Fall 2023, MATH 407, Mid-Term Exam 2

Wednesday, November 15, 2023

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Instructions:

- No books, notes, or help from other people. Think twice (or more) before using a calculator.
- Turn off cell phones.
- Show your work/explain your answers.
- You have 50 minutes to complete the exam.
- There are five problems; 10 points per problem.
- Upload the solutions to GradeScope.

Problem 1. A fair die is rolled until the total sum of all rolls exceeds 350. Compute approximately the probability that at most 100 rolls are necessary. Note that, for a single roll of the die, the expected value and variance of the outcome are 7/2 and 35/12, respectively. Use the continuity correction. Leave the answer in the form $P(Z < r)$ or $P(Z > r)$ (whichever applies), where Z is a standard normal random variable and r is a suitable real number.

`Answer/Soluton:` `This is from Spring 2017. With` X_k `indicating the value of roll number` k `and` $S_n = X_1 + \ldots + X_n$, `the goal is to approximate` $P(S_{100} \geq 351) = P(S_{100} > 350.5)$. `By CLT, and using` $35/12 \approx 1.7$, $P(S_{100} > 350.5) \approx P(Z > (350.5 - 350)/17) = P(Z > 1/34)$, `which is slightly less than` $1/2$.

Problem 2. For a randomly selected group of 50 people, compute the expected number of distinct birthdays (that is, the expected number of the days of the year that are a birthday of at least one person in the group). Assume 365 days in a year.

`Answer/Soluton:` `This is from Fall 2021; the answer is` $365(1 - (364/365)^{50})$ `by indicator method because` $(364/365)^{50}$ `is the probability that a particular day is NOT a birthday of any of the 50 people.`

Problem 3. Customers arrive at a bank according to a Poisson process. Suppose that three customers arrive during the first hour. Compute the probability that at least one arrived during the first 20 minutes.

`Answer/Soluton:` `This is from Fall 2018. We need the probability` p `that, out of three iid unform random variables` U_1, U_2, U_3 `on` $(0, 1)$, `at least one is less than or equal to` $1/3$. `Go by the complement:` $p = 1 - P(\text{all bigger than } 1/3) = 1 - (P(U_1 > 1/3))^3 = 1 - (2/3)^3 = 1 - (8/27) = 19/27$.

Problem 4. Let X and Y be iid standard normal random variables. Determine the value of the number a so that the random variables $X + 2Y$ and $aX + Y$ are independent.

`Answer/Soluton:` `This is from Fall 2022.`
`For Gaussian vector (automatic when start with iid-s), zero correlation implies independence, so need` $E(X + 2Y)(aX + Y) = 0$, `that is,` $a + 2 = 0$ `or` $a = -2$.

Problem 5. The joint probability density function of two random variables X and Y

$$f_{XY}(x, y) = \begin{cases} Cy, & \text{if } x^2 + y^2 \leq 1, \ |x| \leq 1, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Justify your answer.

(b) Compute $E(X|Y)$.

`Answer/Soluton:` `This is from Fall 2013. For (a) the answer is ‘‘no’’ because` $X^2 + Y^2 \leq 1$; `for (b) the answer is` 0 `by symmetry: given` Y, `we have` X `uniform over the corresponding slice of the upper half-disk.`