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An infinite-dimensional model of liquidity in financial markets

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Abstract We consider a dynamic market model of liquidity where unmatched buy and sell limit orders are stored in order books. The resulting net demand surface constitutes the sole input to the model. We model demand using a two-parameter Brownian motion because (i) different points on the demand curve correspond to orders motivated by different information, and (ii) in general, the market price of risk equation of no-arbitrage theory has no solutions when the demand curve is driven by a finite number of factors, thus allowing for arbitrage. We prove that if the driving noise is infinite-dimensional, then there is no arbitrage in the model. Under the equivalent martingale measure, the clearing price is a martingale, and options can be priced under the no-arbitrage hypothesis. We consider several parameterizations of the model and show advantages of specifying the demand curve as a quantity that is a function of price, as opposed to price as a function of quantity. An online appendix presents a basic empirical analysis of the model: calibration using information from actual order books, computation of option prices using Monte Carlo simulations, and comparison with observed data.

Keywords Liquidity modeling, Brownian sheet, Itô-Wentzell formula, No-arbitrage condition, Stochastic partial differential equations

2020 Mathematics Subject Classification Primary 91G80; 91B26; secondary 91G60

1. Introduction

Consider a market model in which equilibrium prices of assets are completely determined by the order flow, viewed as an exogenous process. In particular, there is no specialist, and every trader submits limit orders: for a buy order, the buyer specifies the maximum purchase price, or buy limit price, and, for a sell order, the seller specifies the minimum sale price, or sell limit price[1].

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[1]There is no loss of generality in that statement. A buy market order can be specified in our model as a buy limit order with a limit price equal to infinity. Since we model assets with only positive prices, a sell market order can be specified in our model as a sell limit order with a limit price equal to zero.

If, at a given moment in time, the buyer is unable to complete the entire order due to the shortage of sell orders at the required limit price, the unmatched part of the order is recorded in the order book. A symmetric outcome exists in the case of incoming sell orders. Subsequently, these unmatched buy orders may be matched with new incoming sell orders. We note that this is the operating procedure for many electronic exchanges, such as the New York Stock Exchange Arca. Time-priority is used to break indeterminacies of a match between an incoming buyer at limit price above the ask price, i.e., the lowest limit price in the sell order book. As a result, the equilibrium, or *clearing price* process, is always defined.

Since the matching mechanism does not add any information to the economy, all information about asset prices is included in the order flow. Different orders embed different information received by different traders, and thus, in full generality, a demand curve should have a large number of factors. An infinite-dimensional model makes it possible not only to represent any correlation structure in the net demand curve but also to preserve continuity. Santa Clara and Sornette [1] use a two-parameter Brownian motion, or Brownian sheet, to model the forward rate curve because "a discontinuous with respect to term forward curve $[\cdots]$ is intuitively unlikely". All the uncertainty in the economy is thus contained in this single Brownian sheet. We assume that claims can be replicated by trading at various points on the net demand curve, and thus the market is complete. It turns out that an infinite-dimensional model is not only economically plausible, but also mathematically necessary to prevent arbitrage.

Whether public exchanges should or should not reveal the order book data in real time is an important question in the financial markets community; see [2] and references therein. Our theoretical framework accommodates either viewpoint. However, our model is most useful in an economy where order books are public information, but where the position of the large trader is not known. The latest literature on high-frequency trading [3–5, etc.] confirms our viewpoint that traders (i) are interested in understanding order book information, and (ii) trade on that information.

We do not address in this paper the issue of differential information. The market microstructure literature, such as the Kyle model [6], considers various settings involving multiple uninformed, or noise, traders, and one or several informed traders. A key result of [6] is that, given the information available to noise traders, the resulting price process is a martingale with respect to a suitable measure, whereas this may not be the case for the informed traders. As a consequence, we do not believe that abstracting issues of differential information results in any loss of generality. The order books reflect all the public information. To avoid arbitrage, public information corresponds to the filtration under which the clearing price has an equivalent martingale measure; the clearing price may not be a martingale under that measure if the filtration is enlarged to include private information.

There are two main classes of models in the "no-arbitrage" literature on liquidity. The first class of models [7–16, etc.] considers the action of a large trader who can manipulate the prices in the market; our model belongs to this class. The large trader can employ one of the following two strategies: to "corner the market and squeeze the shorts" or to "front-run one's own trades". While some exchanges have rules to curtail cornering, front-running seems more difficult to control. In discrete-time trading, it is known that absence of arbitrage in periods where the large investor does not trade implies non-existence of a market manipulation strategy [12].

The second class of models, considered, for example, in [17–20], abstracts the issues of market manipulation away, and considers all traders as price-takers. In particular, [18] introduces an exogenous residual supply curve against which an investor trades. The investor trades market orders, the orders are matched instantaneously, and then, as pointed out in [7], it is reasonable to assume that the price effect of an order is limited to the very moment when the order is placed in the market, so that the residual supply curve at a future time is statistically independent from the order just matched.

The paper by Roch [21] attempts to bridge the gap between these two classes of models by analyzing a *linear* impact of the large trader on the demand price. By contrast, our model is not limited to a linear impact. In academic circles, the so-called "Rogers critique" of the [7] model holds that, hypothetically, a speculator can move underlying prices and thus generate unlimited profits in some derivatives markets, which is unrealistic. We are of the opinion that this critique neglects the microstructure of derivatives markets; there are many other arguments in support of the model from [7].

More recently, paper [22] investigates the ability of the large trader to impact stock price, both temporarily and permanently. The idea is to study European option pricing with liquidity risk as a utility maximization problem. Paper [23] extends [7] and allows the large trader to influence the stock price changes not only by the size of the trade, but also by the speed and timing of the execution of the trade. In [24], the financial market is modeled as a two-person stochastic differential zero-sum game, with one major player and N minor but collective players. The cost function satisfies a controlled backward stochastic differential equation, and the game is characterized using the Stackelberg feedback strategy. Other mathematical approaches to model order flow include stochastic partial differential equations (see [25] and references therein) and dimensional analysis [26].

The need for an infinite-dimensional model is immediately apparent if we consider a continuous demand curve, such as in [7], and check for arbitrage opportunity. Denote by $P(x,t)$ the market price at time t given that the large trader's position is x. If we assume that the price curve $P(x,t)$ is a continuous family of semimartingales, then the model becomes

$$\mathrm{d}P(x,t) = \mu_P(x,t)\mathrm{d}t + \sigma_P(x,t)\mathrm{d}B(t),$$

where B is Brownian motion. The market price of risk equation in this case is

$$\mu_P(x,t) = \lambda(t)\sigma_P(x,t),$$

where $\lambda = \lambda(t)$ is the unknown function, called the market price of risk. Unless the functions μ_p and σ_P have a particular form, the above equation has no solution, and thus the corresponding model is generically not free of arbitrage. The same observation applies to multiple factor models. In section 4, we give an explicit construction of an arbitrage strategy in such models.

A second novelty of our model is to consider the net demand as the quantity (number of shares) that is a function of price; traditionally[1], net demand is modeled as price that is a function of quantity. An advantage of our formulation is the ability to derive a risk-neutral pricing formula in a market that the large trader can manipulate using only strategies that are continuous functions of bounded variation; such strategies do not incur liquidity costs. We also show (see Remark 6.3) that there are technical modeling advantages in this approach. Finally, inverting the equations for price makes it possible to remove the monotonicity assumption in the price curve: the assumption is automatically satisfied if the resulting nonlinear stochastic partial differential equation for quantity has a solution. The equation, even in linearized form, is ill-posed, and therefore leads to a generically unstable model. Could it be a mathematical explanation why illiquid markets tend to be unstable [27]? We leave this interesting question for future research.

[1] A notable exception is [10].

We define our model in two steps. Step one is to consider a market with *small*, or *reference*, traders and a large trader, and to develop conditions on the net demand curve of the small traders such that the large trader cannot manipulate the market and, as a result, will refrain from trading large orders, or orders of infinite variation, that generate liquidity costs. Step two is to reduce the admissible strategies of the large trader to processes of finite variation. Generally, large trader strategies are not observable [28], and it seems then that a precise formula for options cannot be achieved. Fortunately, the risk-neutral pricing formula that we obtain, under the assumptions of completeness mentioned above, is the same for every large trader's strategy. Thus, even if the large trader can manipulate the underlying market, this manipulation does not affect option prices beyond the immediate impact of the trade on the current price of the underlying asset. This is an expedient feature of our model: we do not need to know the number of large traders for the option pricing formula to work.

The structure of the paper is as follows. Section 2 covers preliminaries on the market mechanism and two-parameter Brownian motion. Section 3 introduces then infinite-dimensional model, and section 4 illustrates the possibility of arbitrage in a finite-factor model. Section 5 derives general conditions for existence of an equivalent martingale measure. Section 6 introduces an option pricing formula in a market where a large trader can manipulate the price of the underlying asset. An online appendix, available at *https://www.ran-zhao.com/uploads/1/2/4/6/124680022/online_appendix.pdf* demonstrates an implementation of a particular version of the model matched to empirical data. The main objective is to show that a fairly technical infinite-dimensional model can be implemented, and the results are qualitatively correct. We hope that, should this paper become of interest among practitioners, more refined implementations will follow.

2. Preliminaries

2.1 The market mechanism

A buy limit order specifies how many shares a trader wants to buy, and the maximum purchase price per share; we call this price the (buy) *limit price*. A sell limit order specifies how many shares a trader wants to sell and the minimum sell price. We call this price the (sell) *limit price*. The unmatched buy and sell orders are stored in order books, until they are either canceled or matched with an incoming order. An incoming order is matched with the order on the opposite side of the market which has the best price. The clearing price of the transaction is equal to the limit price of the order in the book, and not of the incoming order. Partial execution is allowed, and ties are resolved by time-priority. Here is an example of the matching mechanism in discrete time, where at most one order arrives at time $t \in \{0, 1, 2, \cdots\}$.

Example 2.1 *Suppose that the clearing price at time* 0 *is any price* $P(0) \in [100, 120]$. *After clearing, that is, when* $0 < t < 1$ *we suppose that the order books contain the following orders:*

Buy Order Book	
Price	*Quantity*
100	10

Sell Order Book	
Price	*Quantity*
120	10
130	10

At time $t = 1$ *a buy order arrives with a limit price of \$125, and a quantity of 15. The exchange matches it with the best sell order, i.e., the one with a sell limit price of \$120. However, execution* *is only partial, and the remainder of the buy order is placed in the order book at the limit price of $125, resulting in the following order books:*

Buy Order Book	
Price	*Quantity*
100	10
125	5

Sell Order Book	
Price	*Quantity*
130	10

The clearing price at time 1 is equal to the limit price of the sell order:

$$P(1) = 120.$$

This example illustrates several properties of limit order markets:

- The clearing price is always defined and can assume any positive value[1].
- An incoming order can "cross" the order book, i.e., for the case of a buy order, its limit price is higher than the best sell order limit price (i.e, the best ask), since the buyer does not lose a cent. Crossing the book allows for faster execution. In our example, had the buyer submitted an order at price $130, the buyer would have immediately bought the desired quantity of 15 shares instead of waiting an indeterminate amount of time until enough sell orders arrive at the limit price.
- If several buy orders are simultaneously submitted and demand exceeds supply at the best ask, the buy orders with the highest limit price are executed first. Our own data analysis shows that few orders cross the Arca book. This is consistent with the theory of optimal order book placement suggested by Rosu [29].

2.2 Mathematical modeling

We now move to continuous time, omitting the technicalities related to convergence of a discrete time model. We start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. Equalities of random variables are to be understood almost surely unless stated otherwise. Likewise, after explaining that there exists a modification of a stochastic process that satisfies a certain property, we do not distinguish in the remainder of the text between the original process and the corresponding modification.

All the uncertainty is described by a one-dimensional Brownian sheet $B = B(s,t)$ for $0 \le s \le 1$ and $0 \le t \le T$, and the corresponding filtration $\big\{\mathcal{F}_t \equiv \sigma\big(B(s,t)\big); 0 \le s \le 1\big\}_{0 \le t \le T}$. There are three main approaches to constructing a Brownian sheet and the corresponding stochastic integral; cf. Mueller [30]:

(1) the martingale measure approach (Walsh [31, Chapter 2]),

(2) the Hilbert space approach (Da Prato and Zabczyk [32, Chapter 4]),

(3) the function space approach (Krylov [33, section 8.2]).

The Hilbert space approach covers the martingale measure approach [32, 30]; the function space approach covers the Hilbert space approach [33]. For a more detailed overview, see, for example, [34, sections 1.1.2 and 3.2.2].

To construct the stochastic integral with respect to the Brownian sheet using the function space approach, we take an orthonormal basis $\{\mathfrak{m}_n; n \ge 1\}$ in $L^2[0,1]$ and let $\{w_n,\ n \ge 1\}$ be independent standard Brownian motions on $[0,T]$. Define the random field

[1] We do not consider markets for swaps, where the price can be negative.

$$B(s,t) = \sum_{n=1}^{\infty} w_n(t) \int_0^s \mathfrak{m}_n(r)\mathrm{d}r, \quad s \in [0,1], \ t \in [0,T].$$

It follows that $B = B(s,t)$ is a Gaussian random field with mean zero and covariance

$$\mathbb{E}B(s,t)B(r,u) = \min(r,s) \cdot \min(t,u),$$

and then, by the Kolmogorov continuity criterion, B has a modification that is jointly continuous in (s,t) [31, Proposition 1.4]; this modification is called a Brownian sheet. If $b = b(s,t)$ is a random field such that

$$\int_0^T \int_0^1 \mathbb{E}b^2(s,t)\, \mathrm{d}s\mathrm{d}t < \infty, \tag{2.1}$$

and, for each $s \in [0,1]$, the process $b(s,\cdot)$ is $\mathcal{F}_t-$ adapted, then, by definition,

$$\int_0^t \int_0^s b(r,u)B(\mathrm{d}r,\mathrm{d}u) = \sum_{n\geq 1} \int_0^t \left(\int_0^s b(r,u)\mathfrak{m}_n(r)\mathrm{d}r \right) \mathrm{d}w_n(u). \tag{2.2}$$

Then Girsanov's Theorem (cf. [35, Theorem 2.2] or [32, Theorem 10.14]) can be stated as follows.

Theorem 2.2 *Suppose that $\lambda(s,\cdot)$ is an $\mathcal{F}_t$-predictable process for each $s \in [0,1]$, and that*

$$\mathbb{E}\left[\exp\left(\int_0^T \int_0^1 \lambda(s,u)B(\mathrm{d}s,\mathrm{d}u) - \frac{1}{2}\int_0^T \int_0^1 \lambda^2(s,u)\, \mathrm{d}s\, \mathrm{d}u \right) \right] = 1. \tag{2.3}$$

Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by

$$\mathrm{d}\mathbb{Q} = \exp\left(-\int_0^T \int_0^1 \lambda(s,u)B(\mathrm{d}s,\mathrm{d}u) - \frac{1}{2}\int_0^T \int_0^1 \lambda^2(s,u)\, \mathrm{d}s \right) \mathrm{d}\mathbb{P}. \tag{2.4}$$

Let

$$B^{\mathbb{Q}}(s,t) = B(s,t) + \int_0^t \int_0^s \lambda(r,u)\, \mathrm{d}r\, \mathrm{d}u.$$

Then, for each $s \in [0,1]$, the process $B^{\mathbb{Q}}(s,\cdot)$ is a standard Brownian motion with respect to $\{\mathcal{F}_t\}_{0\leq t\leq T}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

Corollary 2.3 *If $X = X(t), \ t \in [0,T]$, is a stochastic process with representation*

$$X(t) = X(0) + \int_0^t \int_0^1 \sigma_X(s,u)\lambda(s,u)\, \mathrm{d}s\, \mathrm{d}u + \int_0^t \int_0^1 \sigma_X(s,u)B(\mathrm{d}s,\mathrm{d}u),$$

then X is a martingale under the measure $\mathbb{Q}$.

2.3 Itô-wentzell formula

Let $F = F(x,t), \ x \in \mathbb{R}, \ t \in [0,T]$, be a random field, and let $g = g(t), \ t \in [0,T]$, be a stochastic process such that

$$F(x,t) = F(x,0) + \int_0^t \mu_F(x,u)\,\mathrm{d}u + \int_0^t \int_0^1 \sigma_F(x,s,u)B(\mathrm{d}s,\mathrm{d}u),$$
$$g(t) = g(0) + \int_0^t \mu_g(u)\,\mathrm{d}u + \int_0^t \int_0^1 \sigma_g(s,u)B(\mathrm{d}s,\mathrm{d}u).$$

Definition 2.4 *We say that the pair* (F,g) *satisfies the Itô-Wentzell conditions if*

(1) The random variables $F(x,0)$, $x \in \mathbb{R}$ *and* $g(0)$ *are* $\mathcal{F}_0$*-measurable;*

(2) Each of the processes $\mu_g(\cdot)$, $\sigma_g(r,\cdot)$, $\mu_F(x,\cdot)$, $x \in \mathbb{R}$, *and* $\sigma_F(x,r,\cdot)$ is $\mathcal{F}_t$*-adapted;*

(3) The functions F *and* g *are continuous in* t *;*

(4) The function F *is twice continuously differentiable in* x *and the function* σ_F *is continuously differentiable in* x *;*

(5) The following integrability condition holds:

$$\mathbb{E}[\mathcal{I}] < \infty, \tag{2.5}$$

where

$$\begin{aligned}\mathcal{I} = &\int_0^T \Big|\mu_F\big(g(u),u\big)\Big|\,\mathrm{d}u + \int_0^T \int_0^1 \sigma_F^2\big(g(u),s,u\big)\,\mathrm{d}s\,\mathrm{d}u \\ &+ \int_0^T \left|\frac{\partial F}{\partial x}\big(g(u),u\big)\,\mu_g(u)\right|\mathrm{d}u + \int_0^T \int_0^1 \left|\frac{\partial F}{\partial x}\big(g(u),u\big)\,\sigma_g(s,u)\right|^2 \mathrm{d}s\,\mathrm{d}u \\ &+ \int_0^T \int_0^1 \left|\frac{\partial^2 F}{\partial x^2}\big(g(u),u\big)\right| \sigma_g^2(s,u)\,\mathrm{d}s\,\mathrm{d}u \\ &+ \int_0^T \int_0^1 \left|\frac{\partial \sigma_F}{\partial x}\big(g(u),s,u\big)\,\sigma_g(s,u)\right|\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

Theorem 2.5 *If the pair* (F,g) *satisfies the Itô-Wentzell conditions, then*

$$\begin{aligned}F\big(g(t),t\big) - F\big(g(0),0\big) = &\int_0^t \mu_F\big(g(u),u\big)\,\mathrm{d}u + \int_0^t \int_0^1 \sigma_F\big(g(u),s,u\big)B(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t \frac{\partial F}{\partial x}\big(g(u),u\big)\,\mu_g(u)\,\mathrm{d}u \\ &+ \int_0^t \int_0^1 \frac{\partial F}{\partial x}\big(g(u),u\big)\,\sigma_g(s,u)B(\mathrm{d}s,\mathrm{d}u) \\ &+ \frac{1}{2}\int_0^t \int_0^1 \frac{\partial^2 F}{\partial x^2}\big(g(u),u\big)\,\sigma_g^2(s,u)\,\mathrm{d}s\,\mathrm{d}u \\ &+ \int_0^t \int_0^1 \frac{\partial \sigma_F}{\partial x}\big(g(u),s,u\big)\,\sigma_g(s,u)\,\mathrm{d}s\,\mathrm{d}u.\end{aligned} \tag{2.6}$$

Proof This follows by combining [36, Theorem 3.1] with (2.2). Alternatively, one can derive (2.6) from [37, Theorem 3.3.1] by writing

$$F(x,t) = F(x,0) + \int_0^t \mu_F(x,u)\,\mathrm{d}u + M_F(x,t),\ g(t) = g(0) + \int_0^t \mu_g(u)\,\mathrm{d}u + M_g(t),$$

and noting that

$$\langle M_g\rangle(t) = \int_0^t \int_0^1 \sigma_g^2(s,u)\,\mathrm{d}s\,\mathrm{d}u,$$
$$\left\langle \frac{\partial F}{\partial x}, g\right\rangle(t) = \left\langle \frac{\partial M_F}{\partial x}, M_g\right\rangle(t) = \int_0^t \int_0^1 \frac{\partial \sigma_F}{\partial x}\big(g(u),s,u\big)\,\sigma_g(s,u)\,\mathrm{d}s\,\mathrm{d}u,$$

as usual, $\langle\cdot\rangle$ denotes quadratic variation and $\langle\cdot,\cdot\rangle$ denotes quadratic covariation. Note that some form of (2.5) is necessary to define the right side of (2.6). □

3. A market with small traders and a large trader

The position of the large trader at time t is a predictable process $\theta=\theta(t)$, also known as the *large trader trading strategy.* The process θ must be a semimartingale satisfying

$$x_{\min} \le \theta(t) \le x_{\max}. \tag{3.1}$$

We view the collection of small traders as a continuum. Each small trader submits an infinitesimally small order at a price p, and there is no concentration of orders at any particular price. This lack of order concentration distinguishes small traders from the large trader.

Similar to [11, Assumption A1], we assume that the market is frictionless, that is, there are no transaction costs. We also assume that the buy and sell limit price p can take any value between 0 and S, and the orders can be submitted to the market at any time $t\in[0,T]$.

Definition 3.1 *The net demand curve $Q=Q(p,t,\omega)$ is a measurable real-valued function on $[0,S]\times[0,T]\times\Omega$. The number $Q(p,t,\omega)$ is equal to the difference between the total quantity of shares **submitted** for purchase at a price higher than or equal to p and the total quantity of shares **submitted** for sale at a price lower than p between time 0 and time t by small traders.*

Assumption Q1 *For each ω, the function $(p,t)\mapsto Q(p,t,\omega)$ is jointly measurable in (p,t) and, for every $t\in[0,T]$, is twice continuously differentiable and strictly decreasing in p.*

Note that, for a fixed price p, the total quantity of shares submitted, either for purchase or for sale, does not need to be increasing in time. Indeed, cancelled orders are directly subtracted from the submitted orders. As a result, there is no monotonicity condition on $Q(p,t,\omega)$ as a function of t.

In what follows, the elementary outcome ω will be omitted from the notations.

If N is the total number of shares outstanding on the market, then, for all $p\in(0,S)$ and $t\in[0,T]$,

$$-N \le Q(S,t) < Q(p,t) < Q(0,t) \le N. \tag{3.2}$$

The fact that the net demand curve is decreasing as a function of the price p is an immediate consequence of the definition. Indeed, the number of shares of available buy orders is decreasing with price, while the number of shares of available sell orders is increasing: buy low, sell high.

We now define the *clearing price* or *market price* $\pi^\theta(t)$ as the price where small traders trade with each other and/or with the large trader at time t. Since there is by assumption no bid-ask spread, $\pi^\theta(t)$ corresponds to the mid-price. Recall that the large trader orders are not included in Q by definition. Also, by definition of Q, and at time t, there are no outstanding buy limit orders in the order book at a price higher than $\pi^\theta(t)$ and there are no outstanding sell limit orders in the order book at a price lower than $\pi^\theta(t)$. Thus $Q\big(\pi^\theta,t\big)$, which is equal to the difference between *the number of buy orders submitted by t at price $\pi(t)$ or higher* (let us call it Q_B for a moment) and *the number of sell orders submitted by t at price $\pi(t)$ or lower* (call it Q_S) must have served to modify the position of the large trader. This gives us the *market clearing condition*:

$$Q\big(\pi^\theta(t), t\big) + \theta(t) = \theta(0). \tag{3.3}$$

For instance, if the large trader's position increases by $\Delta\theta(t) > 0$, then $Q_S = Q_B + \Delta\theta(t)$, expressing the fact that the excess sell orders (Q_B) have been sold to small buyers (Q_B) and to the large trader ($\Delta\theta(t)$). If θ is constant, the quantity Q is the (accumulated) content of the order book, sometimes called the limit order book imbalance. A sufficient condition for the market clearing condition to hold is that, for all $t \in [0, T]$,

$$Q(S, t) \le \theta(0) - x_{\max} < \theta(0) - x_{\min} \le Q(0, t). \tag{3.4}$$

Link with Bank and Baum (2004)

Denote by $P(x, t)$ the price available on the market at time $t \in [0, T]$ when the large trader's position is $x \in [x_{\min}, x_{\max}]$, with negative values of x corresponding to a short position. By "price available" we mean the price at which an infinitesimally small market order arriving at time t will be matched, whether a buy order or a sell order (recall that the bid/ask spread is always zero in our model). By definition,

$$P(\theta(t), t) = \pi^\theta(t). \tag{3.5}$$

Thus the market clearing condition (3.3) is satisfied if, for every $x \in [x_{\min}, x_{\max}]$,

$$Q\big(P(x, t), t\big) + x = C, \tag{3.6}$$

where C is a constant.

Remark 3.2 Equality (3.6) means that the sum of the number of shares held by the large trader and the current net demand at the corresponding price does not depend on time. From a mathematical point of view, the particular value of the constant on the right side of (3.6) is not important at this point, and we will set it to zero:

$$Q\big(P(x, t), t\big) + x = 0. \tag{3.7}$$

By Assumption **Q1**, $Q(p, t)$ is monotonically decreasing in p, and then (3.7) implies that $P(x, t)$ is monotonically increasing in x, which is exactly Assumption 2 in [7] or Assumption A3 in [11]. By (3.2), $P(x, t) \in [0, S]$ for all $x \in [Q(S, t), Q(0, t)]$, $t \in [0, T]$.

As shown in [7], optimal trading strategies are continuous, so, with no loss of generality, we assume that the process $t \mapsto \theta(t)$ is continuous. Denote by Θ the set of all trading strategies, that is, continuous semimartingales satisfying (3.1). We suppose that the interest rate is zero, as is common in the literature. The large trader (discounted) *holdings in the bank account* are denoted by $\beta^\theta(t)$ and satisfy

$$\beta^\theta(t) = \beta^\theta(0) - \int_0^t P(\theta(u), u)\mathrm{d}\theta(u) - \langle P(\theta, \cdot), \theta\rangle(t).$$

Asset prices are indeed impacted by the large trader strategy before they are actually exercised. As explained in [7], assume that the order is of size $\Delta\theta(t) > 0$. Then the large investor's bank account will be charged

$$\Delta\beta^\theta(t) = -P(\theta(t-), t)\Delta\theta(t) - \Delta P(\theta(t), t)\Delta\theta(t) = -P(\theta(t), t)\Delta\theta(t).$$

Thus the large trader has to pay $P(\theta(t), t)$ for each share. Since $P(x, t)$ is increasing in x, this price will be greater than or equal to the preorder price $P(\theta(t-), t)$. Next, define the *asymptotic liquidation proceeds* of the large trader in a fixed position ϑ at time t by

$$L(\vartheta,t)=\int_0^{\vartheta} P(x,t)\mathrm{d}x. \tag{3.8}$$

In the limit, as both ϑ and the duration for liquidation tend to zero, the quantity $L(\vartheta,t)$ is the position value. As it is more convenient to split the orders into smaller packages over short time periods, we decompose $L\big(\theta(t),t\big)$ into price effects for constant positions and position effects. The former is modeled by the stochastic integral $\int_0^t L\big(\theta(s),\mathrm{d}s\big)$, which is defined as in [37]. For simple integrands $\theta=\sum_i \theta_i 1_{(s_i,s_{i+1}]}$ with $0\le s_0\le\ldots\le s_n\le T$ we have

$$\int_0^t L\big(\theta(s),\mathrm{d}s\big)=\sum_i\{L(\theta(s_i\wedge t),s_{i+1}\wedge t)-L(\theta(s_i\wedge t),s_i\wedge t)\} \qquad (0\le t\le T),$$

and

$$\int_{s_i}^{s_{i+1}} L(\theta(t),\mathrm{d}t)=\int_0^{\theta_i}(P(x,s_{i+1})-P(x,s_i))\mathrm{d}x.$$

This part corresponds only to the change in price of the position, however, the large trader's position also changes. The full expression for the value of the position is given by the Itô-Wentzell formula

$$\begin{aligned}L(\theta(t),t)=L(\theta(0),0)&+\int_0^t L(\theta(s),\mathrm{d}s)+\int_0^t P(\theta(s),s)\mathrm{d}\theta(s)\\&+\left\langle\int_0^{\cdot} P(\theta(s),\mathrm{d}s),\theta\right\rangle(t)+\frac{1}{2}\int_0^t P'(\theta(s),s)\mathrm{d}\langle\theta\rangle(s),\end{aligned}$$

where $\langle\cdot\rangle$ is the quadratic variation and $\langle\cdot,\cdot\rangle$ is the quadratic covariation. The *realizable wealth* of the large trader achieved by the trading strategy θ is denoted by $V^\theta(t)$ and satisfies

$$V^\theta(t)=\beta^\theta(t)+L\big(\theta(t),t\big).$$

Proposition 3.3 *For every* $\theta\in\Theta$,

$$V^\theta(t)-V^\theta(0)=\int_0^t L\big(\theta(u),\mathrm{d}u\big)-\frac{1}{2}\int_0^t\frac{\partial P}{\partial x}\big(\theta(u),u\big)\mathrm{d}\langle\theta\rangle(u).$$

Proof This follows by the Itô-Wentzell formula; see [7, Lemma 3.2] for details. □

Corollary 3.4 *If the process*

$$t\mapsto\int_0^t L\big(\theta(u),\mathrm{d}u\big),\ \ t\in[0,T],$$

is a local martingale under an equivalent martingale measure $\mathbb{Q}$*, then the realizable wealth* V^θ *is a supermartingale under* $\mathbb{Q}$.

Proof Indeed,

$$\int_0^t\frac{\partial P}{\partial x}\big(\theta(u),u\big)\mathrm{d}\langle\theta\rangle(u)\ge 0$$

for all $t\ge0$, because the process $t\mapsto\langle\theta\rangle(t)$ is increasing and, by (3.7), $\partial P(x,t)/\partial x>0$.

□

Definition 3.5 *An arbitrage strategy is a trading strategy* $\theta\in\Theta$ *such that*

$$V^{\theta}(0)=0,\ \mathbb{P}\big(V^{\theta}(T)\geq 0\big)=1,\ \mathbb{P}\big(V^{\theta}(T)>0\big)>0.$$

A market model admits arbitrage if there exists an arbitrage strategy.

4. Arbitrage in finite-factor models

4.1 One-stock example

Take twice continuously differentiable functions F and r such that F is positive and r is increasing. Letting W be a standard Brownian motion, we define

$$P(x,t)=\int_0^t \mu(u)\mathrm{d}u+r(x)F(W(t)).$$

By Itô's lemma,

$$\mathrm{d}P(x,t)=\big(\mu(t)+\frac{1}{2}r(x)F''(W(t))\big)\mathrm{d}t+r(x)F'(W(t))\mathrm{d}W(t).$$

Denote by R the antiderivative of r such that $R(0)=0$. By Fubini's theorem for stochastic integrals [38, Theorem IV.46],

$$\begin{aligned}\int_0^t L(\theta,\mathrm{d}u)&=\int_0^t\left(\int_0^{\theta(u)}\mathrm{d}P(x,u)\right)\mathrm{d}u\\&=\int_0^t\mu(u)\theta(u)\,\mathrm{d}u+\frac{1}{2}\int_0^t F''(W(u))\left(\int_0^{\theta(u)}r(x)\,\mathrm{d}x\right)\mathrm{d}u\\&\quad+\int_0^t F'(W(u))\left(\int_0^{\theta(u)}r(x)\,\mathrm{d}x\right)\mathrm{d}W(u)\\&=\int_0^t\mu(u)\theta(u)\,\mathrm{d}u+\frac{1}{2}\int_0^t F''(W(u))R(\theta(u))\,\mathrm{d}u\\&\quad+\int_0^t F'(W(u)R(\theta(u))\mathrm{d}W(u).\end{aligned}$$

We now make an additional assumption: $R(\bar{x})=0$ for some $\bar{x}\neq 0$. Then we can eliminate all risk terms by setting $\theta(u)=\bar{\theta}(u)\equiv\bar{x}$ so that $R(\bar{\theta}(u))=0$.

Because $\langle\bar{\theta}\rangle=0$, Proposition 3.3 implies $V^{\bar{\theta}}(u)-V^{\bar{\theta}}(0)=\int_0^t L(\bar{\theta},\mathrm{d}u)$, or

$$V^{\bar{\theta}}(u)-V^{\bar{\theta}}(0)=\bar{x}\int_0^t\mu(u)\,\mathrm{d}u.$$

Then the strategy $\bar{\theta}$ is an arbitrage for the large trader as long as $\bar{x}\mu(t)>0$ for $t>0$.

As a concrete example, take $\mu(t)\equiv 1$, $r(x)=x-1$, so that $R(x)=\big((x-1)^2-1\big)/2$, $\bar{x}=2$, and $\bar{\theta}(u)\equiv 2$ is an arbitrage strategy.

While illustrating the main point (existence of arbitrage), the one-stock example is not completely realistic because the volatility of $P(x_1,t)$ has always the opposite sign of the volatility of $P(x_2,t)$ when $x_1<1<x_2$. Accordingly, we will now construct a more realistic example with two stocks.

4.2 A two-stocks example

The following example generalizes [39, Example 5.4.4] in the perfectly liquid case. While in the rest of the article we only consider models with a single stock, it is well-known that arbitrage, should it occur, is a characteristic of the underlying stock model, and, in the perfectly liquid case, one often needs at least two stocks to generate arbitrage. Accordingly, we now consider a two-stock model

$$\begin{aligned}\mathrm{d}P^{(1)}(x,t) &= \mu_P^{(1)}(t)\mathrm{d}t + \sigma_{P,1}^{(1)}(t)\mathrm{d}W_1(t) + \sigma_{P,2}^{(1)}(t)\mathrm{d}W_2(t),\\ \mathrm{d}P^{(2)}(x,t) &= \mu_P^{(2)}(x,t)\mathrm{d}t + \sigma_{P,1}^{(2)}(t)\mathrm{d}W_1(t) + \sigma_{P,2}^{(2)}(x,t)\mathrm{d}W_2(t),\end{aligned}$$

where W_1 and W_2 are independent standard Brownian motions. To make $P^{(2)}$ increasing in x, we take a positive, twice continuously differentiable function $F = F(y)$, $y \in \mathbb{R}$, and set

$$\mu_P^{(2)}(x,t) = \mu^{(2)}(t) + \frac{1}{2}xF''(W_2(t)), \quad \sigma_{P,2}^{(2)}(x,t) = xF'(W_2(t)).$$

We assume that all other coefficients are adapted stochastic processes and $\sigma_{P,1}^{(1)} \neq 0$.

A large trader's strategy is a pair of continuous processes $\theta = (\theta^{(1)}, \theta^{(2)})$, representing the positions in the two stocks. Under normal conditions, realizable wealth of the portfolio is the sum of the realizable wealth of each stock position, and therefore, for every strategy θ of the large trader, Fubini's Theorem implies

$$\begin{aligned}\int_0^t L(\theta,\mathrm{d}u) &= \int_0^t \left(\mu_P^{(1)}(u)\theta^{(1)}(u) + \mu^{(2)}(u)\theta^{(2)}(u) + \frac{1}{4}(\theta^{(2)}(u))^2 F''(W_2(u))\right)\mathrm{d}u\\ &\quad + \int_0^t S_1(u)\mathrm{d}W_1(u) + \int_0^t S_2(u)\mathrm{d}W_2(u),\end{aligned}$$

where

$$\begin{aligned}S_1 &= \sigma_{P,1}^{(1)}\theta^{(1)} + \sigma_{P,1}^{(2)}\theta^{(2)},\\ S_2 &= \sigma_{P,2}^{(1)}\theta^{(1)} + \frac{F'(W_2(t))(\theta^{(2)})^2}{2}.\end{aligned}$$

To achieve arbitrage, we set S_1 and S_2 to zero. Eliminating $\theta^{(1)}$ yields

$$\sigma_{P,2}^{(1)}\sigma_{P,1}^{(2)}\theta^{(2)} - \frac{\sigma_{P,1}^{(1)}F'(W_2(t))(\theta^{(2)})^2}{2} = 0.$$

Under an additional assumption $F'(y) > 0$, a non-zero solution is $\bar\theta = (\bar\theta^{(1)}, \bar\theta^{(2)})$ with

$$\bar\theta^{(1)} = -2\frac{\sigma_{P,2}^{(1)}\left(\sigma_{P,1}^{(2)}\right)^2}{\left(\sigma_{P,1}^{(1)}\right)^2 F'(W_2(t))}, \tag{4.1}$$

$$\bar\theta^{(2)} = 2\frac{\sigma_{P,2}^{(1)}\sigma_{P,1}^{(2)}}{\sigma_{P,1}^{(1)}F'(W_2(t))}. \tag{4.2}$$

For the strategy $\bar\theta$,

$$\int_0^t L(\bar\theta,\mathrm{d}u) = \int_0^t \left(\mu_P^{(1)}(u)\bar\theta^{(1)}(u) + \mu^{(2)}(u)\bar\theta^{(2)}(u) + \frac{1}{4}\left(\bar\theta^{(2)}(u)\right)^2 F''(W_2(u))\right)\mathrm{d}u.$$

Because $\dfrac{\partial P^{(2)}}{\partial x} = F(W_2(t))$,

$$\begin{aligned}V^{\bar\theta}(t) - V^{\bar\theta}(0) &= \int_0^t \left(\bar\theta^{(2)}(u)\left(-\frac{\sigma_{P,1}^{(2)}}{\sigma_{P,1}^{(1)}}\mu_P^{(1)} + \mu^{(2)}\right)(u) + \frac{1}{4}\left(\bar\theta^{(2)}(u)\right)^2 F''(W_2(u))\right)\mathrm{d}u\\ &\quad - \frac{1}{2}\int_0^t F(W_2(u))\mathrm{d}\langle\bar\theta^{(2)}\rangle(u).\end{aligned}$$

Taking, for instance, constant $\sigma^{(2)}_{P,1}, \sigma^{(1)}_{P,1}, \sigma^{(1)}_{P,2}, \mu^{(1)}_P, \mu^{(2)}$, and $F(y) = e^y$, direct computations show that arbitrage occurs if

$$\mu^{(2)} > \frac{\sigma^{(2)}_{P,1}}{\sigma^{(1)}_{P,1}} \mu^{(1)}_P + \frac{\sigma^{(1)}_{P,2}\sigma^{(2)}_{P,1}}{2\sigma^{(1)}_{P,1}}.$$

We note that the above construction of an arbitrage strategy is not possible in an infinite-dimensional model. Indeed, if

$$\begin{aligned} \mathrm{d}P^{(1)}(x,t) &= \mu^{(1)}_{P,1}(t)\mathrm{d}t + \sigma^{(1)}_{P,1}(t) \int_0^1 b^{(1)}_{P,1}(x,s,t) B(\mathrm{d}s,\mathrm{d}t) \\ &\quad + \sigma^{(1)}_{P,2}(t) \int_0^1 b^{(1)}_{P,2}(x,s,t) B(\mathrm{d}s,\mathrm{d}t), \\ \mathrm{d}P^{(2)}(x,t) &= \mu^{(2)}_{P,1}(t)\mathrm{d}t + \sigma^{(2)}_{P,1}(t) \int_0^1 b^{(2)}_{P,1}(x,s,t) B(\mathrm{d}s,\mathrm{d}t) \\ &\quad + \sigma^{(2)}_{P,2}(t) \int_0^1 b^{(2)}_{P,2}(x,s,t) B(\mathrm{d}s,\mathrm{d}t), \end{aligned}$$

then the realizable wealth of the large trader is

$$\begin{aligned} \int_0^t L(\theta,\mathrm{d}u) &= \int_0^t \left(\theta^{(1)}(u)\mu^{(1)}_P(u) + \theta^{(2)}(u)\mu^{(2)}_P(u)\right) \mathrm{d}u \\ &\quad + \int_0^t \int_0^1 S(s,u) B(\mathrm{d}s,\mathrm{d}u), \end{aligned}$$

where

$$\begin{aligned} S(s,\cdot) &= \int_0^{\theta^{(1)}} \left(\sigma^{(1)}_{P,1} b^{(1)}_{P,1}(x,s,\cdot) + \sigma^{(1)}_{P,2} b^{(1)}_{P,2}(x,s,\cdot)\right) \mathrm{d}x \\ &\quad + \int_0^{\theta^{(2)}} \left(\sigma^{(2)}_{P,1} b^{(2)}_{P,1}(x,s,\cdot) + \sigma^{(2)}_{P,2} b^{(2)}_{P,2}(x,s,\cdot)\right) \mathrm{d}x, \end{aligned}$$

and we see that no pair of functions $\theta = (\theta^{(1)}, \theta^{(2)})$ can ensure $S(s,\cdot) = 0$ for all $s \in [0,1]$. The exception is a very special case:

$$\begin{aligned} b^{(1)}_{P,1}(x,s,u) &= b^{(2)}_{P,1}(x,s,u) = 2 \cdot 1_{\left[s \le \frac{1}{2}\right]}, \\ b^{(1)}_{P,2}(x,s,u) &= b^{(2)}_{P,2}(x,s,u) = 2 \cdot 1_{\left[\frac{1}{2} < s\right]}, \end{aligned}$$

which is equivalent to (4.1). An interested reader can analyze this infinite-dimensional two-stock example using the ideas from the next section and, in particular, verify the corresponding no-arbitrage conditions.

4.3 A market model with several stocks

The examples from the previous subsections are not a mere curiosity, but an illustration of a more general phenomenon: the existence of arbitrage in a model with n stocks, n non-necessarily independent Brownian motions $W^{(i)}$, and separable volatility. More specifically, for $i = 1, \cdots, n$, take positive, twice continuously differentiable functions $F^{(i)} = F^{(i)}(y),\ y \in \mathbb{R}$, continuous increasing functions $r^{(i)} = r^{(i)}(x),\ x \in \mathbb{R}$, continuous functions $\mu^{(i)} = \mu^{(i)}(t),\ t > 0$, and define

$$P^{(i)}(x^{(i)}, t) = \int_0^t \mu^{(i)}(u)\mathrm{d}u + r^{(i)}(x^{(i)}) F^{(i)}(W^{(i)}(t)).$$

Suppose that $R^{(i)}$ is the antiderivative of $r^{(i)}$ satisfying $R^{(i)}(0) = 0$, and let $\bar{x}^{(i)}$ satisfy $R^{(i)}(x^{(i)}) = 0$ and $\mu^{(i)}(t)\bar{x}^{(i)} > 0$ for all $t > 0$. Then direct computations, similar to the one- and two-stock examples, show that the model admits arbitrage.

5. Conditions for absence of arbitrage

Assumption Q2 *For every $p \in [0, S]$, the process $t \mapsto Q(p,t)$, $t \in [0,T]$ admits a representation*

$$\mathrm{d}Q(p,t) = \mu_Q(p,t)\mathrm{d}t + \sigma_Q(p,t)\int_0^1 b_Q(p,s,t)B(\mathrm{d}s,\mathrm{d}t), \tag{5.1}$$

$$\int_0^1 b_Q^2(p,s,t)\mathrm{d}s = 1, \tag{5.2}$$

where the processes $\mu_Q(p,\cdot)$, $\sigma_Q(p,\cdot)$, and $b_Q(p,s,\cdot)$ are $\mathbb{R}$-valued and $\mathcal{F}_t$-adapted.

Remark 5.1 *Equality (5.1) is the usual semimartingale condition on the process Q. While not every process (5.1) is monotone in p, a straightforward way to ensure monotonicity is to set*

$$Q(p,t) = \Psi\big(p,t,B\big),$$

for some smooth function Ψ that is strictly decreasing in the first argument. Other examples are below in this section.

Condition (5.2) is a standard normalization, which, with the presence of σ_Q, leads to no loss of generality.

Assumption Q3 *The function $Q = Q(p,t)$ is twice continuously differentiable with respect to p for every t and is continuous in t for every p, and the function*

$$\tilde{\sigma}_Q(p,s,t) = \sigma_Q(p,t)b_Q(p,s,t)$$

is continuously differentiable with respect to p for every s and t.

For notational convenience, we introduce the function

$$C(p,t) = -\frac{1}{\dfrac{\partial Q}{\partial p}(p,t)}\int_0^1 \frac{\partial \tilde{\sigma}_Q(p,s,t)}{\partial p}\tilde{\sigma}_Q(p,s,t)\mathrm{d}s.$$

Theorem 5.2 *Suppose that Assumptions Q1, Q2, and Q3 hold. Then, for every $x \in [x_{\min}, x_{\max}]$, the price process $t \mapsto P(x,\cdot)$ satisfies*

$$\mathrm{d}P(x,t) = \mu_P(x,t)\mathrm{d}t + \sigma_P(x,t)\int_0^1 b_P(x,s,t)B(\mathrm{d}s,\mathrm{d}t), \tag{5.3}$$

where

$$\mu_P(x,t) = -\frac{\mu_Q\big(P(x,t),t\big) + \dfrac{1}{2}\dfrac{\partial^2 Q}{\partial p^2}\big(P(x,t),t\big)\sigma_P^2\big(x,t\big) + C\big(P(x,t),t\big)}{\dfrac{\partial Q}{\partial p}\big(P(x,t),t\big)}, \tag{5.4}$$

$$\sigma_P(x,t) = \frac{\sigma_Q\big(P(x,t),t\big)}{\dfrac{\partial Q}{\partial p}\big(P(x,t),t\big)}, \tag{5.5}$$

$$b_P(x,s,t) = -b_Q\big(P(x,t),s,t\big). \tag{5.6}$$

Proof Monotonicity of Q implies that the process P defined by (3.7) exists and is unique, and $\frac{\partial Q}{\partial p} < 0$, so that (i) the expression (5.4) is well-defined and (ii) $P(x,t) \in [0,S]$. Boundedness of Q implies that (2.5) holds.

Using (2.6), we now verify that (5.3) satisfies the market clearing equation (3.7)

$$\begin{aligned} 0 = \mathrm{d}Q\big(P(x,t),t\big) &= \mu_Q\big(P(x,t),t\big)\mathrm{d}t + \int_0^1 \tilde{\sigma}_Q\big(P(x,t),s,t\big)B(\mathrm{d}s,\mathrm{d}t) \\ &\quad + \frac{\partial Q}{\partial p}\big(P(x,t),t\big)\left(\mu_P(x,t)\mathrm{d}t + \sigma_P(x,t)\int_0^1 b_P(x,s,t)B(\mathrm{d}s,\mathrm{d}t)\right) \\ &\quad + \frac{\sigma_P^2(x,t)}{2}\frac{\partial^2 Q}{\partial p^2}\big(P(x,t),t\big)\mathrm{d}t \\ &\quad + \sigma_P(x,t)\left(\int_0^1 \frac{\partial \tilde{\sigma}_Q}{\partial p}\big(P(x,t),s,t\big)b_P(x,s,t)\mathrm{d}s\right)\mathrm{d}t. \end{aligned} \tag{5.7}$$

Setting the martingale component in (5.7) equal to zero yields (5.5) and (5.6). After that, setting the drift component in (5.7) equal to zero yields (5.4). □

Next, we investigate the conditions for existence of an equivalent martingale measure, that is, the measure under which the price process P is a martingale. For notational convenience, we define

$$A(p,t) = \mu_Q(p,t) + \frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(p,t)\left(\frac{\sigma_Q(p,t)}{\frac{\partial Q}{\partial p}(p,t)}\right)^2 + C(p,t).$$

Definition 5.3 *The market price of risk is a random function $\lambda = \lambda(s,t)$ such that*

$$\int_0^1 \tilde{\sigma}_Q(p,s,t)\lambda(s,t)\mathrm{d}s = A(p,t), \;\; p \in [0,S],\; t \in [0,T], \tag{5.8}$$

and, for every $s \in [0,1]$ and $t \in [0,T]$, the random variable $\lambda(s,t)$ is $\mathcal{F}_t$-measurable. We call equation (5.8) *the market price of risk equation in demand format.*

Remark 5.4 *While we consider Q as the main input (or primitive) for the underlying model, it is possible, by (3.7), to take P instead of Q as the corresponding primitive. Then, by (5.4) and (5.5), formula (5.9) becomes*

$$\int_0^1 \tilde{\sigma}_P(x,s,t)\lambda(s,t)\mathrm{d}s = \mu_P(x,t), \quad x \in [x_{\min}, x_{\max}],\; t \in [0,T], \tag{5.9}$$

with $\tilde{\sigma}_P(x,s,t) = \sigma_P(x,t)b_P(x,s,t)$. We call (5.9) the market price of risk equation in price format.

Theorem 5.5 *In addition to Assumptions **Q1**, **Q2**, and **Q3**, suppose that equation* (5.8) *has a solution $\lambda = \lambda(s,t)$ satisfying* (2.3). *Define the measure $\mathbb{Q}$ according to* (2.4). *Then, for every $x \in [x_{\min}, x_{\max}]$, the price process $t \mapsto P(x,t)$ is a martingale with respect to the measure $\mathbb{Q}$.*

Proof By Theorem 5.2, the process P has the representation

$$\mathrm{d}P(x,t) = -\frac{A\big(P(x,t),t\big)}{\frac{\partial Q}{\partial p}\big(P(x,t),t\big)}\,\mathrm{d}t - \frac{\sigma_Q\big(P(x,t),t\big)}{\frac{\partial Q}{\partial p}\big(P(x,t),t\big)}\int_0^1 b_Q\big(P(x,t),s,t\big)B(\mathrm{d}s,\mathrm{d}t).$$

By Corollary 2.3, this process is a martingale with respect to the measure $\mathbb{Q}$ from (2.4) if equality (5.8) holds; condition (2.3) ensures that the measure $\mathbb{Q}$ is well-defined. □

Combining Theorem 5.5 with [7, Theorem 3.3], we conclude that, under Assumptions Q1, Q2, and Q3, there is no arbitrage in our model.

Equation (5.8) is a Fredholm integral equation of the first kind and has a solution if and only if, for every $t \in [0,T]$, and every ω, the right side $A(p,t)$ is in the range of the integral operator

$$\mathcal{A}_t : f(s) \mapsto \int_0^1 \tilde{\sigma}_Q(p,s,t) f(s) \mathrm{d}s.$$

We now investigate models under which the market price of risk equations admit a solution. In subsections 5.1 and 5.2, the input of the model is the net demand Q, so that we investigate whether (5.8) has a solution, while in subsection 5.3, the input of the model is the price process P, so we investigate whether (5.9) has a solution. While having P as the starting point results in market price of risk equations that are simpler to solve, there is also a subtle drawback of working with P instead of with Q; see Remark 6.3 below.

5.1 A linear model

A sufficient condition for the existence of an equivalent martingale measure is existence of a *bounded* solution λ of (5.8). This condition is easy to verify when the demand curve is linear in p; cf. [21].

Proposition 5.6 *Assume that the demand curve depends linearly on p so that*

$$\frac{\dfrac{\partial \tilde{\sigma}_Q}{\partial p}(p,s,t)}{\dfrac{\partial Q}{\partial p}(p,t)} = h_Q(s,t) \quad \textit{and} \quad \frac{\partial^2 Q}{\partial p^2}(p,t) = 0,$$

with a bounded function h_Q. If there exists a bounded function $\lambda_Q = \lambda_Q(s,t)$ such that, for every $t \in [0,T]$,

$$\mu_Q(p,t) = \int_0^1 \tilde{\sigma}_Q(p,s,t)\lambda_Q(s,t)\mathrm{d}s, \tag{5.10}$$

then equation (5.8) has a bounded solution

$$\lambda(s,t) = \lambda_Q(s,t) - h_Q(s,t). \tag{5.11}$$

Proof Under the assumptions of the proposition, equation (5.8) becomes

$$\int_0^1 \tilde{\sigma}_Q(p,s,t)\lambda(s,t)\mathrm{d}s = \int_0^1 \tilde{\sigma}_Q(p,s,t)\lambda_Q(s,t)\mathrm{d}s - \int_0^1 \tilde{\sigma}_Q(p,s,t)h_Q(s,t)\mathrm{d}s,$$

and then (5.11) follows. □

Combining Proposition 5.6 with [21, Theorem 2.6], we conclude that, in the case of the linear demand curve, there are no arbitrage opportunities as long as condition (5.10) holds. Note that (5.10) is similar to Condition C.4 in the original Heath–Jarrow–Morton model [40].

5.2 A separated model

An extension of a linear demand model is a *separated* demand curve

$$Q(p,t) = \sigma(p) F\left(\int_0^t \int_0^1 b(s,u) B(\mathrm{d}s, \mathrm{d}u)\right).$$

Proposition 5.7 *Assume that*

$$\int_0^1 b^2(s,u)\mathrm{d}s = 1,$$

for all $u \in [0,S]$ *, and also*

$$\begin{aligned} &F(x) \geq \delta > 0,\ |F'(x)| \leq C,\ |F''(x)| \leq C, \\ &\sigma'(p) < 0,\ |b(t,u)| \leq C,\ |F'(x)| \geq \delta > 0, \\ &\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sigma''(p)\sigma(p)}{\big(\sigma'(p)\big)^2}\right) = 0. \end{aligned} \tag{5.12}$$

Then equation (5.8) *has a bounded solution.*

Proof By the Itô formula,

$$\begin{aligned} &\mu_Q(p,t) = \sigma(p)h_2(t),\ \ b_Q(t,s) = b(t,s), \\ &\sigma_Q(p,t) = \sigma(p)h_1(t),\ \ \tilde{\sigma}_Q(p,t,s) = \sigma(p)b(s,t)h_1(t), \end{aligned}$$

where

$$h_1(t) = F'\left(\int_0^t\int_0^1 b(s,u)B(\mathrm{d}s,\mathrm{d}u)\right),\ h_2(t) = \frac{1}{2}F''\left(\int_0^t\int_0^1 b(s,u)B(\mathrm{d}s,\mathrm{d}u)\right).$$

By direct computation, the bounded solution of equation (5.8) is

$$\lambda(s,t) = b(s,t)\left(\frac{h_2(t)}{h_1(t)} + \frac{h_1(t)\sigma_0}{2h_0(t)} - \frac{h_1(t)}{h_0(t)}\right),$$

with $h_0(t) = F\left(\int_0^t\int_0^1 b(s,u)B(\mathrm{d}s,\mathrm{d}u)\right)$ and $\sigma_0 = \dfrac{\sigma''(p)\sigma(p)}{\big(\sigma'(p)\big)^2}$. □

Equation (5.12) defines a three-parameter family of functions σ. This family includes the linear function, corresponding to $\sigma_0 = 0$.

5.3 A lognormal model

In this model, the input is P, the price as a function of quantity, and the main objective becomes analysis of the market price of risk equations in price format (5.9). To solve the resulting Fredholm equation of the first kind, we will transform it to a Volterra equation.

Let $0 < \varepsilon < 1/2$. We define a scale function mapping the quantity variable x to the noise variable s:

$$f(x) = 2\varepsilon + (1-2\varepsilon)\frac{x - x_{\min}}{x_{\max} - x_{\min}},$$

note that $2\varepsilon \leq f(x) \leq 1$ for $x \in [x_{\min}, x_{\max}]$.

Introduce non-random functions $p_0 = p_0(x)$, $\bar{\mu}_p = \bar{\mu}_p(x)$, and $\bar{\sigma} = \bar{\sigma}_p(x,s)$ and suppose that p_0 and $\bar{\mu}_p$ are $\mathcal{C}^1$ (that is, bounded and continuously differentiable, with a bounded derivative) and $\bar{\sigma}$ is $\mathcal{C}^1$ in x, uniformly with respect to s. We also assume that

- $\bar{\sigma}_p(x,s) = 0$ for $s \leq \varepsilon$;
- $\bar{\sigma}_p\big(x, f(x)\big)$ is uniformly bounded from zero for $x \in [x_{\min}, x_{\max}]$;
- $\bar{\sigma}_p(x,s) = 0$ for $s > f(x)$;
- $\int_0^{\varepsilon} \bar{\sigma}_p(x_{\min}, s)\mathrm{d}s \neq 0$;
- $\int_{\varepsilon}^{2\varepsilon} \bar{\sigma}_p(x_{\min}, s)\mathrm{d}s \neq 0$.

Define the *price density function*

$$p(x,t) = p_0(x)\exp\left(\bar{\mu}_p(x)t + \int_\varepsilon^1 \bar{\sigma}_p(x,s)B(ds,t) - \frac{1}{2}\int_\varepsilon^1 \bar{\sigma}_p^2(x,s)\mathrm{d}s\right), \tag{5.13}$$

and then the price process

$$P(x,t) = p(x_{\min},t) + \int_{x_{\min}}^x p(y,t)\mathrm{d}y, \quad x\in[x_{\min},x_{\max}].$$

Proposition 5.8 *Equation* (5.9) *has a bounded solution.*

Proof By (5.13),

$$p(x,t) = p_0(x) + \int_0^t \bar{\mu}_p(x)p(x,r)\mathrm{d}r + \int_0^t\int_\varepsilon^1 \bar{\sigma}_p(x,s)B(\mathrm{d}s,\mathrm{d}r),$$

and therefore the corresponding coefficients in (5.3) are

$$\tilde{\sigma}_P(x,s,t) = \bar{\sigma}_P(x_{\min},s)p(x_{\min},t) + \int_{x_{\min}}^x \bar{\sigma}_p(y,s)p(y,t)\mathrm{d}y,$$
$$\mu_P(x,t) = \bar{\mu}_P(x_{\min})p(x_{\min},t) + \int_{x_{\min}}^x \bar{\mu}_p(y)p(y,t)\mathrm{d}y.$$

We fix $t\in[0,T]$ and construct the solution $\lambda(s,t)$ of (5.9) in three steps: first, for $s\in[\varepsilon,2\varepsilon)$, then, for $s\in(2\varepsilon,1]$, and finally, for $s\in[0,\varepsilon)$.

To begin, differentiate both sides of (5.9) with respect to x:

$$\int_\varepsilon^1 \bar{\sigma}_p(x,s)p(x,t)\lambda(s,t)\mathrm{d}s = \bar{\mu}_p(x)p(x,t), \quad x_{\min}\le x\le x_{\max},\ t\in[0,T].$$

Next, divide both sides by $p(x,t)$:

$$\int_\varepsilon^1 \bar{\sigma}_p(x,s)\lambda(s,t)\mathrm{d}s = \bar{\mu}_p(x), \quad x_{\min}\le x\le x_{\max},\ t\in[0,T].$$

Since $\bar{\sigma}_p(x,s)=0$ for $s>f(x)$,

$$\int_\varepsilon^{f(x)} \bar{\sigma}_p(x,s)\lambda(s,t)\mathrm{d}s = \bar{\mu}_p(x), \quad x_{\min}\le x\le x_{\max},\ t\in[0,T]. \tag{5.14}$$

Evaluating this equation at $x=x_{\min}$ leads to a constant value for $\lambda(s,t)$ when $s\in[\varepsilon,2\varepsilon)$:

$$\lambda(s,t) = \frac{\bar{\mu}_p(x_{\min})}{\displaystyle\int_\varepsilon^{2\varepsilon}\bar{\sigma}_p(x_{\min},s)\mathrm{d}s}.$$

Next, differentiation of (5.14) with respect to x defines $\lambda(s,t)$ for $s\in[2\varepsilon,1]$:

$$\lambda\big(f(x),t\big) = \frac{1}{(1-2\varepsilon)\bar{\sigma}_p\big(x,f(x)\big)}\left(\frac{\partial\bar{\mu}_p(x)}{\partial x} - \int_\varepsilon^{f(x)}\frac{\partial\bar{\sigma}_p(x,s)}{\partial x}\lambda(s,t)\mathrm{d}s\right);$$

recall that $x\mapsto f(x)$ is a bijection from $[x_{\min},x_{\max}]$ to $[2\varepsilon,1]$.

Finally, we define $\lambda(s,t)$ for $s\in[0,\varepsilon)$. To this end, we evaluate (5.9) at $x=x_{\min}$:

$$\int_0^\varepsilon \bar{\sigma}_p(x_{\min},s)p(x_{\min},t)\lambda(s,t)\mathrm{d}s$$
$$= \bar{\mu}_p(x_{\min})p(x_{\min},t) - \int_\varepsilon^1 \bar{\sigma}_p(x_{\min},s)p(x_{\min},t)\lambda(s,t)\mathrm{d}s,$$

and choose a constant solution:

$$\lambda(s,t) = \frac{\bar{\mu}_p(x_{\min}) - \displaystyle\int_\varepsilon^1 \bar{\sigma}_p(x_{\min},s)\lambda(s,t)\mathrm{d}s}{\varepsilon \displaystyle\int_0^\varepsilon \bar{\sigma}_p(x_{\min},s)\mathrm{d}s}, \quad 0 \le s < \varepsilon.$$

□

Remark 5.9 *It is possible to represent the net demand curve "in the risk-neutral measure", like practitioners do to model interest rates in the Heath–Jarrow–Morton framework: combining (5.4) and (5.5) with Theorem 5.5 shows that the process Q under the measure $\mathbb{Q}$ satisfies*

$$\begin{aligned} \mathrm{d}Q(p,t) &= \left(-\frac{1}{2}\frac{\partial^2 Q(p,t)}{\partial p^2} \left(\frac{\sigma_Q(p,t)}{\dfrac{\partial Q(p,t)}{\partial p}} \right)^2 + \frac{1}{2\dfrac{\partial Q}{\partial p}(p,t)} \frac{\partial(\sigma_Q(p,t))^2}{\partial p} \right) \mathrm{d}t \\ &\quad + \sigma_Q(p,t)\int_0^1 b_Q(p,s,t)B^{\mathbb{Q}}(\mathrm{d}s,\mathrm{d}t), \\ Q(p,0) &= Q_0(p). \end{aligned} \tag{5.15}$$

On the one hand, (5.15) is an initial value problem for a backward heat equation and is therefore ill-posed in the sense of Hadamard; cf. [41, section 3.7]. On the other hand, the process Q is originally defined with respect to the physical measure $\mathbb{P}$, which means that, under the conditions of Theorem 5.5, equation (5.15) has a classical solution.

Ill-posed stochastic partial differential equations can be studied using chaos expansion [34, section 5.6.4], but the non-linear structure of (5.15) further complicates the problem. At this point, just about every question about equation (5.15) remains unanswered.

6. Pricing of options in a manipulable market

As before, our market consists of small traders and one large trader. The question we address in this section is how to characterize the price of a derivative security when the large trader can manipulate the market. While it might appear that the price of the derivative security should depend on the future of the large trader's strategy θ, this section shows that it is often not the case.

Denote by Θ^{BV} the subset of Θ (the set of the large trader strategies) consisting of all the functions with bounded variation

$$\theta(t) = \theta(0) + \int_0^t \dot{\theta}(s)\mathrm{d}s, \ \ \dot{\theta} \in L_1((0,T)).$$

By Corollary 3.4 (cf. [7, page 7]), absence of transaction costs for the large trader is equivalent to the condition $\theta \in \Theta^{BV}$.

We define the observable net demand on the market by

$$Q^\theta(p,t) = Q(p,t) + \theta(t),$$

so that

$$\mathrm{d}Q^\theta(p,t) = \big(\mu_Q(p,t) + \dot{\theta}(t)\big)\mathrm{d}t + \int_0^1 \tilde{\sigma}_Q(p,s,t)B(\mathrm{d}s,\mathrm{d}t), \quad p \in [0,S], \ t \in [0,T].$$

Likewise, the corresponding price process $P^\theta = P^\theta(x,t)$ is defined by

$$Q^\theta\big(P^\theta(x,t),t\big) + x = 0,$$

and then, by the Itô-Wentzell formula,

$$\mathrm{d}P^{\theta}(x,t)=\left(\mu_P(x,t)+\frac{\partial P(x,t)}{\partial x}\,\dot{\theta}(t)\right)\mathrm{d}t-\int_0^1\frac{\tilde{\sigma}_Q\big(P^{\theta}(x,t),s,t\big)}{\dfrac{\partial Q}{\partial p}\big(P^{\theta}(x,t),t\big)}B(\mathrm{d}s,\mathrm{d}t),\ t\in[0,T]. \tag{6.1}$$

The quantity x represents the deviation of the trader's position from the strategy θ; the range of admissible values of x will, in general, depend on θ.

We denote the clearing price at time t by $\pi^{\theta}(t)$ so that $Q^{\theta}\big(\pi^{\theta}(t),t\big)=0$. The constant strategy $\theta^c(t):=\theta(0),\ 0\le t\le T,$ will be of special interest.

We need the following market completeness assumption.

Assumption Q4 *For every $\theta\in\Theta^{BV}$, there exists a unique measure $\mathbb{Q}^{\theta}$ such that, for every admissible x, the process $t\mapsto P^{\theta}(x,t)$ is a martingale under $\mathbb{Q}^{\theta}$.*

Theorem 6.1 *If $\theta\in\Theta^{BV}$, then, for every Borel set $A\in\mathcal{B}(\mathcal{C}([0,T]))$,*

$$\mathbb{Q}^{\theta}\big(\pi^{\theta}\in A\big)=\mathbb{Q}^{\theta^c}\big(\pi^{\theta^c}\in A\big).$$

Proof Consider the stochastic process $X=X(t)$ defined by the equation

$$\mathrm{d}X(t)=-\int_0^1\frac{\tilde{\sigma}_Q(X(t),s,t)}{\dfrac{\partial Q}{\partial p}(X(t),t)}B(\mathrm{d}s,\mathrm{d}t),$$

with initial condition $X(0)=P^{\theta}(0,0)$; the properties of $\tilde{\sigma}_Q$ imply existence and uniqueness of the solution.

By construction,

$$\pi^{\theta}(t)=P^{\theta}(0,t). \tag{6.2}$$

Switching from the original measure $\mathbb{P}$ to the measure $\mathbb{Q}^{\theta}$ removes the drift part in (6.1) but does not change the diffusion part. As a result, (6.1) and (6.2) imply

$$\mathbb{Q}^{\theta}\big(\pi^{\theta}\in A\big)=\mathbb{P}(X\in A)=\mathbb{Q}^{\theta^c}\big(\pi^{\theta^c}\in A\big),\ A\in\mathcal{B}\big(\mathcal{C}([0,T])\big),$$

completing the proof. □

We seek to price a contingent claim $H^{\theta}=F\big(\pi^{\theta}(\cdot)\big)$, where F is a continuous functional on $\mathcal{C}((0,T))$. We assume that H^{θ} can be replicated by a trading strategy $\Delta^{\theta}\in\Theta$. Theorem 4.1 in [7] shows that the asymptotic liquidation process generated by a strategy $\Delta^{\theta}\in\Theta$ can be ε-approximated by a strategy $\Delta^{\theta,\varepsilon}\in\Theta^{BV}$. By Proposition 3.3, the corresponding realizable wealth $V^{\Delta^{\theta,\varepsilon}}$ is a $\mathbb{Q}^{\theta}$-martingale, and thus we can define (up to an ε-approximation) the no-arbitrage price of the contingent claim H^{θ} by

$$\nu_H^{\theta}=\mathbb{E}^{\mathbb{Q}^{\theta}}[F(\pi^{\theta})].$$

The following theorem is the main result of this section.

Theorem 6.2 *Suppose that Assumptions Q1–Q4 hold. If $\theta\in\Theta^{BV}$, then the no-arbitrage price of H^{θ} depends only on the initial value of the large trader strategy, and not on its future values:*

$$\nu_H^{\theta}=\mathbb{E}^{\mathbb{Q}^{\theta^c}}[F(\pi^{\theta^c})]. \tag{6.3}$$

Proof With $\theta\in\Theta^{BV}$, (6.3) now immediately follows from Theorem 6.1. □

The online appendix *https://www.ran-zhao.com/uploads/1/2/4/6/124680022/online_appendix.pdf* presents a full implementation of our model, including the use of equation (6.3) to price options.

Remark 6.3 *Theorem 6.2 shows that working with* P *as opposed to* Q *has a subtle drawback. Suppose without loss of generality that today's position of the large trader is* $\theta(0) = x_{\min}$ *. In the lognormal model of section 5.3, the clearing price* π *will thus be equal to* $p(x_{\min})$ *. Looking at the definition of* $p(x_{\min})$ *in equation (5.13), we see that the remainder of the price curve, i.e.,* $p(x)$ *for* $x > x_{\min}$ *has no influence on the dynamics of* $p(x_{\min})$ *, and we are back to the standard Black–Scholes model. Liquidity effects would be present if we specified volatility to be stochastic, but this would further complicate the market price of risk equations.*

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