Fall 2022, MATH 407, Final Exam

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Instructions:

- No books, notes, or help from other people; think twice (or more) before using a calculator.
- Turn off cell phones.
- Show your work/explain your answers.
- You have 120 minutes to complete the exam.
- There are 10 problems; 10 points per problem.
- Upload the solutions to GradeScope.

Standard normal pdf: $(2\pi)^{-1/2}e^{-x^2/2}$; Gamma(a,b) pdf: $b^a\big(\Gamma(a)\big)^{-1}x^{a-1}e^{-bx}$; Exponential with mean θ is Gamma$(1, 1/\theta)$; Beta(a,b) pdf: $\big(B(a,b)\big)^{-1}x^{a-1}(1-x)^{b-1}$; Poisson, mean μ, pmf: $e^{-\mu}k^{\mu}/k!$.

Problem 1. A box contains 5 white balls, 10 blue balls, 15 green balls and 20 red balls, [50 total, well mixed]. Fourteen (14) balls are taken out of the box, all at once.

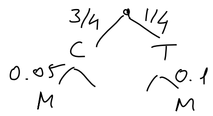
Compute the probability that no green balls were taken out of the box. `For your final answer, do not expand/evaluate any of the binomial coefficients.` $\binom{35}{14}/\binom{50}{14}$

Problem 2. Consider two events A and B such that $P(A) = 0.5, P(B) = 0.6$.

(a) Explain why the events cannot be mutually exclusive. $\mathbb{P}(A)+\mathbb{P}(B) > 1$

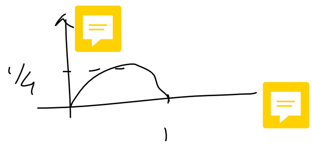
(b) Suppose that the events are independent. Compute $P(A \bigcap B^c)$, where B^c denotes the complement of the event B. In other words, you need the probability that A happens and B does not happen. $A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp B^c \Rightarrow \mathbb{P}(A \cap B^c) = \mathbb{P}(A)(1-\mathbb{P}(B)) = 0.5 \cdot 0.4 = 0.2$

Problem 3. In a certain town, there are three times as many cars as trucks, and 10% of trucks and 5% of cars have manual transmission. A vehicle is selected at random, and it has manual transmission. Compute the probability that the vehicle is a truck. `Simplify your answer as much as possible.` $\mathbb{P}(T|M) = \dfrac{0.1 \cdot 1/4}{0.1 \cdot 1/4 + 0.05 \cdot 3/4} = \dfrac{1}{2.5} = 0.4$



Problem 4. Consider the function

$$f(x) = \begin{cases} 0 & x \le 0 \\ C(x - x^2) & 0 < x < 1 \\ 0 & x \ge 1, \end{cases}$$



where $C > 0$.

(a) Could f be a cumulative distribution function? If yes, determine the value of C; if not, explain why. No: not monotone

(b) Could f be a probability density function? If yes, determine the value of C; if not, explain why.

`Make sure to draw the graph of` f. Yes: $C\int_0^1 (x-x^2)\,dx = 1 \Rightarrow C\left(\frac{1}{2}-\frac{1}{3}\right) = 1 \Rightarrow C = 6$

Problem 5. Let X be a standard exponential random variable. In particular, $X \ge 0$, the pdf of X is e^{-x}, $x \ge 0$, and $\mathbb{P}(X > x) = e^{-x}$, $x \ge 0$. Define the random variable Y by $Y = e^X$. Compute the probability density function of the random variable Y. `Make sure to indicate the range of possible values for` Y. $X \geq 0 \Rightarrow e^X \geq 1$.

$\mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln y) = 1 - e^{-\ln y} = 1 - \frac{1}{y}, \ y \geq 1$

$\Rightarrow f_Y(y) = \begin{cases} 1/y^2, & y \geq 1 \\ 0, & y < 1. \end{cases}$

Problem 6. 10 balls are dropped at random into five boxes so that the balls are dropped independently of one another and each ball is equally likely to land in any of the boxes. Denote by X the number of empty boxes. Compute the expected number $E(X)$ of empty boxes.

$$\mathbb{P}(\text{empty}) = \left(\frac{4}{5}\right)^{10} \Rightarrow \mathbb{E}(X) = \frac{4^{10}}{5^9}$$

Problem 7. A fair die is rolled 100 times. Compute, approximately, the probability that the total sum S of the rolls is exactly 350. For a single roll, the expected value is 3.5 and the standard deviation is (approximately) 1.7. You can leave the answer in the form $P(|Z| < r)$, where Z is standard normal random variable and r is a suitable number.

$$\mathbb{P}(S = 350) = \mathbb{P}(349.5 < S < 350.5) = \mathbb{P}\left(\left|\frac{S-350}{17}\right| < \frac{0.5}{17}\right) \overset{\text{CLT}}{\approx} \mathbb{P}\left(|Z| < \frac{1}{34}\right).$$

$$\approx \mathbb{P}(|Z| < 0.03) \approx 0.024.$$

Problem 8. Let U and V be independent random variables, such that U is uniform on $(0, 1)$ and V is uniform on $(0, 2\pi)$. Define the random variables X and Y by $X = \sqrt{U}\cos(V)$, $Y = \sqrt{U}\sin(V)$. Compute the joint pdf of the random vector (X, Y).

$x = \sqrt{u}\cos v$, $y = \sqrt{u}\sin v$. $f_{U,V}(u,v) = \frac{1}{2\pi}\mathbb{1}\left(\begin{smallmatrix}0<u<1\\0<v<2\pi\end{smallmatrix}\right)$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\left|\begin{matrix}\frac{1}{2\sqrt{u}}\cos v & -\sqrt{u}\sin v\\ \frac{1}{2\sqrt{u}}\sin v & \sqrt{u}\cos v\end{matrix}\right|\right| = \frac{1}{2} \Rightarrow f_{X,Y}(x,y) = \frac{1}{\pi} I(x^2+y^2 \leq 1)$$

$x^2+y^2 \leq 1$ (uniform in unit disk)

Problem 9. Let X and Y be iid standard normal random variables. Determine the value of the number a so that the random variables $X + 2Y$ and $aX - Y$ are independent.

need $\mathbb{E}(X+2Y)(aX-Y) = 0 \Rightarrow a = 2$

Problem 10. Customers arrive at a bank at a Poisson rate λ. Suppose that five customers arrived during the first hour. Compute the probability that at all customers arrived during the last 15 minutes.

$$\underbrace{\mathbb{P}(\text{all in } [\text{[illegible]}])}_{U_{(1)} > 3/4} = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$$