MATH 425B, SPRING 2024

COMPUTER PROJECT   DUE 11:59PM OF THE LAST DAY OF CLASSES [FRIDAY, APRIL 26, 2024]

You are welcome to use any programming language or environment and any help you want.

USE AT LEAST A 10PT FONT, AND DO NOT SUBMIT MORE THAN 10 PAGES. WHEN YOU ARE DONE, PLEASE COMPILE EVERYTHING INTO A **single PDF file and upload the file to Blackboard.**

The general objective of the project is to connect theory and practice of plotting graphs of functions.

Problem 1, where you will investigate functions that are everywhere continuous, nowhere differentiable, and exhibit self-similar (fractal) behavior.

PART 1. Plot the graph of the function

$$h_{10}(x) = \sum_{k=1}^{10} \frac{\sin\Big((k!)^2\, x\Big)}{k!}$$

for $x \in [0,1]$, $x \in [0,0.1]$, $x \in [0,0.01]$, $x \in [0,0.001]$. Make sure to use the right step size for plotting.

What do you notice about the graphs? What *should* you notice?

PART 2. Given positive numbers p, q, consider the function

$$h(x) = \sum_{k=1}^{\infty} \frac{\sin\Big((k!)^p\, x\Big)}{(k!)^q},\ x \in [0,1].$$

For what values of p and q will this function be

(1) defined?
(2) continuous?
(3) differentiable?
(4) continuous but not differentiable?

For each of your answers, provide a convincing *analytical* explanation, as well as an illustrating picture.

If you want to explore further, try to understand the behavior of $\sum_{k=1}^{n} \sin\Big((k!)^p\, x\Big)$, as $n \to +\infty$, for different values of p.

Problem 2, where you will compare Fourier series with the corresponding function and analyze the Gibbs phenomenon.

Let $f = f(x)$ be a 2π-periodic function defined for $x \in (-\pi, \pi]$ by $f(x) = x$. Let $S_n(x) = \sum_{k=1}^{n} b_k \sin(kx)$, where $b_k = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin(kx)dx$.

Do the following:

(1) Plot the graphs of $S_n(x)$ for $n = 10, 50, 100$ and $x \in [-\pi, \pi]$. The choice of the procedure to compute b_k is up to you. Keep in mind that if you divide the interval $[-\pi, \pi]$ to approximate the integral, your step size must be small enough to capture enough of the oscillations of the sines.
(2) Estimate $\max_{x\in[-\pi,\pi]} S_n(x)$ for $n = 10, 50, 100$.
(3) Compute $\lim_{n\to\infty} \frac{S_n(\pi-\pi/n)}{\pi}$ analytically and confirm that the result is consistent with the estimates. This limit is a quantitative measure of the Gibbs phenomenon.

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