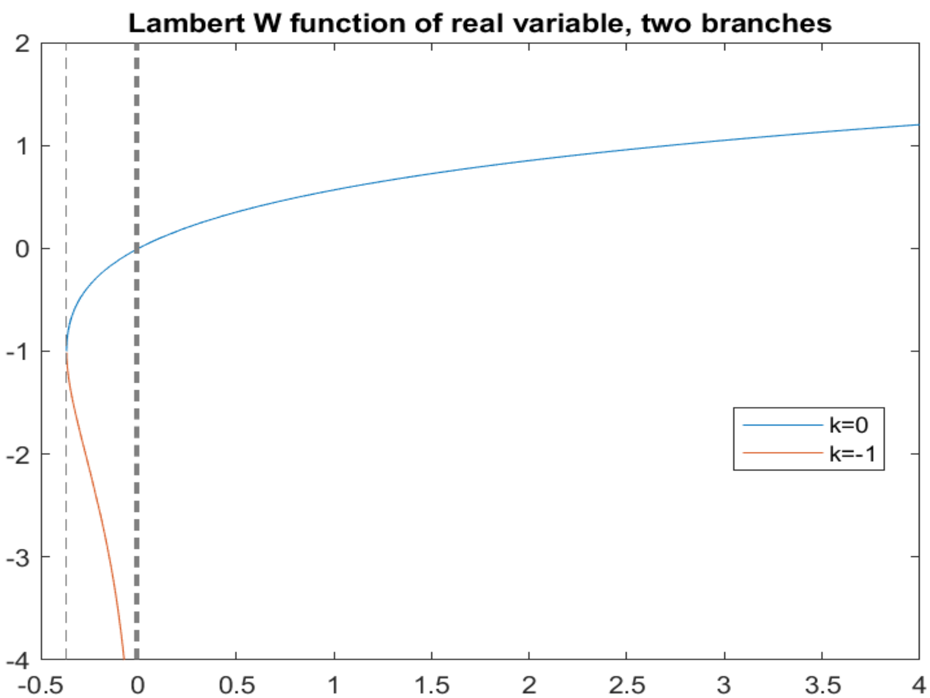
A SUMMARY OF THE LAMBERT W FUNCTION[1]

(Modern) Starting Point/Basic Summary: solving the equation $We^W = x, x > -1/e$; the solution $W = W(x) > -1$ is the Lambert W function, with the following Taylor expansion at zero:

$$(1) \qquad W(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{n^{n-1}}{n!} x^n, \; |x| < \frac{1}{e}.$$

A More Nuanced Approach: The function $x \mapsto xe^x$, $-\infty < x < +\infty$ has a global minimal value $-1/e$ at $x = -1$ and is 2-to-1 for $x < 0$. Accordingly, the second branch [red in the graph below] $W_{-1}(x)$ of the inverse function is defined for $-1/e \le x < 0$; the original branch satisfying (1) [blue in the graph below] is then denoted by $W_0(x)$. In particular, $W_0(xe^x) = x$ for $x \ge -1$ and $W_{-1}(xe^x) = x$ for $x \le -1$.



Some (Immediate) Consequences.

$$W_0(x) + \ln W_0(x) = \ln x, \; x > 0; \quad W_0(x \ln x) = \ln x, \; x \ge \frac{1}{e}; \quad W_0\left(-\frac{\ln x}{x}\right) = -\ln x, \; 0 < x \le e;$$

$$W_0(e) = 1; \; \ln x - \ln\ln x < W_0(x) < \ln x, \; x > e; \; \lim_{x \to +\infty} \frac{W_0(x)}{\ln x} = 1; \quad \lim_{x \to +\infty} \frac{W_0(x) - \ln x}{\ln \ln x} = -1.$$

Some Immediate Applications.

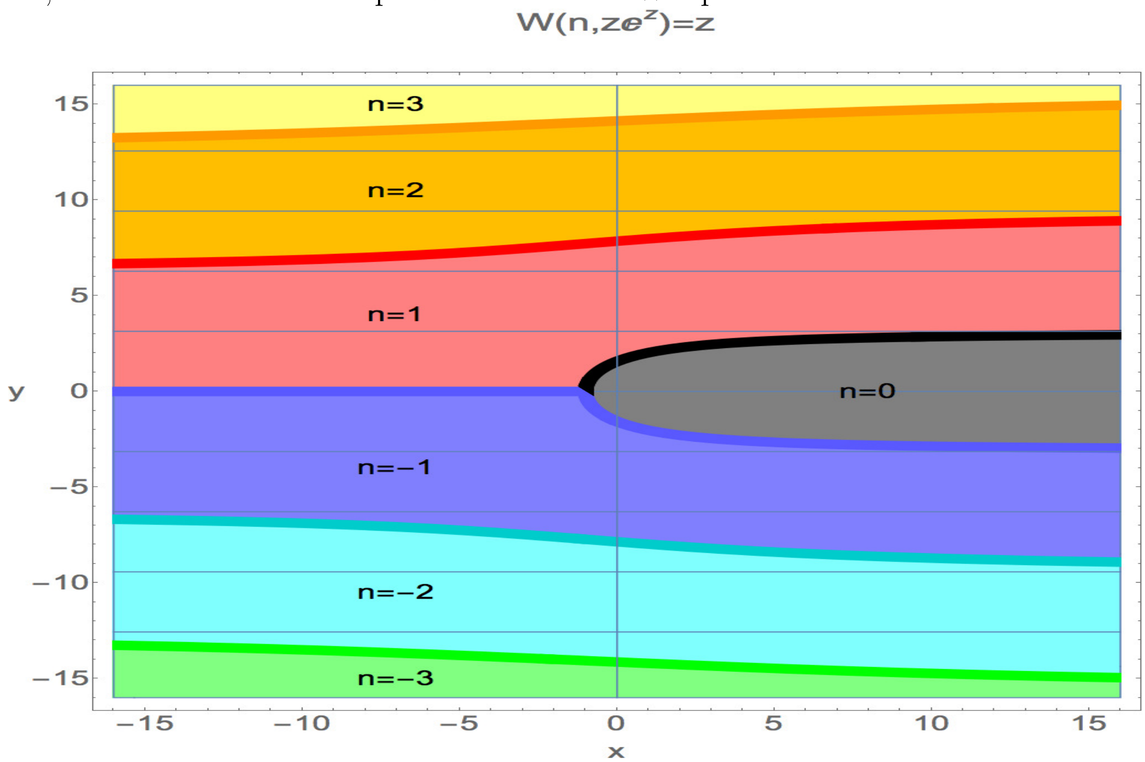
(1) Consider the equation $x \ln x = a$. Setting $x = e^t$ gives $te^t = a$ or $t = W(a)$, that is, $x = e^{W(a)} = W(a)e^{W(a)}/W(a) = a/W(a)$, $a \ge -1/e$. If $-1/e < a \le 0$, then there are two solutions: $a/W_k(a)$, $k = 0, -1$.

(2) Consider the sequence of functions $h_n = h_n(x)$, $x > 0$, $n = 1, 2, \ldots$ defined by $h_1(x) = x$, $h_{n+1}(x) = x^{h_n(x)}$. The limit $h(x) = \lim_{n\to\infty} h_n(x)$, if exists, corresponds to the infinite *power tower* $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. Writing $h(x) = x^{h(x)} = e^{h(x)\ln x}$ or $-\big(h(x)\ln x\big)\, e^{-h(x)\ln x} = -\ln x$, we conclude that, *provided it exists*, the function h satisfies $h(x) = -W_0(-\ln x)/\ln x$. It can be shown that $h(x)$ exists if and only if $e^{-e} \le x \le e^{1/e}$.

[1]Sergey Lototsky, USC, updated on December 18, 2023

(3) The infinite power tower $g(x) = x^{(1/x)^{(1/x)^{\cdot^{\cdot^{\cdot}}}}}$ exists if and only if $x \geq e^{-1/e}$; for $e^{-1/e} \leq x \leq e^e$ we have $g(x) = \ln(x)/W_0(\ln x)$.

The Complex Variable Approach: The function $z \mapsto ze^z$ is analytic in the whole complex plane and has an essential singularity at $z = \infty$. By a theorem of Picard, this function takes every complex value infinitely many times, meaning that the inverse function W has infinitely many *single-valued* branches $W_k(z)$, with alternative notation $W(k, z)$, $k = 0, \pm 1, \pm 2, \ldots$; the `principal branch` is W_0, and, when there is no danger of confusion, W can mean W_0. As a result, $W_k(z)e^{W_k(z)} = z$ for all integer k and all complex z, whereas $W_k(ze^z) = z$ for z in a suitable region of the complex plain, as shown below in the picture lifted from Wikipedia[2].



The paper

Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E. (1996). "On the Lambert W function". Adv. Computational Maths. 5, 329–359

is the standard reference on the subject of the Lambert W function; in particular, it includes some ideas about the origins of the above picture.

Further comments.

- Equality (1) is an illustration of the `Lagrange` (or Bürman-Lagrange) formula for inverting a power series.
- The Lambert W function can go under alternative names, such as `product logarithm` or `omega function`. In particular, $W_0(1) = 0.567143\ldots$ is called the `omega constant`.
- The first software implementation and the name `Lambert W` go back to 1980-s MAPLE®. Currently, we have `lambertw`(k, z)in MATLAB® and `ProductLog`$[k, z]$ in MATHEMATICA®.
- The term `Lambert's series` refers to something very different; a particular case is $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$.

[2] https://en.wikipedia.org/wiki/Lambert_W_function