Math 126

Functions.

FTC: $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ or $\int_0^x f'(t)dt = f(t) - f(0)$.

Inverse function. If $y = f(x)$ is monotone for $x \in [a,b]$ and takes values in $[A,B]$, then $x = f^{-1}(y)$ is monotone for $y \in [A,B]$ and takes values in $[a,b]$; $f^{-1}(f(x)) = x$ for $x \in (a,b)$ and $f(f^{-1}(y)) = y$ for $y \in [A,B]$. By the chain rule,

$$f'(f^{-1}(y))(f^{-1})'(y) = 1 \Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Elementary functions.

Polynomial of degree n: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$.

Rational function: $R(x) = \frac{P(x)}{Q(x)}$, P and Q are polynomials.

Exponential function e^x or $\exp(x)$.

Hyperbolic functions: hyperbolic sine $\sinh x = (e^x - e^{-x})/2$, hyperbolic cosine $\cosh x = (e^x + e^{-x})/2$, hyperbolic tangent $\tanh x = \sinh x/\cosh x$, hyperbolic cotangent $\coth x = \cosh x/\sinh x$.

(Natural) log function $\ln x$

Trigonometric functions: sine $\sin x$, cosine $\cos x$, tangent $\tan x = \sin x/\cos x$, cotangent $\cot x = \cos x/\sin x$, secant $\sec x = 1/\cos x$, cosecant $\csc x = 1/\sin x$.

(The main) Inverse trigonometric functions

- $\sin^{-1}(y) = \arcsin(y) \in [-\pi/2, \pi/2]$, $y \in [-1,1]$.
- $\cos^{-1}(y) = \arccos(y) \in [0,\pi]$, $y \in [-1,1]$.
- $\tan^{-1}(y) = \arctan(y) \in (-\pi/2, \pi/2)$, $-\infty < y < +\infty$.

For the purposes of integration:

$$\sin^2 x + \cos^2 x = 1,\ \tan^2 x + 1 = \sec^2 x;$$
$$(\cos x)' = -\sin x,\ (\sin x)' = \cos x,\ (\tan x)' = \sec^2 x,\ (\sec x)' = \sec x \tan x;$$
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}},\ (\arctan x)' = \frac{1}{1+x^2}.$$

Partial fractions for a rational function $P(x)/Q(x)$ with degree of Q bigger than degree of P:

- Represent $Q(x)$ as a product of terms $(x-a)^k$ and $((x+b)^2+c^2)^m$;
- Each $(x-a)^k$ produces

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k}.$$

- Each $((x+b)^2+c^2)^m$ produces

$$\frac{B_1 x + C_1}{(x+b)^2+c^2} + \frac{B_2 x + C_2}{((x+b)^2+c^2)^2} + \ldots + \frac{B_m x + C_m}{((x+b)^2+c^2)^m}.$$

In general, to get the undetermined coefficients, proceed in this order: (a) guess, (b) use "cover-up method," (c) plug in other "convenient" values of x, (d) only as a last resort, put to the common denominator and solve the system [typically, after going through (a)–(c), you get some of the coefficients, so that, if it comes to (d), there will be fewer unknowns in the system].

For example [keeping in mind that $x^2 + 2x + 3 = (x+1)^2 + 2$],

$$\frac{x^5 + 4x^3 + 1}{(x-1)(x+1)^3(x^2+2)(x^2+2x+3)^2}$$
$$= \frac{A}{x-1}$$
$$+ \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3}$$
$$+ \frac{Bx+C}{x^2+2}$$
$$+ \frac{B_1 x + C_1}{(x+1)^2+2} + \frac{B_2 x + C_2}{((x+1)^2+2)^2}.$$

Not especially pleasant, but the "cover-up method" immediately gives

$$A = \frac{1+4+1}{2^3 \cdot (1+2) \cdot (1+2+3)^2} = \frac{6}{24 \cdot 6^2} = \frac{1}{144},$$
$$A_3 = \frac{(-1) + (-4) + 1}{(-2) \cdot 3 \cdot 2^2} = \frac{1}{6}.$$

[To get A, multiply both side by $(x-1)$ and then put $x = 1$; to get A_3, multiply both sides by $(x+1)^3$ and then put $x = -1$.]