Math 126
Limits.

Time-saving notations for limit comparison. With f and g representing either functions or sequences,

- $f \sim g$ if $f \geq 0$, $g \geq 0$, $\lim(f/g) = 1$;
- $f \simeq g$ if $f \geq 0$, $g \geq 0$, $\lim(f/g) = c \in (0, +\infty)$;
- $f = o(g)$ if $f \geq 0$, $g \geq 0$, $\lim(f/g) = 0$;
- $f = \omega(g)$ if $f \geq 0$, $g \geq 0$, and $g = o(f)$;
- $f = O(g)$ if $f \geq 0$, $g \geq 0$, and $f \leq Cg$, $C \in (0, +\infty)$;
- $f = \Omega(g)$ if $f \geq 0$, $g \geq 0$, and $g = O(f)$;
- $f = \Theta(g)$ if $f \geq 0$, $g \geq 0$, and $f = O(g)$ and $g = O(f)$.

Squeeze (sandwich, pinch) theorem for limits: if $g \leq f \leq h$ and $\lim g = \lim h = L$, then $\lim f = L$. A typical application is when $g = 0$ and $L = 0$.

Limit comparison: if $f \simeq g$ then either both converge or both diverge. Only works for absolute convergence.

To establish `absolute convergence` of an integral or a series using **simple comparison**, look for a $\big|\cdot\big| \leq \odot$-type inequality, so that the the thing on the right is positive and `converges`.

To establish `absolute divergence` of an integral or a series using **simple comparison**, look for a $\big|\cdot\big| \geq \odot$-type inequality, so that the thing on the right is positive and `diverges`.

Using the notations:

- If $f = \Theta(g)$, then either both converge or both diverge;
- If $f = o(g)$ or $f = O(g)$, then "f is not bigger than g" and convergence of g implies convergence of f;
- If $f = \omega(g)$ or $f = \Omega(g)$, then "f is not smaller than g" and divergence of g implies divergence of f.

To keep in mind:

- as $x \to 0+$,

$$|\ln x|^a = o(x^{-r}) \text{ for all } a > 0,\ r > 0, \text{ e.g. } \lim_{x\to 0+} x^{0.001}|\ln x|^{1000} = 0;$$

$$x^r = o(x^q),\ r > q: \quad \text{at } 0, \quad \text{smaller power means bigger function;}$$

- as $x \to +\infty$ or $n \to +\infty$, "$\ln n < n < a^n < n!$" if $a > 1$ [for x, replace $n!$ with $\Gamma(x)$]:

$$(\ln x)^a = o(x^r) \text{ for all } a > 0,\ r > 0, \text{ e.g. } \lim_{x\to+\infty} x^{-0.001}(\ln x)^{1000} = 0;$$

$$x^a = o(e^{cx}) \text{ for all } a > 0,\ c > 0, \text{ e.g. } \lim_{x\to+\infty} x^{10000}e^{-0.00001x} = 0;$$

$$e^{cn} = o((n!)^r) \text{ for all } c > 0,\ r > 0, \text{ e.g. } \lim_{n\to+\infty} \frac{e^{10000n}}{(n!)^{0.00001}} = 0.$$

Also remember that

$$q^n = e^{n\ln q}$$

and

$$\ln x < 0 \ \Leftrightarrow\ 0 < x < 1$$
$$\ln x > 0 \ \Leftrightarrow\ x > 1.$$