INTRUSIVE AND NON-INTRUSIVE CHAOS APPROXIMATION FOR A TWO-DIMENSIONAL STEADY STATE NAVIER-STOKES SYSTEM WITH RANDOM FORCING

S. V. LOTOTSKY, R. MIKULEVICIUS, AND B. L. ROZOVSKY

Abstract. While convergence of a chaos approximation for linear equations is relatively well understood, a lot less is known for non-linear equations. The paper investigates this convergence, by establishing the corresponding a priori error bounds, for a particular equation with quadratic nonlinearity and for two different approximations: stochastic Galerkin and discrete projection. Stochastic Galerkin approximation reduces the stochastic equation to a system of deterministic equation to compute the coefficients in the chaos expansion. The approximation is called intrusive because the resulting system of equations is highly coupled and is harder to solve than the original system; there is also a special condition for uniqueness of solution. An alternative approximation of the chaos coefficients, using the discrete projection version of the stochastic collocation method, is non-intrusive and requires the solution of the original equation for specially chosen realizations of the random input. Compared to the Galerkin approximation, this non-intrusive procedure is easier to analyze and implement, but the resulting approximation error and computational costs can be higher.

December 15, 2021

1. Introduction

There are two main ways to study an equation with random input. One way is to use deterministic tools for each particular realization of randomness; in what follows, we call it `path-wise approach`. An alternative, which we call `random field approach`, is to consider the random input as an additional independent variable in the equation, along with space and/or time.

Questions such as existence/uniqueness/regularity of the solution are often addressed by combining the two approaches; cf. [18, 19] for ordinary differential equations and [17, 24, 25] for equations with partial derivatives.

The difference between the two approaches becomes noticeable in numerical computations; see, for example, [13, 35]. Path-wise approach leads to repeated numerical solutions of the underlying equation for various realizations of the random input; a

2020 *Mathematics Subject Classification.* 35Q30, 35R60, 60H35, 65N15, 65N30, 65N35, 76D05, 76M22.

Key words and phrases. Gauss Quadrature, Generalized Polynomial Chaos, Stochastic Galerkin Approximation.

SVL: Research supported by ARO Grant W911N-16-1-0103 and AFSOR Grant FA9550-21-1-0015.

RM and BLR: Research supported by ARO Grant W911N-16-1-0103.

typical example is Monte Carlo simulations. In computational terms, this approach is `non-intrusive`, because no new numerical procedures are required to solve the equation compared to the deterministic case.

Numerical procedures based on the random field approach lead to a stochastic Galerkin approximation [35, Chapter 6]: starting with a chaos expansion of the solution, using a basis in the underlying space of random variables, the problem is reduced to a system of deterministic equations. In computational terms, such procedures are `intrusive`, because the resulting system is more complicated than the original equation and requires different algorithms to obtain a solution.

The stochastic collocation method [35, Chapter 7], with sampling at specially selected realizations of the random input, somewhat bridges the gap between pure random sampling (Monte Carlo) and complete elimination of randomness (stochastic Galerkin approximation). In computational terms, the method is non-intrusive [3, 31]. We will consider the *discrete projection*, or *pseudo-spectral* version of the method, when the sampled solution is used to approximate the coefficients in the chaos expansion via Gauss quadrature.

For many, although apparently not all [8], equations, various empirical studies [15, 28, 32, etc.] suggest that stochastic Galerkin approximation, with a fixed computational cost, can be a much more efficient way to study statistical properties of the solution than Monte Carlo or stochastic collocation methods. In the case of nonlinear equations, this experimental success has yet to be fully justified theoretically; for linear equations, the picture is rather clear; see, for example, [9, 20, 21] as well as [22, Chapter 5] and [30, Section 8.3].

The objective of this paper is to carry out a comparative theoretical analysis of an intrusive and a non-intrusive approximations for a particular nonlinear equation. Specifically, we consider the stationary Navier-Stokes system in a smooth bounded planar domain with zero boundary conditions and with randomness in the external force, and we establish *a priori* error bounds for both approximations.

The paper is organized as follows. Section 2 describes the model and introduces the necessary function spaces. Sections 3 and 4 investigate stochastic Galerkin approximation and a non-intrusive pseudo-spectral approximation, respectively. Section 5 puts the results in a broader context.

Throughout the paper, G is a bounded domain in $\mathbb{R}^2$ with area $|G|$ and sufficiently regular (e.g. locally Lipschtiz) boundary ∂G. We use the following convention with the notations of various function spaces and their elements: if X denotes a space of scalar fields f on G, then $\mathbf{X}$ denotes the corresponding space of vector fields $\mathbf{f}$, and $\mathbb{X}$ denotes the collection of $\mathbf{X}$-valued random elements $\boldsymbol{f}$.

2. The Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that the $L^2(\Omega)$ is a separable Hilbert space having a complete orthogonal basis $\{\mathfrak{P}_n,\ n \geq 0\}$: with

$$c(n) = \mathbb{E}\mathfrak{P}_n^2,$$

every $\zeta \in L^2(\Omega)$ has a `chaos expansion`

$$\zeta = \sum_{k\geq 0} \frac{\mathbb{E}\big(\zeta \mathfrak{P}_k\big)}{c(k)} \mathfrak{P}_k. \tag{2.1}$$

Typically, the sigma-algebra $\mathcal{F}$ is generated by a collection Ξ of random variables, and each $\mathfrak{P}_k$ is a function of a finite number of random variables from Ξ. In the particular case, when all $\mathfrak{P}_k$ are polynomials, equality (2.1) is called a `generalized polynomial chaos` expansion of ζ.

We assume that the basis $\{\mathfrak{P}_n,\ n \geq 0\}$ has the following property: for every $m, n \geq 0$, there are *finitely many* real numbers $A_{m,n;l},\ l \geq 0$, such that

$$\mathfrak{P}_m \mathfrak{P}_n = \sum_{l\geq 0} A_{m,n;l} \mathfrak{P}_l; \tag{2.2}$$

in that case

$$A_{m,n;l} = \frac{\mathbb{E}\big(\mathfrak{P}_m \mathfrak{P}_n \mathfrak{P}_l\big)}{c(l)}.$$

Property (2.2) holds for generalized polynomial chaos.

Denote by $\mathcal{P}^N$ the orthogonal projection in $L^2(\Omega)$ on the subspace spanned by $\{\mathfrak{P}_k,\ k = 0, \ldots, N\}$.

Consider a steady-state Navier-Stokes system with random forcing in a bounded domain $G \subset \mathbb{R}^2$ with sufficiently regular boundary ∂G:

$$\begin{aligned} \nu \Delta \boldsymbol{u}(x) &= \big(\boldsymbol{u} \cdot \nabla\big)\boldsymbol{u} + \nabla p(x) + \boldsymbol{f}(x),\ x \in G, \\ \operatorname{div} \boldsymbol{u}(x) &= 0,\ x \in G,\ \boldsymbol{u}|_{\partial G} = 0. \end{aligned} \tag{2.3}$$

In equation (2.3),

- $\nu > 0$ is the kinematic viscosity coefficient, $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator, and ν is constant;
- $\boldsymbol{u}(x) = (u^1(x), u^2(x))$ is the (unknown) velocity and
$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \frac{\partial u^1}{\partial x_1} + \frac{\partial u^2}{\partial x_2},\quad \big(\boldsymbol{u} \cdot \nabla\big) u^i = u^1 \frac{\partial u^i}{\partial x_1} + u^2 \frac{\partial u^i}{\partial x_2},\ i = 1, 2;$$
- $p = p(x)$ is the (unknown scalar) pressure and $\big(\nabla p\big)^i = \frac{\partial p}{\partial x_i},\ i = 1, 2$;
- $\boldsymbol{f}$ is the random forcing.

Two standard references for the deterministic counterpart of (2.3) are [11, Chapter IX] and [33, Chapter II].

For the sake of simplicity, the only source of randomness in (2.3) is $\boldsymbol{f}$; analysis of equation (2.3) with random ν is similar and is briefly discussed in Section 5.

We will use the following function spaces:

- $\mathcal{C}_0^\infty(G)$, the collection of infinitely differentiable real-valued functions on G with compact support in G;
- $\mathcal{D}(G) = \{\boldsymbol{\varphi} = (\varphi^1, \varphi^2),\ \varphi^i \in \mathcal{C}_0^\infty(G),\ i = 1, 2 : \operatorname{div} \boldsymbol{\varphi} = 0\}$;

- $L^r(G)$, $1 \le r < +\infty$, the collection of measurable functions g on G such that
$$|g|_{L^r} = \left(\int_G |g(x)|^r \, dx \right)^{1/r} < \infty;$$
for $g, f \in L^2(G)$, we write
$$(f,g)_0 = \int_G f(x) g(x) \, dx;$$
- $\mathbf{L}^r(G)$, the collection of vector fields $\mathbf{g} = (g^1, g^2)$ on G such that $g^1, g^2 \in L^r(G)$, and endowed with norm
$$|\mathbf{g}|_{\mathbf{L}^r} = \left(\left|g^1\right|_{L^r}^r + \left|g^2\right|_{L^r}^r \right)^{1/r};$$
for $\mathbf{g}, \mathbf{f} \in \mathbf{L}^2(G)$, we write
$$(\mathbf{f}, \mathbf{g})_0 = \int_G \Big(f^1(x) g^1(x) + f^2(x) g^2(x) \Big) \, dx;$$
- $\mathbb{L}^2(G) = L^2\big(\Omega; \mathbb{L}^2(G)\big)$, that is, the collection of $\mathbf{L}^2(G)$-valued random elements
$$\boldsymbol{g}(\omega, x) = \big(g^1(\omega, x), g^2(\omega, x) \big)$$
such that
$$|\boldsymbol{g}|_{\mathbb{L}^2} = \big(\mathbb{E} \, |\boldsymbol{g}|_{\mathbf{L}^2}^2 \big)^{1/2} < \infty;$$
- $H_0^{1,2}(G)$, the completion of $\mathcal{C}_0^\infty(G)$ with respect to the norm
$$|g|_{1,2} = \left(\int_G |\nabla g(x)|^2 \, dx \right)^{1/2} = \left(\int_G \left(\left| \frac{\partial g(x)}{\partial x_1} \right|^2 + \left| \frac{\partial g(x)}{\partial x_2} \right|^2 \right) dx \right)^{1/2};$$
note that $|\cdot|_{1,2}$ is indeed a norm on $\mathcal{C}_0^\infty(G)$ because, by a version of the Poincaré inequality, if $g \in \mathcal{C}_0^\infty(G)$ and $|G|$ is the Lebesgue measure (area) of G, then (cf. [11, Exercise II.5.4])
$$|g|_{L^2}^2 \le \frac{|G|}{2} \, |g|_{1,2}^2; \tag{2.4}$$
- $\mathbf{H}_0^{1,2}(G)$, the collection of vector fields $\mathbf{g} = (g^1, g^2)$ on G such that $g^1, g^2 \in H_0^{1,2}(G)$, and endowed with norm
$$|\mathbf{g}|_{1,2} = \left(\left|g^1\right|_{1,2}^2 + \left|g^2\right|_{1,2}^2 \right)^{1/2};$$
for $\mathbf{f}, \mathbf{g} \in \mathbf{H}_0^{1,2}(G)$, we write
$$\big(\nabla \mathbf{f}, \nabla \mathbf{g} \big)_0 = \sum_{i,j=1}^2 \int_G \left(\frac{\partial f^i(x)}{\partial x_j} \, \frac{\partial g^i(x)}{\partial x_j} \right) dx,$$
so that
$$|\mathbf{g}|_{1,2}^2 = \big(\nabla \mathbf{g}, \nabla \mathbf{g} \big)_0;$$
- $\mathbb{H}_0^{1,2}(G) = L^2\big(\Omega; \mathbf{H}_0^{1,2}(G)\big)$;
- $\widehat{\mathbf{H}}_0^{1,2}(G)$, the completion of $\mathcal{D}(G)$ with respect to the norm $|\cdot|_{1,2}$;
- $\widehat{\mathbb{H}}_0^{1,2}(G) = L^2\big(\Omega; \widehat{\mathbf{H}}_0^{1,2}(G)\big)$;

- $H_0^{-1,2}(G)$, the completion of $L^2(G)$ with respect to the norm
$$|g|_{-1,2} = \sup\left\{\int_G g(x)\varphi(x)\,dx :\ \varphi \in H_0^{1,2}(G),\ |\varphi|_{1,2} \le 1,\right\};$$
- $\mathbf{H}_0^{-1,2}(G)$, the collection of vector fields $\mathbf{g} = (g^1, g^2)$ such that $g^1, g^2 \in H_0^{-1,2}(G)$, and endowed with norm
$$|\mathbf{g}|_{-1,2} = \left(\left|g^1\right|_{-1,2}^2 + \left|g^2\right|_{1,2}^2\right)^{1/2};$$
- $\mathbb{H}_0^{-1,2}(G) = L^2\big(\Omega; \mathbf{H}_0^{-1,2}(G)\big)$.

The (Banach space) dual of $H_0^{1,2}(G)$ is isomorphic to $H_0^{-1,2}(G)$: see [11, Theorem II.3.5]. We denote the corresponding duality by $\langle f, g\rangle_1$, $f \in H_0^{-1,2}(g), g \in H_0^{1,2}(G)$. Similarly, the dual of $\mathbf{H}_0^{1,2}(G)$ is isomorphic to $\mathbf{H}_0^{-1,2}(G)$ and the duality is denoted by $\langle \mathbf{f}, \mathbf{g}\rangle_1$, $\mathbf{f} \in \mathbf{H}_0^{-1,2}(G)$, $\mathbf{g} \in \mathbf{H}_0^{1,2}(G)$.

For $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{H}_0^{1,2}(G) \times \mathbf{H}_0^{1,2}(G) \times \mathbf{H}_0^{1,2}(G)$, we define the tri-linear form

$$\mathfrak{a}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \big((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}\big)_0, \quad \text{where } \big(\mathbf{u}\cdot\nabla\big)v^i = u^1\frac{\partial v^i}{\partial x_1} + u^2\frac{\partial v^i}{\partial x_2},\ i = 1, 2. \tag{2.5}$$

Lemma 2.1. *The trilinear form $\mathfrak{a}$ has the following properties:*

(1) *If $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{H}_0^{1,2}(G) \times \mathbf{H}_0^{1,2}(G) \times \mathbf{H}_0^{1,2}(G)$, then*
$$|\mathfrak{a}(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le \frac{\sqrt{|G|}}{2}\,|\mathbf{u}|_{1,2}\,|\mathbf{v}|_{1,2}\,|\mathbf{w}|_{1,2}, \tag{2.6}$$
$$|\mathfrak{a}(\mathbf{u}, \mathbf{u}, \mathbf{w}) - \mathfrak{a}(\mathbf{v}, \mathbf{v}, \mathbf{w})| \le \frac{\sqrt{|G|}}{2}\,|\mathbf{u}-\mathbf{v}|_{1,2}\big(|\mathbf{u}|_{1,2} + |\mathbf{v}|_{1,2}\big)\,|\mathbf{w}|_{1,2};$$

(2) *If $\mathbf{u} \in \widehat{\mathbf{H}}_0^{1,2}(G)$, then*
$$\mathfrak{a}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \tag{2.7}$$
and
$$\mathfrak{a}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\mathfrak{a}(\mathbf{u}, \mathbf{w}, \mathbf{v}).$$

For the proofs, see [11, Lemma IX.1.1] and [11, Lemma IX.2.1], respectively. Note that (2.6) follows from the Hölder inequality

$$|\mathfrak{a}(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le |\mathbf{u}|_{\mathbf{L}^q}\,|\mathbf{v}|_{1,2}\,|\mathbf{w}|_{\mathbf{L}^r}, \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \tag{2.8}$$

by taking $q = r = 4$ and using a suitable embedding theorem; for other versions of (2.6) and (2.8), see, for example, [11, Exercise IX.2.1].

In particular, with $\mathbf{w} = \mathbf{u} - \mathbf{v}$,

$$\mathfrak{a}(\mathbf{u}, \mathbf{u}, \mathbf{w}) - \mathfrak{a}(\mathbf{v}, \mathbf{v}, \mathbf{w}) = \mathfrak{a}(\mathbf{w}, \mathbf{u}, \mathbf{w}) + \mathfrak{a}(\mathbf{v}, \mathbf{w}, \mathbf{w}),$$

so that, if $\mathbf{v} \in \widehat{\mathbf{H}}_0^{1,2}(G)$, then (2.6) and (2.7) imply

$$|\mathfrak{a}(\mathbf{u}, \mathbf{u}, \mathbf{u}-\mathbf{v}) - \mathfrak{a}(\mathbf{v}, \mathbf{v}, \mathbf{u}-\mathbf{v})| \le \frac{\sqrt{|G|}}{2}\,|\mathbf{u}-\mathbf{v}|_{1,2}^2\,|\mathbf{u}|_{1,2}. \tag{2.9}$$

Similar to [11, Definition IX.1.1], we have

Definition 2.2. *Let $\boldsymbol{f} \in \mathbb{H}_0^{-1,2}(G)$. A random vector field $\boldsymbol{u} \in \widehat{\mathbb{H}}_0^{1,2}(G)$ is called a solution to* (2.3) *if, for every $\boldsymbol{\varphi} \in \mathcal{D}(G)$,*

$$\mathbb{P}\Big\{ \nu \left(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}\right)_0 + \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) = -\left\langle \boldsymbol{f}, \boldsymbol{\varphi} \right\rangle_1 \Big\} = 1, \tag{2.10}$$

where $\mathfrak{a}$ is the tri-linear form (2.5).

By applying [11, Lemma IX.1.2] to (2.3) with a particular realization of $\boldsymbol{f}$, we get the following result.

Lemma 2.3. *If $\boldsymbol{f} \in \mathbb{H}_0^{-1,2}(G)$, $\boldsymbol{u} \in \widehat{\mathbb{H}}_0^{1,2}(G)$, and* (2.10) *holds, then there exists a $p \in \mathbb{L}^2(G)$ with $\mathbb{P}\left(\int_G p(x)\,dx = 0\right) = 1$ such that, for every $\boldsymbol{\varphi} \in \mathbf{H}_0^{1,2}(G)$,*

$$\mathbb{P}\Big\{ \nu \left(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}\right)_0 + \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) = (p, \operatorname{div} \boldsymbol{\varphi})_0 - \left\langle \boldsymbol{f}, \boldsymbol{\varphi} \right\rangle_1 \Big\} = 1.$$

Similarly, applying [11, Theorems IX.2.1 and IX.3.2] to (2.3), we get the basic existence and uniqueness result.

Theorem 2.4. *If $\boldsymbol{f} \in \mathbb{H}_0^{-1,2}(G)$, then, with probability one, equation* (2.3) *has a solution and*

$$\mathbb{P}\big(\nu |\boldsymbol{u}|_{1,2} \le |\boldsymbol{f}|_{-1,2}\big) = 1.$$

If, in addition, there exists a non-random $\theta \in (0,1)$ such that

$$\mathbb{P}\left(|\boldsymbol{f}|_{-1,2} \le \frac{2\theta\nu^2}{\sqrt{|G|}} \right) = 1, \tag{2.11}$$

then the solution is unique and satisfies

$$\mathbb{P}\left(|\boldsymbol{u}|_{1,2} \le \frac{2\nu\theta}{\sqrt{|G|}} \right) = 1. \tag{2.12}$$

Intuitively, once we know the velocity field $\boldsymbol{u}$, we should be able to recover pressure p from the original equation (2.3). Lemma 2.3 confirms this intuition; see also [33, Proposition I.1.1]. As a result, in what follows, we only consider the function $\boldsymbol{u}$.

Sometimes it is convenient to work with alternative characterizations of the solution of (2.3).

Proposition 2.5. *Let $\mathbf{f} \in \mathbb{H}_0^{-1,2}(G)$, $\boldsymbol{u} \in \widehat{\mathbb{H}}_0^{1,2}(G)$, and let $\mathfrak{a}$ be the tri-linear form* (2.5). *Then $\boldsymbol{u}$ is a solution to* (2.3) *if and only of, for every $\boldsymbol{w} \in \widehat{\mathbb{H}}_0^{1,2}(G)$,*

$$\mathbb{P}\Big\{ \nu \left(\nabla \boldsymbol{u}, \nabla \boldsymbol{w}\right)_0 + \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = -\left\langle \boldsymbol{f}, \boldsymbol{w} \right\rangle_1 \Big\} = 1, \tag{2.13}$$

or

$$\nu\, \mathbb{E}\big(\nabla \boldsymbol{u}, \nabla \boldsymbol{w}\big)_0 + \mathbb{E}\mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = -\mathbb{E}\left\langle \boldsymbol{f}, \boldsymbol{w} \right\rangle_1. \tag{2.14}$$

Proof. By construction,

$$(2.10) \;\Rightarrow\; (2.13) \;\Rightarrow\; (2.14).$$

To establish (2.14) $\Rightarrow$ (2.10), take $\boldsymbol{w} = \boldsymbol{\varphi}\,\zeta$ with $\boldsymbol{\varphi} \in \mathcal{D}(G)$ and a bounded random variable ζ. □

Corollary 2.6. *Let $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{H}_0^{-1,2}(G)$ and let $\boldsymbol{u}, \boldsymbol{v} \in \widehat{\mathbb{H}}_0^{1,2}(G)$ be the corresponding solutions of* (2.3). *If* (2.11) *holds, then*

$$\mathbb{P}\left(|\boldsymbol{u} - \boldsymbol{v}|_{1,2} \leq \frac{|\boldsymbol{f} - \boldsymbol{g}|_{-1,2}}{\nu(1-\theta)}\right) = 1. \tag{2.15}$$

Proof. By (2.13), we have, with probability one,

$$\nu\left(\nabla(\boldsymbol{u} - \boldsymbol{v}), \nabla \boldsymbol{w}\right)_0 + \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - \mathfrak{a}(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) = -\left\langle \boldsymbol{f} - \boldsymbol{g}, \boldsymbol{w}\right\rangle_1. \tag{2.16}$$

Taking $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and using (2.9), we re-write (2.16) as

$$\nu|\boldsymbol{u} - \boldsymbol{v}|_{1,2}^2 - \frac{\sqrt{|G|}}{2}\,|\boldsymbol{u} - \boldsymbol{v}|_{1,2}^2\,|\boldsymbol{u}|_{1,2} \leq \langle \boldsymbol{f} - \boldsymbol{g}, \boldsymbol{u} - \boldsymbol{v}\rangle_1 \leq |\boldsymbol{f} - \boldsymbol{g}|_{-1,2}\,|\boldsymbol{u} - \boldsymbol{v}|_{1,2},$$

and then (2.15) follows from (2.12).

□

3. Stochastic Galerkin Approximation

Recall that $\{\mathfrak{P}_k,\ k \geq 0\}$ is an orthogonal basis in $L^2(\Omega)$ and $\mathcal{P}^N$ is the orthogonal projection on the linear span of $\{\mathfrak{P}_k,\ 0 \leq k \leq N\}$.

For $N \geq 1$, consider the equation

$$\begin{aligned} \nu\Delta \boldsymbol{v}_N &= \mathcal{P}^N\Big((\boldsymbol{v}_N \cdot \nabla)\boldsymbol{v}_N\Big) + \nabla p_N + \mathcal{P}^N \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v}_N &= 0, \quad \boldsymbol{v}_N\big|_{\partial G} = 0. \end{aligned} \tag{3.1}$$

Similar to Definition 2.2, we have

Definition 3.1. *Given $\boldsymbol{f} \in \mathbb{H}_0^{-1,2}(G)$, a random vector field $\boldsymbol{v}_N \in \mathcal{P}^N\Big(\widehat{\mathbb{H}}_0^{1,2}(G)\Big)$ is called a solution of* (3.1), *if, for every $\boldsymbol{\varphi} \in \mathcal{D}(G)$,*

$$\mathbb{P}\left(\nu\left(\nabla \boldsymbol{v}_N, \nabla \boldsymbol{\varphi}\right)_0 + \mathcal{P}^N \mathfrak{a}(\boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{\varphi}) = -\left\langle \mathcal{P}^N \boldsymbol{f}, \boldsymbol{\varphi}\right\rangle_1\right) = 1.$$

We call $\boldsymbol{v}_N$ a `stochastic Galerkin approximation` of the solution $\boldsymbol{u}$ of equation (2.3).

Similar to (2.13) and (2.14), we will establish two alternative characterizations of the solution of (3.1).

If $\boldsymbol{u} \in \mathbb{H}_0^{1,2}(G)$, $\boldsymbol{v} \in \mathbb{H}_0^{1,2}(G)$, and $\boldsymbol{w} \in \mathcal{P}^N\Big(\mathbb{H}_0^{1,2}(G)\Big)$, then $\mathcal{P}^N \boldsymbol{w} = \boldsymbol{w}$ and therefore

$$\mathbb{E}\big(\mathcal{P}^N(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}\big) = \mathbb{E}\big((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \mathcal{P}^N \boldsymbol{w}\big) = \mathbb{E}\,\mathfrak{a}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}). \tag{3.2}$$

As a result, direct computations lead to the first alternative characterization of the solution of (3.1).

Proposition 3.2. *A random vector field* $\boldsymbol{v}_N \in \mathcal{P}^N\left(\widehat{\mathbb{H}}_0^{1,2}(G)\right)$ *is a solution of* (3.1) *if and only if, for every* $\boldsymbol{w} \in \mathcal{P}^N\left(\widehat{\mathbb{H}}_0^{1,2}(G)\right)$,

$$\nu\,\mathbb{E}\left(\nabla \boldsymbol{v}_N, \nabla \boldsymbol{w}\right) + \mathbb{E}\,\mathfrak{a}(\boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{w}) = -\mathbb{E}\left\langle \boldsymbol{f}, \boldsymbol{w}\right\rangle_1. \tag{3.3}$$

In particular, if a solution $\boldsymbol{v}_N$ exists, then, taking $\boldsymbol{w} = \boldsymbol{v}_N$ and using (2.7), we find

$$\nu|\boldsymbol{v}_N|_{\mathbb{H}_0^{1,2}} \le |\boldsymbol{f}|_{\mathbb{H}_0^{-1,2}}. \tag{3.4}$$

Equality (3.3) confirms that $\boldsymbol{v}_N$ is indeed a stochastic Galerkin approximation of $\boldsymbol{u}$.

To derive yet another form of (3.1), start by writing

$$\boldsymbol{v}_N = \sum_{l=0}^{N} \mathbf{v}_N^l \mathfrak{P}_l, \qquad \mathcal{P}^N \boldsymbol{f} = \sum_{l=0}^{N} \mathbf{f}^l \mathfrak{P}_l.$$

Then, using the numbers $A_{m,k;l}$ defined in (2.2), we compute

$$\begin{aligned}\left((\boldsymbol{v}_N \cdot \nabla)\boldsymbol{v}_N\right) &= \sum_{m,k=0}^{N} (\mathbf{v}_N^m \cdot \nabla)\mathbf{v}_N^k \mathfrak{P}_m \mathfrak{P}_k = \sum_{m,k=0}^{N} (\mathbf{v}_N^m \cdot \nabla)\mathbf{v}_N^k \sum_{l=0}^{m+k} A_{m,k;l}\mathfrak{P}_l \\ &= \sum_{l=0}^{2N}\left(\sum_{m,n=0}^{N} A_{m,k;l}\,(\mathbf{v}_N^m \cdot \nabla)\mathbf{v}_N^k\right)\mathfrak{P}_l,\end{aligned} \tag{3.5}$$

that is,

$$\mathcal{P}^N\left((\boldsymbol{v}_N \cdot \nabla)\boldsymbol{v}_N\right) = \sum_{l=0}^{N}\left(\sum_{m,n=0}^{N} A_{m,k;l}\,(\mathbf{v}_N^m \cdot \nabla)\mathbf{v}_N^k\right)\mathfrak{P}_l.$$

As a result, (3.1) is equivalent to the following system of equations for the non-random vector functions $\mathbf{v}_N^l$, $l = 0, \dots, N$:

$$\nu\Delta \mathbf{v}_N^l = \sum_{m,n=0}^{N} A_{m,k;l}\,(\mathbf{v}_N^k \cdot \nabla)\mathbf{v}_N^m + \nabla p_N^l + \mathbf{f}^l. \tag{3.6}$$

This system is more complicated than (2.3) and will require more sophisticated numerical procedures to compute a solution, whence the term "intrusive" in connection with stochastic Galerkin approximation.

For example, if ξ is a standard normal random variable and $\mathcal{F} = \sigma(\xi)$, $\boldsymbol{f}(x) = \mathbf{f}(\xi, x) = (f^1(\xi, x), f^2(\xi, x))$ for a non-random vector field $\mathbf{f}$, then $\mathfrak{P}_n = \mathrm{H}_n(\xi)$, where

$$\mathrm{H}_n(x) = (-1)^n e^{x^2/2}\frac{d^n e^{-x^2/2}}{dx^n}$$

is n-th Hermite polynomial, $c(n) = n!$, and

$$\mathfrak{P}_m\mathfrak{P}_n = \sum_{k=0}^{\min(m,n)} \frac{m!\,n!}{(m-k)!\,(n-k)!\,k!}\,\mathfrak{P}_{m+n-2k}$$

(cf. [35, Formula (6.7)]); after some algebraic manipulations, (3.6) becomes

$$\nu\Delta\mathbf{v}_N^l = \sum_{n=0}^{N}\frac{1}{n!}\sum_{\substack{k+m=l,\\ k+n\le N, m+n\le N}}\frac{(k+n)!}{k!}\frac{(m+n)!}{m!}\,(\mathbf{v}_N^{k+n}\cdot\nabla)\mathbf{v}_N^{m+n} + \nabla p_N^l + \mathbf{f}^l.$$

Combining (3.6) with Proposition 3.2, we get the second alternative characterization of the solution of (3.1).

Proposition 3.3. *A collection of functions $\mathbf{v}_N^l$, $l=0,\dots,N$, with each $\mathbf{v}_N^l \in \widehat{\mathbf{H}}_0^{1,2}(G)$, is a solution of (3.6) if and only if, for every collection of functions $\{\mathbf{w}^l,\ l=0,\dots,N\}$, $\mathbf{w}^l\in\mathcal{D}(G)$, the following equality holds:*

$$\begin{aligned}\nu\sum_{l=0}^{N}c(l)\left(\nabla\mathbf{v}_N^l,\nabla\mathbf{w}^l\right)_0 + \sum_{l=0}^{N}c(l)\sum_{m,n=0}^{N}A_{m,k;l}\,\mathfrak{a}(\mathbf{v}_N^k,\mathbf{v}_N^m,\mathbf{w}^l)\\ = -\sum_{l=0}^{N}c\,(l)\left\langle\mathbf{f}^l,\mathbf{w}^l\right\rangle_1,\quad c(l)=\mathbb{E}\mathfrak{P}_l^2.\end{aligned}\tag{3.7}$$

Given $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{w}}$ in $\left(\mathbf{H}_0^{1,2}(G)\right)^{N+1}$, with $\bar{\mathbf{u}}=(\mathbf{u}^0,\dots,\mathbf{u}^N)$ and similarly for $\bar{\mathbf{v}},\bar{\mathbf{w}}$, define

$$\mathfrak{A}(\bar{\mathbf{u}},\bar{\mathbf{v}},\bar{\mathbf{w}}) = \sum_{l=0}^{N}c(l)\sum_{m,n=0}^{N}A_{m,k;l}\,\mathfrak{a}(\mathbf{u}^k,\mathbf{v}^m,\mathbf{w}^l).$$

Then we can re-write (3.7) as

$$\nu\sum_{l=0}^{N}c(l)\left(\nabla\mathbf{v}_N^l,\nabla\mathbf{w}^l\right)_0 + \mathfrak{A}(\bar{\mathbf{v}}_N,\bar{\mathbf{v}}_N,\bar{\mathbf{w}}) = -\sum_{l=0}^{N}c\,(l)\left\langle\mathbf{f}^l,\mathbf{w}^l\right\rangle_1.\tag{3.8}$$

Furthermore, given $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{w}}$ in $\left(\mathbf{H}_0^{1,2}(G)\right)^{N+1}$, define

$$\boldsymbol{u}=\sum_{k=0}^{N}\mathbf{u}^k\mathfrak{P}_k,\quad \boldsymbol{v}=\sum_{k=0}^{N}\mathbf{v}^k\mathfrak{P}_k,\quad \boldsymbol{w}=\sum_{k=0}^{N}\mathbf{w}^k\mathfrak{P}_k.$$

Then equality (3.2) implies

$$\mathfrak{A}(\bar{\mathbf{u}},\bar{\mathbf{v}},\bar{\mathbf{w}}) = \mathbb{E}\mathfrak{a}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}).$$

In particular, by (2.7),

$$\mathfrak{A}(\bar{\mathbf{u}},\bar{\mathbf{v}},\bar{\mathbf{v}}) = 0\tag{3.9}$$

provided $\mathbf{u}^k\in\widehat{\mathbf{H}}_0^{1,2}(G)$ for all $k=0,\dots,N$.

We now use (3.7) to establish a basic solvability result for equation (3.1).

Theorem 3.4. *For every $\boldsymbol{f}\in\mathbb{H}_0^{-1,2}(G)$ and $N\ge 1$, equation (3.1) has a solution $\boldsymbol{v}_N$ and (3.4) holds.*

The solution is unique if there exists a non-random number $\varepsilon_N\in(0,1)$ such that

$$\mathbb{P}\Big(|\boldsymbol{v}_N|_{1,2}\le\frac{2\nu(1-\varepsilon_N)}{\sqrt{|G|}}\Big)=1.\tag{3.10}$$

Proof. For $M \geq 1$ and $l = 0, \dots, N$, define

$$\mathbf{v}^l_{M,N} = \sum_{k=0}^{M} z^{k,l}_{M,N} \mathbf{h}_k, \tag{3.11}$$

where $z^{k,l}_{M,N} \in \mathbb{R}$ and the functions $\mathbf{h}_k$ have the following properties:

(1) $\mathbf{h}_k \in \widehat{\mathbf{H}}^{1,2}_0(G)$, $k \geq 0$;
(2) Finite linear combinations of $\mathbf{h}_k$ are dense in the space $\widehat{\mathbf{H}}^{1,2}_0(G)$;
(3) $|\mathbf{h}_k|_{\mathbf{L}^2} = 1,\ (\mathbf{h}_k, \mathbf{h}_m)_0 = 0,\ k \neq m$.

A possible choice is the normalized eigenfunctions of the Stokes operator [33, Section I.2.6].

Also, we will use the notations

$$\bar{\mathbf{v}}_{M,N} = (\mathbf{v}^0_{M,N}, \dots, \mathbf{v}^N_{M,N}), \quad \boldsymbol{v}_{M,N} = \sum_{l=0}^{N} \mathbf{v}^l_{M,N} \mathfrak{P}_l.$$

Consider the system of equations

$$\nu\big(\nabla \mathbf{v}^l_{M,N}, \nabla \mathbf{h}_k\big) + \sum_{m,n=0}^{N} A_{m,n;l} \left((\mathbf{v}^m_{M,N} \cdot \nabla)\mathbf{v}^n_{M,N}, \mathbf{h}_k\right) + \langle \mathbf{f}^l, \mathbf{h}_k \rangle_1 = 0, \tag{3.12}$$

$k = 0, \dots, M,\ l = 0, \dots, N$; with (3.11) in mind, we think of (3.12) as a system of equations for the numbers $z^{k,l}_{M,N}$.

To show that (3.12) has a solution for every $M \geq 1$, we introduce the following notations:

$$\bar{\mathbf{z}} = \big(z^{0,0}_{M,N}, \dots, z^{M,0}_{M,N}, z^{0,1}_{M,N} \dots, z^{M,1}_{M,N}, \dots, z^{0,N}_{M,N}, \dots, z^{M,N}_{M,N}\big),$$

$$Q_{k,l}(\bar{\mathbf{z}}) = \nu\big(\nabla \mathbf{v}^l_{M,N}, \nabla \mathbf{h}_k\big) + \sum_{m,n=0}^{N} A_{m,n;l} \left((\mathbf{v}^m_{M,N} \cdot \nabla)\mathbf{v}^n_{M,N}, \mathbf{h}_k\right) + \langle \mathbf{f}^l, \mathbf{h}_k \rangle_1,$$

$$F(\bar{\mathbf{z}}) = \sum_{k,l} c(l) Q_{k,l}(\bar{\mathbf{z}}) z^{k,l}_{M,N}.$$

Combining (3.11), (3.8), and (3.9),

$$\begin{aligned} F(\bar{\mathbf{z}}) &= \nu \sum_{l=0}^{N} c(l) \left|\mathbf{v}^l_{M,N}\right|^2_{1,2} + \mathfrak{A}(\bar{\mathbf{v}}_{M,N}, \bar{\mathbf{v}}_{M,N}, \bar{\mathbf{v}}_{M,N}) + \sum_{l=0}^{N} c\,(l) \left\langle \mathbf{f}^l, \mathbf{v}^l_{M,N} \right\rangle_1 \\ &= \nu \sum_{l=0}^{N} c(l) \left|\mathbf{v}^l_{M,N}\right|^2_{1,2} + \sum_{l=0}^{N} c\,(l) \left\langle \mathbf{f}^l, \mathbf{v}^l_{M,N} \right\rangle_1. \end{aligned} \tag{3.13}$$

We now show that if

$$\frac{2\nu^2}{|G|} \sum_{k,l} c(l) \big|z^{k,l}_{M,N}\big|^2 = |\boldsymbol{f}|^2_{\mathbb{H}^{-1,2}_0}, \tag{3.14}$$

then

$$F(\bar{\mathbf{z}}) \geq 0.$$

Indeed, by the Cauchy-Schwarz inequality,

$$F(\bar{\mathbf{z}}) \geq |\boldsymbol{v}_{M,N}|_{\mathbb{H}_0^{1,2}}\Big(\nu|\boldsymbol{v}_{M,N}|_{\mathbb{H}_0^{1,2}} - |\boldsymbol{f}|_{\mathbb{H}_0^{-1,2}}\Big),$$

whereas the Poincaré inequality (2.4) implies

$$|\boldsymbol{v}_{M,N}|^2_{\mathbb{H}_0^{1,2}} \geq \frac{2|\boldsymbol{v}_{M,N}|^2_{\mathbb{L}^2}}{|G|} = \frac{2}{|G|}\sum_{k,l} c(l)\big|z^{k,l}_{M,N}\big|^2,$$

so that, under (3.14),

$$\nu^2|\boldsymbol{v}_{M,N}|^2_{\mathbb{H}_0^{1,2}} \geq |\boldsymbol{f}|^2_{\mathbb{H}_0^{-1,2}}.$$

By a multi-dimensional version of the intermediate value theorem [11, Lemma IX.3.1], we conclude that there exists a $\bar{\mathbf{z}}^*$ with

$$\frac{2\nu^2}{|G|}\sum_{k,l} c(l)\big|z^{*,k,l}_{M,N}\big|^2 \leq |\boldsymbol{f}|^2_{\mathbb{H}_0^{-1,2}}$$

such that $Q_{k,l}(\bar{\mathbf{z}}^*) = 0$ for all k, l. In other words, we now have existence of solution of (3.12) for every $M \geq 1$. Moreover, by (3.13) the solution satisfies

$$\nu\,\big|\boldsymbol{v}_{M,N}\big|^2_{\mathbb{H}_0^{1,2}} = -\mathbb{E}\langle \boldsymbol{f}, \boldsymbol{v}_{M,N}\rangle_1.$$

Then the Cauchy-Schwarz inequality implies

$$\nu\,\big|\boldsymbol{v}_{M,N}\big|_{\mathbb{H}_0^{1,2}} \leq |\boldsymbol{f}|_{\mathbb{H}_0^{-1,2}},$$

which, together with compactness of the embedding $\mathbf{H}_0^{1,2}(G) \subset \mathbf{L}^2(G)$, allows us to pass to the limit $M \to \infty$ and get a solution of (3.7) in the same way as in [11, pp. 600–601].

To establish uniqueness, let $\boldsymbol{v}_N$ be the solution of (3.7) satisfying (3.10) and let $\tilde{\boldsymbol{v}}$ be the difference between $\boldsymbol{v}_N$ and any other possible solution of (3.7). By (3.3),

$$\nu\,\mathbb{E}(\nabla\tilde{\boldsymbol{v}}, \nabla\boldsymbol{w})_0 + \mathbb{E}\mathfrak{a}(\boldsymbol{v}_N, \tilde{\boldsymbol{v}}, \boldsymbol{w}) - \mathbb{E}\mathfrak{a}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} - \boldsymbol{v}_N, \boldsymbol{w}) = 0. \tag{3.15}$$

Because $\mathcal{P}^N\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}} \in \widehat{\mathbb{H}}_0^{1,2}(G)$, we can put $\boldsymbol{w} = \tilde{\boldsymbol{v}}$ in (3.15) and then use (2.7) to conclude that

$$\nu\,\mathbb{E}|\tilde{\boldsymbol{v}}|^2_{1,2} + \mathbb{E}\mathfrak{a}(\tilde{\boldsymbol{v}}, \boldsymbol{v}_N, \tilde{\boldsymbol{v}}) = 0,$$

which, together with (2.6) implies

$$\mathbb{E}\left(|\tilde{\boldsymbol{v}}|^2_{1,2}\left(\nu - \frac{\sqrt{|G|}}{2}|\boldsymbol{v}_N|_{1,2}\right)\right) \leq 0. \tag{3.16}$$

If (3.10) holds, then

$$\nu - \frac{\sqrt{|G|}}{2}|\boldsymbol{v}_N|_{1,2} \geq \nu\varepsilon_N > 0,$$

and (3.16) is only possible when $\mathbb{P}\big(|\tilde{\boldsymbol{v}}|^2_{1,2} = 0\big) = 1$, that is, when $\boldsymbol{v}_N$ is the unique solution of (3.7). □

Similar to (2.11), we need (3.10) to guarantee uniqueness of the stochastic Galerkin approximation. In fact, without (2.11), uniqueness can fail for the original equation (2.3) [11, Section IX.2]. Even though system of equations (3.7) has been successfully used for numerical simulations [14, 32], it is not immediately clear how condition (3.10) can be verified.

The following theorem is the first key result of the paper and shows that, under (2.11) and (3.10), stochastic Galerkin approximation $\boldsymbol{v}_N$ is indeed an approximation of the orthogonal projection $\mathcal{P}^N\boldsymbol{u}$. In particular, if ε_N does not depend on N, then stochastic Galerkin approximation is asymptotically equivalent to the orthogonal projection, in the sense that, as $N \to \infty$, both converge to the true solution at the same rate.

Theorem 3.5. *Assume that* (2.11) *holds so that* (2.3) *has a unique solution* $\boldsymbol{u}$*, and let* $\boldsymbol{v}_N$ *be the unique solution of* (3.1) *satisfying* (3.10)*. Then*

$$|\mathcal{P}^N\boldsymbol{u} - \boldsymbol{v}_N|_{\mathbb{H}_0^{1,2}} \le \frac{\theta + 1 - \varepsilon_N}{\varepsilon_N}\, |\boldsymbol{u} - \mathcal{P}^N\boldsymbol{u}|_{\mathbb{H}_0^{1,2}}, \tag{3.17}$$

$$|\boldsymbol{u} - \boldsymbol{v}_N|_{\mathbb{H}_0^{1,2}} \le \left(1 + \frac{\theta + 1 - \varepsilon_N}{\varepsilon_N}\right)\, |\boldsymbol{u} - \mathcal{P}^N\boldsymbol{u}|_{\mathbb{H}_0^{1,2}}. \tag{3.18}$$

Proof. To make the formulas shorter, we write

$$\boldsymbol{u}^N = \mathcal{P}^N\boldsymbol{u},\ \boldsymbol{u}_N = \boldsymbol{u}^N - \boldsymbol{v}_N, \quad \text{and} \quad \boldsymbol{w}^N = \mathcal{P}^N\boldsymbol{w} \quad \text{for} \quad \boldsymbol{w} \in \widehat{\mathbb{H}}_0^{1,2}(G)$$

Using (2.14) and (3.2),

$$\nu\,\mathbb{E}\big(\nabla\boldsymbol{u}^N, \nabla\boldsymbol{w}^N\big)_0 + \mathbb{E}\mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^N) = -\mathbb{E}\langle \boldsymbol{f}, \boldsymbol{w}^N\rangle_1,$$

and, after subtracting (3.3),

$$\nu\,\mathbb{E}\big(\nabla\boldsymbol{u}_N, \nabla\boldsymbol{w}^N\big)_0 + \mathbb{E}\mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^N) - \mathbb{E}\mathfrak{a}(\boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{w}^N) = 0.$$

Next,

$$\begin{aligned}\mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^N) - \mathfrak{a}(\boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{w}^N) &= \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}_N, \boldsymbol{w}^N) + \mathfrak{a}(\boldsymbol{u} - \boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{w}^N)\\ &= \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^N, \boldsymbol{w}^N) + \mathfrak{a}(\boldsymbol{u}, \boldsymbol{u}^N - \boldsymbol{v}_N, \boldsymbol{w}^N)\\ &\quad + \mathfrak{a}(\boldsymbol{u} - \boldsymbol{u}^N, \boldsymbol{v}_N, \boldsymbol{w}^N) + \mathfrak{a}(\boldsymbol{u}^N - \boldsymbol{v}_N, \boldsymbol{v}_N, \boldsymbol{w}^N).\end{aligned}$$

Taking $\boldsymbol{w}^N = \boldsymbol{u}_N$ leads to

$$\nu\,\mathbb{E}|\boldsymbol{u}_N|_{1,2}^2 + \mathbb{E}\mathfrak{a}(\boldsymbol{u}_N, \boldsymbol{v}_N, \boldsymbol{u}_N) + \mathbb{E}\mathfrak{a}(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^N, \boldsymbol{u}_N) + \mathbb{E}\mathfrak{a}(\boldsymbol{u} - \boldsymbol{u}^N, \boldsymbol{v}_N, \boldsymbol{u}_N) = 0.$$

Then (2.6), (2.12), (3.10), and the Cauchy-Schwarz inequality [for expectations] imply

$$\nu\varepsilon_N|\boldsymbol{u}_N|_{\mathbb{H}_0^{1,2}}^2 \le \nu(\theta + 1 - \varepsilon_N)|\boldsymbol{u}_N|_{\mathbb{H}_0^{1,2}}\ |\boldsymbol{u} - \boldsymbol{u}^N|_{\mathbb{H}_0^{1,2}}.$$

We now get (3.17), and then, by triangle inequality, (3.18). □

4. A Discrete Projection Approximation Using Gauss Quadrature

Stochastic Galerkin approximation can be constructed and analyzed for a very general random force $\boldsymbol{f}$. In particular, the `stochastic dimension` of the problem, that is, the number of random variables generating the sigma-algebra of $\boldsymbol{f}$, can be infinite. To study a discrete projection version of the stochastic collocation method, we have

to assume that the stochastic dimension of the problem is finite; for the sake of simplicity, we take it equal to one.

The precise setting is as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a random variable ξ and let $\mathcal{F}_\xi$ be the $\mathbb{P}$-completion of the sigma algebra generated by ξ. We assume that the moment generating function $\lambda \mapsto \mathbb{E}e^{\lambda\xi}$ is defined in some neighborhood of $\lambda = 0$. Under this assumption, given a collection $\{P_n,\ n \geq 0\}$ of orthogonal polynomials corresponding to the distribution of ξ, the collection of random variables

$$\mathfrak{P}_n = P_n(\xi),\ n \geq 0,$$

is an orthogonal basis in $L^2(\Omega, \mathcal{F}_\xi, \mathbb{P})$. Denote by $\mathcal{P}^N$ the orthogonal projection in $L^2(\Omega, \mathcal{F}_\xi, \mathbb{P})$ on the subspace spanned by $\{\mathfrak{P}_k,\ k = 0, \dots, N\}$. Let

$$c(n) = \mathbb{E}\mathfrak{P}_n^2,$$

so that every $\zeta \in L^2(\Omega, \mathcal{F}_\xi, \mathbb{P})$ has a (generalized) polynomial chaos expansion

$$\zeta = \sum_{k\geq 0} \frac{\mathbb{E}\big(\zeta\mathfrak{P}_k\big)}{c(k)}\,\mathfrak{P}_k.$$

In this section, we assume that the random forcing in equation (2.3) has a special form

$$\boldsymbol{f}(x) = \mathbf{f}(\xi, x) = \big(f^1(\xi, x), f^2(\xi, x)\big), \tag{4.1}$$

where $\mathbf{f}$ is a non-random vector field. In other words, the sigma algebra of $\boldsymbol{f}$ is $\mathcal{F}_\xi$, and the stochastic dimension of the problem is equal to 1. Given the tools developed in [27], extension to any finite stochastic dimension is straightforward.

If $\boldsymbol{u} = \boldsymbol{u}(\xi)$ is a solution of (2.3) corresponding to the particular realization of ξ and $\boldsymbol{u}^N = \mathcal{P}^N\boldsymbol{u}$, then

$$\boldsymbol{u}(\xi) \approx \boldsymbol{u}^N(\xi), \quad \boldsymbol{u}^N(\xi) = \sum_{k=0}^{N} \mathbf{u}_k \frac{\mathfrak{P}_k(\xi)}{c(k)}, \quad \mathbf{u}_k = \mathbb{E}\big(\boldsymbol{u}\mathfrak{P}_k\big).$$

To compute the coefficients $\mathbf{u}_k,\ k = 0, \dots, N$, we use the *Gauss quadrature* approximation $\mathbf{u}_k \approx \mathbf{u}_k^{(N)}$, where

$$\mathbf{u}_k^{(N)} = \sum_{j=1}^{N+1} w_{j,N}\boldsymbol{u}(\xi_{j,N})\mathfrak{P}_k(\xi_{j,N}), \tag{4.2}$$

$\xi_{j,N},\ j = 1, \dots, N+1$, are the roots of P_{N+1}, and $w_{j,N}$ are the corresponding weights; cf. [12, Section 1.4]. The resulting *discrete projection* or *pseudo-spectral* approximation,

$$\boldsymbol{u}^{(N)}(\xi) = \sum_{k=0}^{N} \mathbf{u}_k^{(N)} \frac{\mathfrak{P}_k(\xi)}{c(k)}, \tag{4.3}$$

requires the solution $\boldsymbol{u}(\xi_{j,N})$ of (2.3) for $N+1$ distinct values of ξ.

To simplify the formulas, it is convenient to introduce the square matrix $\mathfrak{W} = \big(\mathfrak{W}_{kj},\ k = 0, \dots, N,\ j = 1, \dots, N+1\big)$, with

$$\mathfrak{W}_{kj} = w_{j,N}\mathfrak{P}_k(\xi_{j,N}).$$

Then (4.2) becomes

$$\mathbf{u}_k^{(N)} = \sum_{j=1}^{N+1} \mathfrak{W}_{kj} \boldsymbol{u}(\xi_{j,N}). \tag{4.4}$$

The basic property of the Gauss quadrature is that the equality

$$\mathbb{E}h(\xi) = \sum_{j=1}^{N+1} w_{j,N} h(\xi_{j,N})$$

holds for all functions $h = h(\xi)$ that are polynomials in ξ of degree at most $2N+1$; cf. [12, Theorem 1.45]. In particular, for every $k, m = 0, \dots, N$,

$$\sum_{j=1}^{N+1} \mathfrak{W}_{kj} \mathfrak{P}_m(\xi_{j,N}) = \sum_{j=1}^{N+1} w_{j,N} \mathfrak{P}_k(\xi_{j,N}) \mathfrak{P}_m(\xi_{j,N}) = \mathbb{E}\big(\mathfrak{P}_k \mathfrak{P}_m\big) = \begin{cases} c(k) > 0, & \text{if } k = m, \\ 0, & \text{if } k \neq m, \end{cases}$$

which means that the matrix $\mathfrak{W}$ is non-singular.

With the above choice of the sampling points $\xi_{j,N}$, the discrete projection (4.3) is equivalent to interpolation:

Proposition 4.1. *The equality*

$$\boldsymbol{u}(\xi_{j,N}) = \boldsymbol{u}^{(N)}(\xi_{j,N}) \tag{4.5}$$

holds for all $j = 1, \dots, N+1$.

Proof. Equality (4.3) implies that $\boldsymbol{u}^{(N)}$ is a polynomial in ξ of order at most N, so that each product $\boldsymbol{u}^{(N)} \mathfrak{P}_k$, $k = 0, \dots, N$, is a polynomial in ξ or order at most $2N$. Then

$$\mathbb{E}\big(\boldsymbol{u}^{(N)} \mathfrak{P}_k\big) = \sum_{j=1}^{N+1} \mathfrak{W}_{kj} \boldsymbol{u}^{(N)}(\xi_{j,N}), \ k = 0, \dots, N. \tag{4.6}$$

On the other hand, (4.3) also implies

$$\mathbb{E}\big(\boldsymbol{u}^{(N)} \mathfrak{P}_k\big) = \mathbf{u}_k^{(N)}, \tag{4.7}$$

and then (4.5) follows from (4.4) and non-degeneracy of the matrix $\mathfrak{W}$. □

The following theorem is the second key result of the paper and gives an upper bound on the approximation error $\mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^{(N)}|_{1,2}^2$. Recall that $\boldsymbol{u}^N = \mathcal{P}^N \boldsymbol{u}$.

Theorem 4.2. *Define*

$$\delta_N = \sup_{\xi} |\boldsymbol{u}(\xi) - \boldsymbol{u}^N(\xi)|_{1,2}. \tag{4.8}$$

Then

$$\mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^{(N)}|_{1,2}^2 \leq \mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^N|_{1,2}^2 + N\big(\delta_N\big)^2. \tag{4.9}$$

Proof. By orthogonality,

$$\begin{aligned} \mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^{(N)}|_{1,2}^2 &= \mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^N|_{1,2}^2 + \mathbb{E}|\boldsymbol{u}^N - \boldsymbol{u}^{(N)}|_{1,2}^2 \\ &= \mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^N|_{1,2}^2 + \sum_{k=0}^{N} \frac{|\mathbf{u}_k - \mathbf{u}_k^{(N)}|_{1,2}^2}{c(k)}. \end{aligned} \tag{4.10}$$

Combining (4.2), (4.6), and (4.7) results in

$$\mathbf{u}_k - \mathbf{u}_k^{(N)} = \sum_{j=1}^{N+1} w_{j,N}\big(\boldsymbol{u}^N(\xi_{j,N}) - \boldsymbol{u}(\xi_{j,N})\big)\mathfrak{P}_k(\xi_{j,N}),$$

or, using the Cauchy-Schwarz inequality and $w_{j,N} > 0$,

$$|\mathbf{u}_k - \mathbf{u}_k^{(N)}|_{1,2}^2 \le \left(\sum_{j=1}^{N+1} w_{j,N}|\boldsymbol{u}^N(\xi_{j,N}) - \boldsymbol{u}(\xi_{j,N})|_{1,2}^2\right)\left(\sum_{j=1}^{N+1} w_{j,N}\mathfrak{P}_k^2(\xi_{j,N})\right).$$

Properties of the Gauss quadrature imply

$$\sum_{j=1}^{N+1} w_{j,N}\mathfrak{P}_k^2(\xi_{j,N}) = \mathbb{E}\mathfrak{P}_k^2 = c(k),\ k = 0,\dots,N,\ \text{and}\ \sum_{j=1}^{N+1} w_{j,N} = 1,$$

whereas (4.8) implies

$$\sum_{j=1}^{N+1} w_{j,N}|\boldsymbol{u}^N(\xi_{j,N}) - \boldsymbol{u}(\xi_{j,N})|_{1,2}^2 \le \big(\delta_N\big)^2 \sum_{j=1}^{N+1} w_{j,N}.$$

As a result,

$$|\mathbf{u}_k - \mathbf{u}_k^{(N)}|_{1,2}^2 \le \big(\delta_N\big)^2 c(k),$$

and (4.9) follows from (4.10). □

Of course, $\mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^N|_{1,2}^2 \le \big(\delta_N\big)^2$, leading to a somewhat weaker form of (4.9): $\mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^{(N)}|_{1,2}^2 \le \big(\delta_N\big)^2(1+N)$.

Remark 4.3. Both intrusive and non-intrusive approximations require some kind of an L^∞-bound, either in the form of (3.10) or (4.8), to establish an L^2-bound on the approximation error; for (4.9) to be useful, one additionally needs to establish

$$\lim_{N\to\infty} \sqrt{N}\delta_N = 0. \tag{4.11}$$

On the one hand, condition (4.11) is easier to verify than condition (3.10). On the other hand, under condition (3.10), the error bound (3.18) can be better than (4.9), and this difference can become even more pronounced as the stochastic dimension of the problem (the number of independent random variables in the input) grows.

Theorem 4.2 is rather general, and the proof does not use the fact that $\boldsymbol{u}$ solves (2.3): only exactness of the quadrature rule on polynomials of degree up to $2N$ is required. On the flip side, without (4.11), the conclusion of the theorem is not especially useful.

To establish (4.11), we need additional information about the polynomials P_n and about the dependence of $\boldsymbol{u}$ on ξ, which means additional assumptions about equation (2.3).

As an example, consider the random variable ξ that is uniform on $[-1,1]$. Then $\mathfrak{P}_n = P_n(\xi)$, where P_n is nth Legendre polynomial; the standard normalization [2,

equation (6.4.4.)] is $P_n(1) = 1$, and then

$$c_n = \frac{1}{2}\int_{-1}^{1} P_n^2(x)\,dx = \frac{1}{2n+1}.$$

Theorem 4.4. *Assume that, in* (4.1)*, the random variable* ξ *is uniform on* $[-1,1]$ *and the function* $\mathbf{f}$ *is Lipschitz continuous as a function of* ξ*: there exists a positive number* C_f *such that, for all* $\xi_1, \xi_2 \in [-1,1]$*,*

$$|\mathbf{f}(\xi_1,\cdot) - \mathbf{f}(\xi_2,\cdot)|_{-1,2} \le C_f|\xi_1 - \xi_2|.$$

If (2.11) *holds and* $\boldsymbol{u} = \boldsymbol{u}(\xi)$ *is the corresponding unique solution of* (2.3)*, then*

$$\sup_{\xi} |\boldsymbol{u}(\xi) - \boldsymbol{u}^N(\xi)|_{1,2} \le CN^{-3/4} \tag{4.12}$$

for some C *depending only on* C_f, ν*, and* θ*. In particular, we have* (4.11)*.*

Proof. By (2.15),

$$|\boldsymbol{u}(\xi_1) - \boldsymbol{u}(\xi_2)|_{1,2} \le \frac{|\mathbf{f}(\xi_1,\cdot) - \mathbf{f}(\xi_2,\cdot)|_{-1,2}}{\nu(1-\theta)} \le \frac{C_f}{\nu(1-\theta)}\,|\xi_1 - \xi_2|. \tag{4.13}$$

For the rest of the proof, C denotes positive number depending only on C_f, ν, and θ. The value of C can be different in different formulas.

Let $\mathcal{E}_N$ be the error of the best uniform approximation of $\boldsymbol{u}$ by an element of $\widehat{\mathbb{H}}_0^{1,2}(G)$ that is a polynomial of degree at most N in ξ:

$$\mathcal{E}_N(\boldsymbol{u}) = \inf\Big(\max_{\xi\in[-1,1]} |\boldsymbol{u}(\xi) - \boldsymbol{v}(\xi)|_{1,2} : \boldsymbol{v} \in \mathcal{P}^N\big(\widehat{\mathbb{H}}_0^{1,2}(G)\big)\Big).$$

Then

- Jackson's Theorem [29, Theorem 1.4], together with (4.13), implies
$$\mathcal{E}_N(\boldsymbol{u}) \le \frac{C}{N}; \tag{4.14}$$
- Combining (4.13) with [34, Theorem 2.1] yields
$$\mathbb{E}|\boldsymbol{u} - \boldsymbol{u}^{(N)}|_{1,2}^2 \le \frac{C}{N^3}; \tag{4.15}$$
- Combining (4.14) and (4.15) with [5, Theorem 1 (p=2)] leads to (4.12) and completes the proof.

□

5. Summary and Discussion

Within the general framework of numerical analysis, this paper studies *a priori* error bounds, as opposed to *a posteriori* error analysis that requires some basic knowledge about convergence of the numerical procedure; cf. [1, Section 9.3]. Comparing the (intrusive) stochastic Galerkin approximation [Theorem 3.5] and a (non-intrusive) stochastic collocation/Gauss quadrature approximation [Theorem 4.2] for equation (2.3), we see that

- The intrusive approximation works for a broader class of random input and can, in principle, achieve an asymptotically optimal rate of convergence;
- The non-intrusive approximation is easier to study, both analytically and numerically.

While the setting in the paper, a stationary two-dimensional Navier-Stokes system with zero boundary conditions and additive random perturbation, is intentionally simple to isolate the effects of non-linearity (the convection term) on the stochastic Galerkin approximation, some of the results are rather universal and can be used for many other equations with a quadratic-type nonlinearity. The key is equality (3.5) describing the product of two chaos expansions.

For example, (3.5) implies that equations (3.6) describe the stochastic Galerkin approximation for the stationary Navier-Stokes system in any number of space dimensions and with randomness in both boundary conditions and the external force; after minor modifications, time-dependent problems with a random initial condition will also be covered. The stochastic dimension of the problem does not matter either, as long as the corresponding orthogonal basis $\{\mathfrak{P}_n,\ n \geq 0\}$ with property (2.2) can be constructed [27].

The main technical difficulty, both in the general analysis of stochastic Galerkin approximation and in the particular proof of Theorem 3.5, is related to the fact that, for the solution $\boldsymbol{v}$ of a nonlinear equation,

$$\mathcal{P}^N \boldsymbol{v} \neq \boldsymbol{v}_N.$$

The two possible sources of non-linearity are (a) the structure of the underlying deterministic equation, and (b) the way the random perturbation enters the equation. For example, consider the system (2.3) with random viscosity

$$\nu = \sum_{k=0}^{\infty} \nu_k \mathfrak{P}_k.$$

Equality (3.5) shows that the stochastic Galerkin approximation in this case will be

$$\sum_{m,k=0}^{N} A_{m,k;l}\, \nu_k\, \Delta \mathbf{v}_N^m = \sum_{m,k=0}^{N} A_{m,k;l}\, (\mathbf{v}_N^k \cdot \nabla)\mathbf{v}_N^m + \nabla p_N^l + \mathbf{f}^l,\ l = 0, \ldots, N. \tag{5.1}$$

Equation (5.1) illustrates the effects of the two sources of nonlinearity: the convection term leads to the coupling of the functions $\mathbf{v}_N^m$ on the right-hand side, whereas random viscosity leads to a similar coupling on the left-hand side. While not very different from (3.6), analysis of (5.1) must be carried out from scratch and, for now, is left to an interested reader.

It is the "deterministic nonlinearity" that leads to hard-to-verify conditions of the type (3.10). Linear equations with random coefficients, such as heat equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = a\, \Delta \boldsymbol{v}$$

with random diffusion coefficient a, exhibit some non-linear features when it comes to stochastic Galerkin approximation, but are much more manageable when it comes to

the corresponding error analysis [6, 7]. The non-linear effects can be further mitigated by replacing the usual product with the Wick product, a convolution operation $\diamond$ such that $\mathfrak{P}_m \diamond \mathfrak{P}_n = \alpha_{mn}\,\mathfrak{P}_{m+n}$, $\alpha_{mn} \in \mathbb{R}$ [16, 26, 27].

To conclude, let us note that there are many equivalent ways to write Navier-Stokes equations: even the basic velocity-pressure formulation (2.3) admits at least four alternative forms [10, Section 5], not to mention alternative variables, such stream function and vorticity [4, 23]. For the purpose of our investigation, it appears that none of the alternatives will lead to any major simplifications, but, as reference [10] suggests, one should keep those alternatives in mind for further analysis of various approximations of (2.3).

References

[1] M. Ainsworth and J. T. Oden, *A posteriori error estimation in finite element analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2000.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.

[3] M. Arnst and J.-P. Ponthot, *An overview of nonintrusive characterization, propagation, and sensitivity analysis of uncertainties in computational mechanics*, Int. J. Uncertain. Quantif. **4** (2014), no. 5, 387–421.

[4] K. E. Barrett, *A variational principle for the stream function-vorticity formulation of the Navier-Stokes equations incorporating no-slip conditions*, J. Computational Phys. **26** (1978), no. 2, 153–161.

[5] I. Bashmakova and S. Rafalson, *On the approximation of continuous functions by Fourier-Legendre sums*, J. Approx. Theory **86** (1996), no. 2, 197–215.

[6] A. Bespalov, C. E. Powell, and D. Silvester, *A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data*, SIAM J. Numer. Anal. **50** (2012), no. 4, 2039–2063.

[7] A. Bespalov and F. Xu, *A posteriori error estimation and adaptivity in stochastic Galerkin FEM for parametric elliptic PDEs: beyond the affine case*, Comput. Math. Appl. **80** (2020), no. 5, 1084–1103.

[8] M. Branicki and A. J. Majda, *Fundamental limitations of polynomial chaos for uncertainty quantification in systems with intermittent instabilities*, Commun. Math. Sci. **11** (2013), no. 1, 55–103.

[9] A. Budhiraja and G. Kallianpur, *Approximations to the solution of the Zakai equations using multiple Wiener and Stratonovich integral expansions*, Stochastics Stochastics Rep. **56** (1996), no. 3–4, 271–315.

[10] S. Charnyi, T. Heister, M. A. Olshanskii, and L. G. Rebholz, *On conservation laws of Navier-Stokes Galerkin discretizations*, J. Comput. Phys. **337** (2017), 289–308.

[11] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*, second ed., Springer Monographs in Mathematics, Springer, New York, 2011.

[12] W. Gautschi, *Orthogonal polynomials: computation and approximation*, Oxford University Press, 2004.

[13] R. G. Ghanem and P. D. Spanos, *Stochastic finite elements: a spectral approach*, Springer-Verlag, New York, 1991.

[14] M. D. Gunzburger, L. Hou, and J. Ming, *Stochastic steady-state Navier-Stokes equations with additive random noise*, J. Sci. Comput. **66** (2016), no. 2, 672–691.

[15] T. Y. Hou, W. Luo, B. Rozovskii, and H.-M. Zhou, *Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics*, J. Comput. Phys. **216** (2006), no. 2, 687–706.

[16] S. Kaligotla and S. V. Lototsky, *Wick product in the stochastic Burgers equation: a curse or a cure?*, Asymptot. Anal. **75** (2011), no. 3-4, 145–168.

[17] N. V. Krylov, *An analytic approach to SPDEs*, Stochastic Partial Differential Equations. Six Perspectives, Mathematical Surveys and Monographs (B. L. Rozovsky and R. Carmona, eds.), AMS, 1999, pp. 185–242.
[18] N. V. Krylov and A. J. Veretennikov, *On explicit formula for solutions of stochastic equations*, Math. USSR Sbornik **29** (1976), no. 2, 239–256.
[19] N. V. Krylov and A. K. Zvonkin, *On strong solutions of stochastic differential equations*, Sel. Math. Sov. **1** (1981), 19–61.
[20] S. V. Lototsky, R. Mikulevicius, and B. L. Rozovsky, *Nonlinear filtering revisited: A spectral approach*, SIAM J. Contr. Optim. **35** (1997), no. 2, 435–461.
[21] S. V. Lototsky and B. L. Rozovsky, *Wiener chaos solutions of linear stochastic evolution equations*, Ann. Probab. **34** (2006), no. 2, 638–662.
[22] S. V. Lototsky and B. L. Rozovsky, *Stochastic partial differential equations*, Universitext, Springer, Cham, 2017.
[23] F. J. McGrath, *Nonstationary plane flow of viscous and ideal fluids*, Arch. Rational Mech. Anal. **27** (1967), 329–348.
[24] R. Mikulevicius and B. L. Rozovskii, *Stochastic Navier-Stokes equations for turbulent flows*, SIAM J. Math. Anal. **35** (2004), no. 5, 1250–1310.
[25] ———, *Global L_2-solutions of stochastic Navier-Stokes equations*, Ann. Probab. **33** (2005), no. 1, 137–176.
[26] ———, *On unbiased stochastic Navier-Stokes equations*, Probab. Theory Related Fields **154** (2012), no. 3-4, 787–834.
[27] ———, *On distribution free Skorokhod-Malliavin calculus*, Stoch. Partial Differ. Equ. Anal. Comput. **4** (2016), no. 2, 319–360.
[28] C. E. Powell and D. J. Silvester, *Preconditioning steady-state Navier-Stokes equations with random data*, SIAM J. Sci. Comput. **34** (2012), no. 5, A2482–A2506.
[29] T. J. Rivlin, *An introduction to the approximation of functions*, Dover Books on Advanced Mathematics, Dover Publications, Inc., New York, 1981.
[30] B. L. Rozovsky and S. V. Lototsky, *Stochastic evolution systems: Linear theory and applications to non-linear filtering, Second edition*, Probability Theory and Stochastic Modelling, vol. 89, Springer, Cham, 2018.
[31] M. Sinsbeck and W. Nowak, *An optimal sampling rule for nonintrusive polynomial chaos expansions of expensive models*, Int. J. Uncertain. Quantif. **5** (2015), no. 3, 275–295.
[32] B. Sousedík and H. C. Elman, *Stochastic Galerkin methods for the steady-state Navier-Stokes equations*, J. Comput. Phys. **316** (2016), 435–452.
[33] R. Temam, *Navier-Stokes equations: theory and numerical analysis*, revised ed., Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam-New York, 1979.
[34] H. Wang and S. Xiang, *On the convergence rates of Legendre approximation*, Math. Comp. **81** (2012), no. 278, 861–877.
[35] D. Xiu, *Numerical methods for stochastic computations: a spectral method approach*, Princeton University Press, Princeton, NJ, 2010.

Declarations. The authors have no conflicts of interest to declare that are relevant to the content of this article.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Current address, S. V. Lototsky: Department of Mathematics, USC, Los Angeles, CA 90089

Email address, S. V. Lototsky: lototsky@usc.edu

URL: https://dornsife.usc.edu/sergey-lototsky/

Current address, R. Mikulevicius: Department of Mathematics, USC, Los Angeles, CA 90089

Email address, R. Mikulevicius: mikulvcs@usc.edu

URL: https://dornsife.usc.edu/cf/faculty-and-staff/faculty.cfm?pid=1003536

Current address, B. L. Rozovsky: Division of Applied Mathematics, Brown University, Providence, RI 02912

Email address, B. L. Rozovsky: Boris_Rozovsky@Brown.edu

URL: www.dam.brown.edu/people/rozovsky/rozovsky.htm