Math 126

Some pure math applications.

If N is large, then the probability that two randomly selected numbers from $1, \dots, N$ are relatively prime, that is, do not have any prime divisors in common, is approximately

$$\frac{6}{\pi^2} = \frac{1}{\sum_{k\geq 1} k^{-2}} \approx 0.6079\cdots .$$

This also turns out to be the (approximate) probability that a randomly selected number from $1, \dots, N$ is square-free, that is, not divisible by a square of any prime number.

The probability that $m \geq 2$ numbers out of $1, \dots, N$ are relatively prime is approximately

$$\frac{1}{\sum_{k\geq 1} k^{-m}}$$

In what follows, p denotes a prime number, i.e. one of $2, 3, 5, 7, 11, 13, \dots$.

THE PRIME NUMBER THEOREM: the number $\pi(n)$ of primes that are less than or equal to n is

$$\pi(n) = \sum_{p\leq n} 1 \sim \frac{n}{\ln n} \sim \int_2^n \frac{dx}{\ln x}, \ n \to +\infty.$$

Equivalently, n-th prime number p_n satisfies

$$p_n \sim n \ln n.$$

THREE THEOREMS OF MERTEN:

$$\sum_{p\leq n} \frac{\ln p}{p} = \ln n + O(1), \ n \to +\infty;$$

$$\sum_{p\leq n} \frac{1}{p} = \ln(\ln n) + O(1), \ n \to +\infty;$$

$$\lim_{n\to+\infty} \ln n \prod_{p\leq n} \left(1 - \frac{1}{p}\right) = e^{-\gamma}.$$

EULER: for every $r > 1$,

$$\prod_p \left(1 - \frac{1}{p^r}\right) = \frac{1}{\sum_{k=1}^{\infty} k^{-r}}.$$

HARDY-RAMANUJAN: for large n, the number of distinct prime divisors of n is "typically around" $\ln(\ln n)$.