Fall 2023, MATH 407, Final Exam

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Instructions:

- No books, notes, or help from other people; think twice (or more) before using a calculator.
- Turn off cell phones.
- Show your work/explain your answers.
- You have 120 minutes to complete the exam.
- There are 10 problems; 10 points per problem.
- Upload the solutions to GradeScope.

Standard normal pdf: $(2\pi)^{-1/2}e^{-x^2/2}$; Gamma(a,b) pdf: $b^a\big(\Gamma(a)\big)^{-1}x^{a-1}e^{-bx}$; Exponential with mean θ is Gamma$(1, 1/\theta)$; Beta(a,b) pdf: $\big(B(a,b)\big)^{-1}x^{a-1}(1-x)^{b-1}$; Poisson, mean μ, pmf: $e^{-\mu}k^{\mu}/k!$.

Problem 1. A box contains 5 white balls, 10 blue balls, and 15 green balls [30 total, well mixed]. Eleven (11) balls are taken out of the box, all at once.

Compute the probability that five (5) of those 11 balls are blue, and six (6) are green. *For your final answer, do not expand/evaluate any of the binomial coefficients.*

`Solution/Answer:` $\dfrac{\binom{10}{5}\binom{15}{6}}{\binom{30}{11}}$

Problem 2. A fair coin is tossed six times. Compute the probability that there were more heads than tails. *Simplify your answer to an ordinary fraction.*

`Solution/Answer:` $P(\mathcal{B}(6,1/2)=4,5,6) = 2^{-6}\left(\binom{6}{4}+\binom{6}{5}+\binom{6}{6}\right) = (15+6+1)/64 = 11/32$.

Problem 3. In a certain town, there are twice as many cars as trucks, and 15% of trucks and 5% of cars have manual transmission. A vehicle is selected at random, and it has manual transmission. Compute the probability that the vehicle is a truck. *Simplify your answer to an ordinary fraction.*

`Solution/Answer: 0.15*(1/3) / (0.15*1/3 + 0.05*(2/3)) = 15/25=0.6.`

Problem 4. A charitable lottery has 10,000 tickets, of which 400 win prizes and the rest win nothing. You buy 50 tickets. Compute the number of the prize-winning tickets you expect to find. *Simplify your answer to an integer.*

`Solution/Answer:` $(400/10000)\cdot 50 = 2$

Problem 5. Let X be a standard normal random variable. Define the random variable Y by $Y=|X|$. Compute the probability density function of the random variable Y.

`Solution/Answer: with` Φ `denoting the standard normal cdf,` $P(Y\le y) = 2P(0<X<y) = 2\Phi(y)-1$`, so the pdf of` Y `is` $\sqrt{2/\pi}e^{-y^2/2}$ `(twice the normal pdf) for` $y>0$ `and zero for` $y<0$.

Problem 6.

$$f_{X,Y}(x,y) = \begin{cases} Cx^2, & \text{if } x^2+y^2 \le 1,\ x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Justify your answer.

No: you can ~~also~~ see that $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$, and also $X^2+Y^2 \le 1$.

(b) Compute $E(Y|X)$.

Given X, Y is uniform over a symmetric range (no dependence on y in conditional distribution), so the answer is 0

Problem 7. A fair die is rolled until the total sum of all rolls exceeds 360. Compute approximately the probability that at most 100 rolls are necessary. Note that, for a single roll of the die, the expected value and variance of the outcome are $7/2 = 3.5$ and $35/12 \approx (1.7)^2$, respectively. Use the continuity correction. Leave the answer in the form $P(Z < r)$ or $P(Z > r)$ (whichever applies), where Z is the standard normal random variable and r is a suitable real number expressed as an ordinary fraction.

With X_k indicating the value of roll number k and $S_n = X_1 + \ldots + X_n$, the goal is to approximate $P(S_{100} \ge 361) = P(S_{100} > 360.5)$. By CLT, $P(S_{100} > 360.5) \approx P(Z > (360.5 - 350)/17) = P(Z > 21/34)$

Problem 8. For a randomly selected group of 80 people, compute the expected number of distinct birthdays (that is, the expected number of the days of the year that are a birthday of at least one person in the group). Assume 365 days in a year.

The answer is $365(1-(364/365)^{80})$ by indicator method, because $(364/365)^{80} = \left(1 - \frac{1}{365}\right)^{80}$ is the probability that a particular day is NOT a birthday of any of the 80 people.

Problem 9. Customers arrive at a bank according to a Poisson process. Suppose that four customers arrive during the first hour. Compute the probability that at least one arrived during the last 20 minutes. *Simplify your answer to an ordinary fraction.*

We need the probability p that, out of four iid unform random variables U_1, U_2, U_3, U_4 on $(0,1)$, at least one is bigger than or equal to $2/3$. Go by the complement: $p = 1 - P(\text{all less than } 2/3) = 1 - (P(U_1 < 2/3))^4 = 1 - (2/3)^4 = 1 - (16/81) = 65/81$.

Problem 10. Let X and Y be iid standard normal random variables. Determine the values of the number a so that the random variables $4X - aY$ and $X + aY$ are independent.

For Gaussian vector (automatic when start with iid-s), zero correlation implies independence, so need $E(4X - aY)(X + aY) = 0$, or, because $E(XY) = 0$ and $EX^2 = EY^2 = 1$, $4 - a^2 = 0$ or $a = \pm 2$