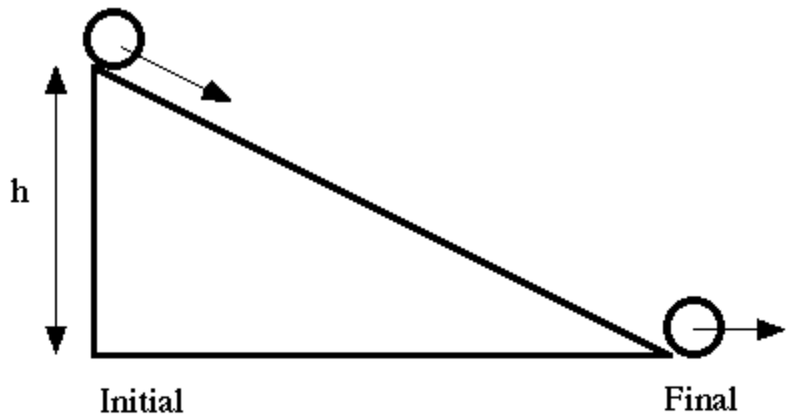
The Rolling Object Derby

Cylindrically symmetrical objects (balls, hoops, cylinders, spherical shells) rolling down an incline for Larry Brown:



Start with an object initially at rest at the top of the ramp, calculate the final linear velocity at the bottom of the ramp. We will write the moment of inertia in a generalized form for convenience later on:

$$I = Amr^2$$

Where A is 1 for a hoop, 1/2 for a cylinder or disk, 3/5 for a hollow sphere and 2/5 for a solid sphere.

Now we solve by energy conservation. At the top of the ramp, the object is at rest, so:

$$U_{initial} = mgh$$

$$KE_{trans_{initial}} = \tfrac{1}{2}mv_{initial}^2 = \tfrac{1}{2}m(0)^2 = 0$$

$$KE_{rot_{initial}} = \tfrac{1}{2}I\omega_{initial}^2 = \tfrac{1}{2}Amr^2(0)^2 = 0$$

At the bottom of the ramp, the object is rolling at a final velocity having fallen the full height of the ramp so and is rolling without slipping so:

$$U_{final} = mgh = mg(0) = 0$$

$$KE_{trans_{final}} = \tfrac{1}{2} m v_{final}^2 = \tfrac{1}{2} m v_{final}^2$$

$$KE_{rot_{final}} = \tfrac{1}{2} I \omega_{final}^2 = \tfrac{1}{2}(Amr^2)\left(\frac{v_{final}}{r}\right)^2$$

Setting energy at the top and bottom of the ramp equal (assuming no frictional losses) we can solve for final velocity:

$$U_{initial} + KE_{trans_{initial}} + KE_{rot_{initial}} = U_{final} + KE_{trans_{final}} + KE_{rot_{final}}$$

$$mgh + 0 + 0 = 0 + \tfrac{1}{2} m v_{final}^2 + \tfrac{1}{2} Amr^2\left(\frac{v_{final}}{r}\right)^2$$

$$mgh = \tfrac{1}{2} m v_{final}^2 + \tfrac{1}{2} Amv_{final}^2 = \tfrac{1}{2} m v_{final}^2 (1 + A)$$

$$v_{final} = \sqrt{\frac{2gh}{1 + A}}$$

So given a perfectly uniform sphere (no density variation throughout the material) or a perfectly hoop like hoop (all of the material located at the rim which is as thin as possible), the final velocity at the bottom of the ramp will be independent of the radius or the mass, and only depends on the arrangement of the mass, which is described by the constant A. All cylinders will have the same final velocities regardless of their mass or radius, as will all spheres and all hoops. Furthermore, all spheres will beat all disks which will beat all shells which will beat all hoops for our geometrically perfect objects in the absence of friction.

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