ if and only if $x = 0$, $f(ax) = |a|f(x)$ for every real number a, and $f(x + y) \le f(x) + f(y)$. Denote by $\|x\|$ the usual Euclidean norm of x. Identify two positive numbers c_f and C_f so that the inequalities $c_f\|x\| \le f(x) \le C_f\|x\|$ hold for all $x \in \mathbb{R}^n$. In other words, all norms on $\mathbb{R}^n$ are *equivalent*. [For example, you can argue that the set $\{x \in \mathbb{R}^n : f(x) = 1\}$ is compact and does not contain zero; therefore the function $x \mapsto \|x\|$, being continuous, achieves minimal and maximal values there; keeping in mind that every nonzero $x \in \mathbb{R}^n$ can be written as $\|x\|(x/\|x\|)$, these values then lead to the numbers c_f and C_f. Alternatively, you can argue that f is continuous and look at its minimal and maximal values on the unit sphere in $\mathbb{R}^n$.]

PROBLEM 2. Confirm that, for a real-valued function f defined on an open interval (a, b), the following three properties are equivalent (and each can be used to define convexity):

(a) For every $x, y \in (a, b)$ and $\lambda \in (0, 1)$, $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ [that is, the graph of f is always below a secant line].

(b) For every $x_0 \in (a, b)$, the function $x \mapsto (f(x_0 + x) - f(x_0))/x$ is non-decreasing, as long as $x_0 + x \in (a, b)$ [if f is twice differentiable, then this is equivalent to having the second derivative of f non-negative].

(c) For every $x_0 \in (a, b)$, there exists a real number C_0 such that, for all $x \in (a, b)$ we have $f(x) \ge f(x_0) + C_0(x - x_0)$ [if f is differentiable, then $C_0 = f'(x_0)$ so that the graph of f is above the tangent line at $(x_0, f(x_0))$.

(d) For every real number r, the set $\{x : f(x) < r\}$ is connected (with empty set and a single point assumed to be connected).

One approach is to derive (b) from (a) directly; then use (b) to argue existence of *one-sided* derivatives $f'_\pm$ of f everywhere in (a, b) so that (c) follows with C_0 any number in the closed interval $[f'_-(x_0), f'_+(x_0)]$.

Problem 3. Confirm that if f is continuous on an open interval (a,b) and $f((x+y)/2) \le (f(x)+f(y))/2$, then f is convex. [Continuity condition can be relaxed, but some assumption is necessary: there exist *really wild* functions satisfying $f(x+y)=f(x)+f(y)$.]

Problem 4. Let $f=f(x)$, $x>a$, be a twice-differentiable function such that

$$\sup_{x>a}|f(x)| = M_0,\ \ \sup_{x>a}|f'(x)| = M_1,\ \ \sup_{x>a}|f''(x)| = M_2$$

for some positive numbers M_0, M_1, M_2.

(a) Show that $M_1^2 \le 4M_0M_2$ [for example, starting with Taylor formula $f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(y)$, $h>0$, $y\in[x,x+2h]$, deduce $f'(x) = (2h)^{-1}(f(x+2h)-f(x)) - hf''(y)$ so that $M_1 \le M_0/h + hM_2$; then minimize with respect to h].

(b) Use the result from (a) to explain why $\lim_{x\to\infty}|f(x)|=0$ and $|f''(x)|\le C$ imply $\lim_{x\to\infty}|f'(x)| = 0$.

(c) Construct an infinitely differentiable function $f=f(x)$ such that $\lim_{x\to+\infty}|f(x)|=0$ but $\limsup_{x\to\infty}|f'(x)| = +\infty$ and confirm that there are no contradictions with part (b). [$f(x) = \frac{\sin(h(x))}{1+x^2}$ with a suitable function h might work; of course, $f''(x)$ will be unbounded.]

Problem 5. [Basic calculus exercises]

(a) Compute the following limits as $x\to 0$:

$$(\tan x)(\ln x);\ x\ln x;\ \frac{\sin 5x}{\tan 3x};\ \frac{e^x-1-x}{x^2};\ (\sin x)^x;\ \left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\frac{\tan^2 x + 2x}{x^2+x};\ \frac{e^x+1}{x^2};\ \frac{\cos(2x)-1}{\sinh(x^2)};\ (\cos x)^{1/x^2};\ \frac{e^x-1-x}{\cos(2x)-\cos(3x)};\ \frac{\sin(3x)}{5x+7x^2};$$

$$\frac{\sin x - x}{\sqrt{1+x^2}-1};\ (1+3x+5x^2)^{1/x};\ \big(1+\tan(2x)\big)^{3/\sin x};\ \big(1+\sin(3x)\big)^{2/x}.$$

(b) Compute the following limits as $x\to+\infty$:

$$xe^{1/x}-x;\ \left(\frac{x+3}{x}\right)^{x+1};\ (3+x)^{1/x};\ x(e^{5/x}-1);$$

$$\left(\frac{2x+3}{2x+5}\right)^{3x};\ x\ln(x+2)-x\ln x;\ x^2\ln\left(1+\frac{1}{x}\right).$$

(c) Explain why the following inequalities are true [you can follow the suggestions]:

$$\frac{x}{1+x^2} < \arctan(x) < x,\ x>0,\ \ [\arctan x = \frac{x}{1+\xi^2(x)},\ \ 0\le\xi(x)\le x \text{ by MVT}]$$

$$1+x<e^x<2xe^x+e^{-x},\ x\ne 0,$$

$$[\text{for upper bound, use } f(x)=2xe^x+e^{-x}-e^x : f'(x)>0, x>0; f'(x)<0, x<0]$$

$$e^x > x^e,\ x\ne e\ \ [f(x)=x-e\ln x \text{ is convex; horizontal tangent at } (e,0)].$$

Homework 4.

Problem 1. Write $\sum_{k=1}^{\infty}(-1)^{k+1}/k$ as a Riemann-Stieltjes integral $\int_a^b f(x)d\alpha(x)$ [you need to identify limits a,b, and the function f,α.]

PROBLEM 2. Compute the limits by recognizing a suitable Riemann sum:

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\left(1+\frac{1}{n}\right)^3\left(1+\frac{2}{n}\right)^3\cdots\left(1+\frac{n}{n}\right)^3\right),$$

$$\lim_{n\to\infty} \frac{\Big(1!2!\cdots n!\Big)^{1/(n(n+1))}}{\sqrt{n}},$$

$$\lim_{n\to\infty} \frac{1}{n^4}\prod_{k=1}^{2n}(n^2+k^2)^{1/n},$$

$$\lim_{n\to\infty} \frac{\Big((n+1)\cdots(n+n)\Big)^{1/n}}{n},$$

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}}.$$

The answers and the corresponding integrals are as follows: $3\int_0^1 \ln(1+x)\,dx = 3(2\ln 2 - 1)$; $e^{-3/4}$, $(1/n)\sum_n(1-((k/(n+1)))\ln(k/n) \to \int_0^1(1-x)\ln x = -3/4$, equality $1+2+...+n = n(n+1)/2$ can also help; $\exp(\int_0^2 \ln(1+x^2)dx)) = 25\exp(2\arctan(2)-4)$; $\exp(\int_0^1 \ln(1+x)\,dx) = 4/e$; $\exp(-\int_0^1 \ln x)\,dx = e$, can use Stirling *to confirm* the answer.

PROBLEM 3. For $r>0$, define the function $f_r(x) = x^r\mathfrak{S}(x)$, where $\mathfrak{S}$ is from (1). Determine, with an explanation, the values of r for which the function f is Riemann integrable on (a) $[0,1]$ (b) $[1,\infty)$ (c) $[0,\infty)$. Then answer the same questions for the derivative f' of f. As usual, the *improper* Riemann integral is defined as the corresponding limit of proper integrals (over bounded intervals for unbounded interval, and over intervals where the function is bounded for unbounded integrands).

PROBLEM 4. Let $f:[a,b]\to\mathbb{R}$ be a continuously differentiable function such that $f(a)=f(b)=0$ and

$$\int_a^b f^2(x)\,dx = 1.$$

Confirm that $\int_a^b xf'(x)f(x)\,dx = -1/2$ [integrate by parts using $2ff'\,dx = d(f^2)$], and then use the Cauchy-Schwarz inequality to show that

$$\left(\int_a^b |f'(x)|^2\,dx\right)\left(\int_a^b x^2f^2(x)\,dx\right) > \frac{1}{4}.$$

Then comment on the following: (a) Is equality possible? [this would require $f' = cxf$] (b) Can the interval (a,b) be infinite? [most probably, as long as integration by parts is still possible] (c) Is there a connection with the Heisenberg uncertainly principle from quantum mechanics? [most probably, yes...]

PROBLEM 5. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function such that $|f'|^p$ is integrable for some $p>1$. Use the FTC and Hölder's inequality to show that

$$|f(x)-f(y)| \le |x-y|^{1/q}\left(\int_0^1 |f'(x)|^p\right)^{1/p}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

[This is a gateway to the Sobolev embedding theorems...]

Homework 5.

PROBLEM 1. Determine the range of values of the parameter p for which the following series (a) converge (b) converge absolutely. In each case, provide a convincing reason why your conclusion is correct.

(a) $\displaystyle\sum_{n\ge1}\frac{(-1)^n}{n^p}.$

(b) $\sum_{n\geq 1} \frac{(-1)^{n^2}}{n^p}$.

(c) $\sum_{n\geq 1} \frac{(-1)^n}{n^{1+n^{-p}}}$ [this does not converge absolutely for any $p>0$ by the Cauchy condensation test].

Problem 2. Determine the radius of convergence of the following power series:

(a) $\sum_{n\geq 0} n\left(\frac{3-\mathrm{i}}{3+3\mathrm{i}}\right)^n z^{2n}$; $\mathrm{i}=\sqrt{-1}$.

(b) $\sum_{n\geq 0}(5+\mathrm{i}+(-1)^n)^n z^{2n}$; $\mathrm{i}=\sqrt{-1}$.

(c) $\sum_{n\geq 0}\frac{(n!)^2}{(2n)!}z^{3n+1}$.

(d) $\sum_{n\geq 0}\frac{n!}{n^n}z^{2n+1}$.

(e) $\sum_{n\geq 1}\frac{z^{2n+1}(4n)!}{(3n)^{4n}(17+(-1)^n)^n}$.

Problem 3. Write the Taylor series of the given function $f=f(z)$ at the given point z_0 and determine the radius of convergence of the series.

(a) $f(z)=\frac{1}{2z+3}$, $z_0=0$.

(b) $f(z)=\frac{z^2+1}{z+1}$, $z_0=1$.

(c) $f(z)=\frac{1}{(2z-1)^2}$, $z_0=0$.

(d) $f(z)=\frac{1}{z^2+2z+2}$, $z_0=0$.

Problem 4. (I) For each differential equation below, (a) write the general solution as a power series; (b) determine if the equation has a solution that is a polynomial. Everywhere, $y=y(x)$. The name of the equation is indicated for easier on-line search.

(a) $y''=xy$ [Airy]

(b) $xy''+(5-x)y'+2y=0$ [Laguerre]

(c) $y''-2xy'+10y=0$ [Hermite]

(d) $(1-x^2)y''-2xy'+n(n+1)y=0$, $n=0,1,2,3,\dots$ [Legendre]

(II) Given real numbers $\sigma_0,\sigma_1,\sigma_2$, τ_0,τ_1, and λ, confirm that the ordinary differential equation

$$(\sigma_0+\sigma_1x+\sigma_2x^2)y''(x)+(\tau_0+\tau_1x)y'(x)=\lambda y(x)$$

has a polynomial solution if [and only if?]

$$\lambda=n\tau_1+n(n-1)\sigma_2$$

for some non-negative integer n, and then n is the degree of this polynomial.

Problem 5. Compute the power series expansion of the solutions of the following equations:

(a) $w''(z)-zw(z)=0$, $w(0)=0$, $w'(0)=1$.

(b) $z^2w''(z)+zw'(z)+z^2w(z)=0$, $w(0)=1$, $w'(0)=0$.

(c) $w''(z)-zw'(z)+2w(z)=0$, $w(0)=-1$, $w'(0)=0$.

Problem 6. Determine the values of the parameter λ for which the following equations have polynomial solutions:

(a) $w''(z)-2zw'(z)+\lambda w(z)=0$.

(b) $(1-z^2)w''(z)-zw'(z)+\lambda w(z)=0$.

(c) $(1-z^2)w''(z)-2zw'(z)+\lambda w(z)=0$.

(d) $zw''(z)+(1-z)w'(z)+\lambda w(z)=0$.

Homework 6.

PROBLEM 1. For each sequence of functions below, identify the limit and determine, with an explanation, whether the convergence is uniform:

(a) x^n, $x \in (0,1)$
(b) $\sin(x/n)$, $x \in (-\pi, \pi)$
(c) $nx/(1+nx)$, $x \in (0,1)$
(d) $nx^2/(n+x)$, $x \in (0,1)$

PROBLEM 2. Determine whether each of the following series converges (a) absolutely for each x in the indicated interval (b) uniformly over the indicated interval.

(a) $\sum_{n\geq 0} nx^n$, $|x| < 1$.
(b) $\sum_{n\geq 1} \frac{\cos(nx)}{n^2}$, $-\infty < x < +\infty$.
(c) $\sum_{n\geq 1} \frac{x^n}{n^{3/2}}$, $|x| < 1$.
(d) $\sum_{n\geq 1} \frac{x^n}{n!}$, $-\infty < x < +\infty$.

PROBLEM 3. We say that $F = F(x)$, $x \in \mathbb{R}$, is a `cumulative distribution function` if F has the following properties:

- for every x, F is right-continuous and has a limit from the left ($F(x) = F(x+)$, $F(x-)$ exists);
- F is non-decreasing (for every $x < y$, $F(x) \leq F(y)$);
- $\lim_{x\to -\infty} F(x) = 0$, $\lim_{x\to +\infty} F(x) = 1$.

Let $f_n = f_n(x)$, $x \in \mathbb{R}$, $n \geq 1$, and $f = f(x)$ be cumulative distribution functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$, and assume that the limit function f is continuous. Explain why the convergence of f_n to f is uniform on $\mathbb{R}$. [Make an effort to provide all the details how, given $\varepsilon > 0$, there exists an N_ε such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N_\varepsilon$ and all $x \in \mathbb{R}$. You can start by noticing that common limits at infinity effectively reduce the problem to a compact interval, and then uniform continuity of f and monotonicity of everything will finish the job. The main challenge is to keep track of all the epsilons while ensuring that the inequalities are in the right direction].

PROBLEM 4. Let $f = f(x)$, $x \in [-1,1]$, be a three times continuous differentiable function. For $n = 1, 2, \ldots$, define

$$a_n = n\Big(f(1/n) - f(-1/n)\Big) - 2f'(0).$$

Explain why the series $\sum_n a_n$ converges absolutely.

PROBLEM 5. Given a sequence of numbers a_n with $|a_n| < 1$, we say that the `infinite product` $\prod_n (1+a_n)$ converges if $\lim_{N\to\infty} \prod_{n=1}^N (1+a_n) = (1+a_1)\cdot \ldots \cdot (1+a_N)$ exists and is strictly positive; we say that $\prod_n (1+a_n)$ converges absolutely if $\prod_n (1+|a_n|)$ converges. Using the Taylor expansion of the function $f(x) = \ln(1+x)$, confirm that

(i) If $\prod_n (1+a_n)$ converges, then $\lim_{n\to\infty} a_n = 0$ (consider $\sum_n \ln(1+a_n)$);
(ii) $\prod_n (1+a_n)$ converges absolutely if and only if $\sum_n |a_n|$ converges;
(iii) If $a_n > 0$ or if $\sum_n |a_n|^2$ converges, then $\prod_n (1+a_n)$ and $\sum_n a_n$ either both converge or both diverge (consider $\sum_n \ln(1+a_n)$);
(iv) If $0 < a_n < 1$ and $\sum_n a_n$ diverges, then $\lim_{N\to\infty} \prod_{n=1}^N (1-a_n) = 0$.
(v) Without assuming that $a_n > 0$, it is possible to have $\sum_n a_n$ converging but $\prod_n (1+a_n)$ diverging [try $a_n = (-1)^n/\sqrt{n+1}$; the partial products will converge to zero] and it is possible to have $\sum_n a_n$ diverging but $\prod_n (1+a_n)$ converging [try $a_{2n} = -n^{-1/2}$, $a_{2n+1} = n^{-1} + n^{-1/2}$].
(vi) As concrete examples, confirm that

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3},$$

[use ($n^3 \pm 1 = (n\pm 1)(n(n\mp 1)+1)$ and the idea of telescoping], and that the infinite product

$$\prod_{k=1}^{\infty} \left(1+\frac{x}{k}\right) e^{-x/k}$$

converges uniformly in x on compact sub-sets of $\mathbb{R}$ [use Taylor expansion].

Homework 7.

PROBLEM 1. By considering the Taylor expansion of the function $f(t) = \int_0^1 \frac{dx}{x^{tx}}$ at the point $t = 1$, confirm the identity

$$\int_0^1 \frac{dx}{x^x} = \sum_{k\geq 1} \frac{1}{k^k}.$$

Here is an outline of the solution: writing $f(1) = \sum_{n\geq 0} f^{(n)}(0)/k!$, you need to confirm that

$$(-1)^n \int_0^1 (x \ln x)^n dx = \frac{n!}{(n+1)^{n+1}},$$

which you do by substitutions $x = e^{-y}$, $t = (n+1)y$, and then noticing that $\int_0^\infty t^n e^{-t} dt = \Gamma(n+1) = n!$.

PROBLEM 2. Consider the sequences

$$a_n = \frac{(n/e)^n}{n!}, \; b_n = \sqrt{n}\, a_n, \; n \geq 1.$$

Confirm that the sequence $\{a_n\}$ is monotonically decreasing and the sequence $\{b_n\}$ is monotonically increasing. Verify that $|b_n| \leq 2$ for all n. Conclude that $\lim_{n\to\infty} b_n$ exists and is finite, that $\lim_{n\to\infty} a_n = 0$, and then that the series $\sum_{n\geq 1}(-1)^n a_n$ converges conditionally.

In fact, $\lim_{n\to\infty} b_n = 1/\sqrt{2\pi}$, which is another way to state the Stirling formula. Moreover, $1/\sqrt{2\pi} = 0.39894228\ldots$, whereas $b_1 = 1/e = 0.367879441\ldots$, $b_2 = 2\sqrt{2}b_1^2 = 0.382785986\ldots$, and $b_3 = 4.5\sqrt{3}\, b_1^3 = 0.388051794\ldots$.

PROBLEM 3. [A little taste of Bessel functions]

(a) Determine the values of the parameter q so that `Bessel's equation` $x^2y'' + xy' + (x^2 - q)y = 0$ has a real analytic solution.

(b) Use the power series representation of the Bessel functions

$$J_N(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+N)!} \left(\frac{z}{2}\right)^{2k+N}, \; N = 0, 1, 2, 3, \ldots,$$

to confirm that

(1) $J_N(-z) = (-1)^N J_N(z)$,
(2) $(z^N J_N(z))' = z^N J_{N-1}(z)$,
(3) $(z^{-N} J_N(z))' = -z^{-N} J_{N+1}(z)$,
(4) $zJ_N'(z) = NJ_N(z) - zJ_{N+1}(z) = -NJ_N(z) + zJ_{N-1}(z)$,

(5) $\int_0^x y^N J_{N-1}(y)dy = x^N J_N(x)$.

PROBLEM 4. Confirm that

$$\lim_{n\to\infty} n^{1/2} \int_{-1}^1 (1-x^2)^n \, dx = \sqrt{\pi}.$$

A possible outline: by symmetry, $\int_{-1}^1 (1-x^2)^n \, dx = 2\int_0^1 (1-x^2)^n \, dx$, then change variables $u = x^2$ to get a Beta integral, evaluate it using Gamma functions, and then apply Stirling.

PROBLEM 5. Use the properties of the Gamma and Beta functions to confirm the following equalities:

$$\int_0^1 (1-x^3)^{1/7} \, dx = \int_0^1 (1-x^7)^{1/3} \, dx, \quad \text{then suggest a more general identity;}$$

$$\int_0^1 x^{1/3}(1-x)^{2/3} \, dx = \frac{2\pi\sqrt{3}}{27}.$$

Homework 8.

Problem 1. For functions f, g below, find a, b, c, d so that $g(x) = a + bf(cx + d)$.

(a) $f(x) = x, 0 \le x \le \pi$, f is even and 2π-periodic; $g(x) = 2x, 0 \le x \le 1/2; g(x) = 2 - 2x, 1/2 \le x \le 1$, g is odd and is periodic with period 2.
(b) $f(x) = 1, 0 \le x < \pi; f(x) = 0, \pi \le x < 2\pi$, f is 2π-periodic; $g(x) = 1, 0 \le x < 1/2; g(x) = -1, 1/2 \le x < 1$, g is periodic with period 1.
(c) $f(x) = x, -\pi \le x < \pi$, f is 2π-periodic; $g(x) = x, 0 \le x < 1$, g has period 1.

Problem 2. Let $f(x) = 2x$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f and evaluate (a) $S_f(3)$ (b) $S_f(5/2)$.

Problem 3. Compute the Fourier series expansion of each of the six functions in problem 4. (You will have to compute some integrals, but not for all six functions). Use the results to evaluate the following infinite sums:

(a) $\sum_{k\ge 0} \frac{(-1)^k}{2k+1}$
(b) $\sum_{k\ge 0} \frac{1}{(2k+1)^4}$
(c) $\sum_{k\ge 0} \frac{1}{(2k+1)^2}$
(d) $\sum_{k\ge 1} \frac{1}{k^2}$
(e) $\sum_{k\ge 1} \frac{(-1)^{k+1}}{k^2}$

Problem 4. The function $f(x) = x^2$, $0 < x < 1$, is to be expanded in a Fourier series. You have three options:
(a) take the periodic extension of the function with period 1.
(b) take the periodic extension of the even extension of the function with period 2.
(c) take the periodic extension of the odd extension of the function with period 2.
Draw the pictures of the sum of the resulting Fourier series in each case. Which option would you choose and why? (Note: to answer these questions you do not need to compute the Fourier coefficients of the function.)

Problem 5. Solve the following initial-boundary value problems:
Heat equation

$$\begin{aligned} u_t &= u_{xx},\ u = u(x,t),\ x \in (0,1),\ t > 0, \\ u(x,0) &= \sum_{n\ge 0} \frac{\sin(2\pi(2n+1)x)}{(2n+1)^2}, \\ u(0,t) &= 0, \\ u(1,t) &= 0; \end{aligned}$$

Wave equation

$$\begin{aligned} u_{tt} &= 4u_{xx},\ u = u(t,x),\ t > 0,\ x \in (0,\pi), \\ u(0,x) &= 0, \\ u_t(0,x) &= \sin(2x) + 3\sin(5x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

Homework 9.

Problem 1. Let the function $x = x(r)$ be defined in some neighborhood of $r = 0$ by $(x-1)(x^2+1) = r$. Determine the numbers c_0, c_1, c_2, c_3 so that

$$\lim_{r\to 0} \frac{x(r) - c_0 - c_1 r - c_2 r^2 - c_3 r^3}{r^3} = 0.$$

Problem 2. In what follows, W_0 is the principal branch of the Lambert W function.

(a) Confirm that the unique solution of $x^{x^a} = b$, $a > 0, b > 1$, is

$$x = e^{W_0(a \ln b)/a} = \left(\frac{a \ln b}{W_0(a \ln b)}\right)^{1/a}$$

and that the result is consistent with the special case $a = b$, when the solution is (obviously) $x = a^{1/a}$. Then investigate the cases $0 < b < 1$ and/or $a < 0$.

(b) Confirm that, for $0 < a < 1$ and $b \in \mathbb{R}$, the unique solution of $a^x = x + b$ is

$$x = -\frac{b + W_0(-a^{-b} \ln a)}{\ln a}.$$

[A comment: the equation is equivalent to $e^{(x+b)\ln a} = a^b(x + b)$]. Then investigate the case $a > 1$, followed by the complex-valued case.

(c) Confirm that the solution of the initial value problem $y' = y^2(1-y)$, $t > 0$, $y(0) = a \in (0, 1)$, is

$$y(t) = \frac{1}{1 + W_0(re^{r-t})}, \quad r = \frac{1-a}{a}.$$

Then investigate the case $a \notin (0, 1)$. [A comment: $\frac{1}{y^2(1-y)} = \frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y}$.]

Problem 3. [Examples of the Feynman trick]
(a) Confirm that

$$\int_0^1 \frac{x-1}{\ln x}\, dx = \ln 2.$$

[Set $F(t) = \int_0^1 (x^t - 1)/\ln x \;\; dx$, argue that $F(0+) = 0$, $F(1)$ is what you need, and $F'(a) = 1/(1+a)$; also explain how the argument breaks down if, instead of $x^t - 1$ you consider $x^t - 2$].
(b) Confirm that

$$\int_0^{+\infty} \ln x \, e^{-x^2}\, dx = -\frac{\sqrt{\pi}(\gamma + 2\ln 2)}{2},$$

where γ is the Euler-Mascheroni constant [set $F(t) = \int_0^\infty x^t e^{-x^2} dx$, argue that $F(t) = (1/2)\Gamma((t + 1)/2)$ and the integral you want is $F'(0) = (1/4)\Gamma'(1/2)$; then use the duplication formula for the Gamma function, together with $\Gamma(1) = 1, \Gamma'(1) = -\gamma, \Gamma(1/2) = \sqrt{\pi}$.]

Problem 4. [This is about contractions and related ideas]
(a) Construct a mapping f of a closed bounded interval into itself so that $|f(x) - f(y)| < |x - y|$ for all x, y but there is no $c \in (0, 1)$ with the property $|f(x) - f(y)| < c|x - y|$ for all x, y. Does the mapping have a fixed point? If so, how fast is it reached? Are there any other fixed points in your example?
(b) Assume that X is a *compact* metric space with metric d and the mapping $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$. Prove that f has a unique fixed point $x*$ and the point is the limit of the sequence $y_n = f(y_{n-1})$, $n \to \infty$, for every starting point $y_0 \in X$.
(c) Explain why a contracting mapping (satisfying $d(f(x), f(y)) \le cd(x, y)$ for some $c \in (0, 1)$) of a compact metric space cannot be invertible.
(d) Construct a metric space X, with metric d, and a mapping $f : X \to X$ such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, but f has no fixed points.

Problem 5. Recall that a sequence of continuous functions $\{f_n = f_n(x)\}_{n\ge1}$ on $[0, 1]$ is *equicontinuous* on $[0, 1]$ if for every $\varepsilon > 0$ there is $\delta > 0$ so that for all x, y from the interval $[0, 1]$ satisfying $|x - y| < \delta$ and for all $n \ge 1$ it holds that $|f_n(x) - f_n(y)| < \varepsilon$.

Let $\{h_n = h_n(x)\}_{n\ge1}$ be a sequence of (Riemann) integrable functions defined on $[0, 1]$ and $|h_n(x)| \le 1$ for all $x \in [0, 1]$ and all $n \ge 1$. Let $K = K(x, t)$ be a continuous function on $[0, 1] \times [0, 1]$. Define $f_n(x) = \int_0^1 K(x, t)h_n(t)dt$. Prove that the sequence $\{f_n\}$ is equicontinuous on $[0, 1]$. Can the same conclusion hold without assuming *joint* continuity of K in x and t?

Homework 10.

Problem 1. What can you say about wedge product of (a) two closed forms (b) two exact forms (c) an exact form and a closed form (d) an arbitrary form with itself (that is, $\omega \wedge \omega$)? [There is no guarantee that $\omega \wedge \omega = 0$: consider $\omega = dx \wedge dy + dz \wedge dw$ in $\mathbb{R}^4$. Can you construct a similar example in $\mathbb{R}^3$ or $\mathbb{R}^5$? How about $\mathbb{R}^6$? Can you state and prove some general statements based on these observations?]

PROBLEM 2. Verify the following identities:

$$\begin{aligned}
\nabla(fg) = g\nabla f + f\nabla g,\ \nabla\cdot(f\vec{\boldsymbol{F}}) = \vec{\boldsymbol{F}}\cdot\nabla f + f\nabla\cdot\vec{\boldsymbol{F}},\\
\nabla\times(f\vec{\boldsymbol{F}}) = (\nabla f)\times\vec{\boldsymbol{F}} + f\nabla\times\vec{\boldsymbol{F}},\\
\nabla\times(\nabla f) = \vec{\boldsymbol{0}},\ \nabla\cdot(\nabla\times\vec{\boldsymbol{F}}) = 0,\\
\nabla\cdot(\vec{\boldsymbol{F}}\times\vec{\boldsymbol{G}}) = \vec{\boldsymbol{G}}\cdot(\nabla\times\vec{\boldsymbol{F}}) - \vec{\boldsymbol{F}}\cdot(\nabla\times\vec{\boldsymbol{G}}),\ \nabla\cdot(\nabla f\times\nabla g) = 0.
\end{aligned}$$

When possible, use the language of differential forms.

PROBLEM 3. Confirm that the two forms below are closed but not exact:

$$\begin{aligned}
\omega &= -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \text{ in } \mathbb{R}^2 \text{ with } (0,0) \text{ removed;}\\
\omega &= \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dy\wedge dz + \frac{y}{(x^2+y^2+z^2)^{3/2}}\,dz\wedge dx\\
&+ \frac{z}{(x^2+y^2+z^2)^{3/2}}\,dx\wedge dy \text{ in } \mathbb{R}^3 \text{ with } (0,0,0) \text{ removed.}
\end{aligned}$$

PROBLEM 4. Evaluate the following integrals:

(a) $\displaystyle\int_C (2y^2+2xz)dx + 4xydy + x^2dz$, where C is the path $x(t) = \cos t, y(t) = \sin t, z(t) = t,\quad 0 \le t \le 2\pi$.

(b) $\displaystyle\oint_C y^2dx + x^2dy$, where C is the boundary of the rectangle with vertices $(1,0)$, $(3,0)$, $(3,2)$, $(1,2)$, oriented counterclockwise.

(c) $\displaystyle\oint_C ydx - zdy + ydz$, where C is the ellipse $x^2+y^2 = 1; 3x+4y+z = 12$ oriented counterclockwise as seen from the point $(0,0,1000)$.

PROBLEM 5. Compute the following quantities using a suitable integral:

(a) The mass of the curve shaped as a helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $t \in [0,2\pi]$, if the density at every point is the square of the distance of the point to the origin.
(b) The area between x-axis and the curve $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$, $t \in [0,2\pi]$.
(c) The average distance to the (x,y) plane of the points on the hemisphere $z = \sqrt{1-x^2-y^2}$.
(d) The flux of the vector field $\mathbf{F} = \langle x,y,z\rangle$ through the lateral surface of the cylinder $x^2+y^2 = 1, z \in [0,2]$.

Homework 11.

PROBLEM 1. Explain why all three functions from (2) are Lebesgue-integrable on $[0,1]$ and all three integrals are equal to zero.

PROBLEM 2. Let $f = f(x)$, $x \in \mathbb{R}$, be a Lebesgue-integrable function.

(a) Given $\beta \in (0,1)$, explain why the integral $\int_0^{+\infty} \frac{|f(x)|}{|x-y|^\beta}\,dx$ is finite for almost all $y \in \mathbb{R}$. [One way: integrate with respect to y].

(b) Assuming $0 < f(x) < 1$, explain why, for every $C > 0$, there exists a $y \in [0,1]$ such that $\int_0^1 \frac{dx}{|f(x)-y|} > C$. [One possibility: assume otherwise, integrate with respect to y, see if there is a contradiction].

PROBLEM 3. Explain why

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Note that the sequence $(1+(1/n))^{n+1}$ is monotonically decreasing to e; you can try and use it without a proof, but eventually make sure you can prove it too.

How will the result and/or argument change if we replace the power $-(n+1)$ with $-n$?

PROBLEM 4. Let f be a Lebesque-integrable function on $(0,+\infty)$ and define the function

$$g(x)=\int_0^{+\infty}\frac{f(y)}{x+y},\ dy,\ x>0.$$

(a) Explain why the function g is differentiable for all $x>0$. [Justify differentiation under the integral sign.]

(b) Does $\lim_{x\to 0+} g(x)$ always exist? Explain your conclusion. [Probably not: take $f(y)=1,\ 0\le y\le 1$ and $f(y)=0,\ y>1$].

(c) Is it possible to have $g(0+)$ finite but no $\lim_{x\to 0+} g'(x)$? [Probably not: take $f(y)=y,\ 0\le y\le 1$ and $f(y)=0,\ y>1$].

PROBLEM 5. [This is a look into infinite dimensions]

(a) Explain why the closed unit ball in $L_2((0,1))$ is not compact.

(b) Explain why there is no analog of the Lebesgue measure on $L_2((0,1))$ (that it, there is no countably additive measure that is positive on open balls and is translation-invariant

[In both cases, a possible argument relies on an infinite orthonormal collection, which does not contain a converging subsequence and can be separated by non-overlapping open balls.]

Homework 12.

PROBLEM 1. Compute the Fourier transform of the function f in each of the following cases:

(a) $f(x)=e^{-2x}, x>0,\ f(x)=0$ otherwise.

(b) $f(x)=x, a<x<b,\ f(x)=0$ otherwise.

(c) $f''(x)-f(x)=u(x)$, where $u(x)=1, |x|<1,\ u(x)=0$ otherwise.

(d) $f''(x)=xf(x)$.

PROBLEM 2. Compute the Fourier transform of the function $f(x)=e^{-|x|}$ and use the result to evaluate the integrals $\displaystyle\int_0^{\infty}\frac{\cos(wx)}{1+w^2}\,dw$ and $\displaystyle\int_0^{\infty}\frac{dw}{(1+w^2)^2}$.

PROBLEM 3. (a) The Fourier transform of the function $f(x)=e^{-x^2/2}$ is $\widehat{f}(\omega)=e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x)=e^{-ax^2}$, $a>0$.

(b) In the previous problem you learned that the Fourier transform of the function $f(t)=e^{-|t|}$ is $\widehat{f}(\omega)=\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{1+\omega^2}$. Use the result to compute the Fourier transform of the following functions (i) $f(t)=1/(1+t^2)$; (ii) $f(t)=a/(b+ct^2),\ a,b,c>0$.

PROBLEM 4. Confirm that if

$$h(x)=\int_{-\infty}^{+\infty} f(x-y)g(y)dy,$$

then $\hat{h}(w)=\sqrt{2\pi}\hat{f}(w)\hat{g}(w)$.

PROBLEM 5. Let $f=f(x)$ be a bounded continuous function such that $\int_{-\infty}^{+\infty}|f(x)|dx<\infty$ and $\int_{-\infty}^{+\infty}f^2(x)dx=1$. For $t>0$ define

$$u(t,x)=\frac{1}{\sqrt{4\pi\mathrm{i}t}}\int_{-\infty}^{\infty} e^{\mathrm{i}(x-y)^2/(4t)}f(y)dy,\quad \mathrm{i}=\sqrt{-1}.$$

Verify that, for all $t>0$,

$$\int_{-\infty}^{\infty}|u(t,x)|^2dx=1.$$

PROBLEM 6. Is it possible that f is a bounded function and $\int_{-\infty}^{+\infty}|f(x)|dx<\infty$, but $\int_{-\infty}^{+\infty}f^2(x)dx=\infty$? Is it possible that f is bounded function and $\int_{-\infty}^{+\infty}f^2(x)dx<\infty$, but $\int_{-\infty}^{+\infty}|f(x)|dx=\infty$? What if we drop the assumption that f is bounded? In each case, either construct an example (if your answer is positive) or outline an argument supporting your negative answer.

PROBLEM 7. Define the function

$$B(t) = \int_0^{+\infty} \frac{x^3 \sin(tx)}{x^4+4}\, dx, \ t > 0. \tag{3}$$

Confirm that $B(t) = \frac{\pi}{2} e^{-t} \cos t$ by filling in the details below:

(i) If $C(t) = \int_0^{+\infty} \frac{\cos(tx)}{x^4+4}\, dx$, then $C(0) = \pi/8$ and $C'''(t) = B(t)$.

(ii) The function $C = C(t)$ is the *bounded solution* of the ODE $C''''(t) + 4C(t) = 0$ satisfying $C(0) = \pi/8$ and $C'(0) = 0$.

Explain why $\lim_{t \to 0+} B(t) = \pi/2$ and we cannot simply pass to the limit on the right-hand side of (3). Here, the following observation can be helpful:

$$\frac{x^3}{x^4+4} = \frac{1}{x}\left(1 - \frac{4}{x^4+4}\right), \quad \int_0^{+\infty} \frac{\sin(tx)}{x}\, dx = \frac{\pi}{2},$$

so that

$$B(t) = \frac{\pi}{2} - 4\int_0^{+\infty} \frac{\sin(tx)}{x(x^4+4)}\, dx.$$

PROBLEM 8. Confirm that, for $s > 1$,

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx = \zeta(s)\Gamma(s),$$

where ζ is the Riemann zeta function and Γ is the (Euler) Gamma function.

One way is to write $1/(e^x - 1) = e^{-x}/(1 - e^{-x}) = e^{-x} \sum_{k=0}^{\infty} e^{-kx}$, integrate term-by-term, and re-arrange the result; the main challenge is to justify the steps related to swapping integration and summation.