GAUSSIAN FIELDS AND STOCHASTIC HEAT EQUATIONS

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Abstract. The objective of the paper is to characterize the Gaussian free field as a stationary solution of the heat equation with additive space-time white noise. In the whole space, the investigation leads to other types of Gaussian fields, as well as interesting phenomena in dimensions one and two.

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1. Introduction

It is well-known, for example by the Donsker theorem [12, Corollary VII.3.11], that a suitably scaled simple symmetric random walk on $[0,1]$ converges to the standard Brownian motion. When pinned (conditioned to hit zero) at the right point $x=1$, the same random walk converges to the Brownian bridge $\bar{W}=\bar{W}(x)$, a Gaussian process on $[0,1]$ with mean zero and covariance

$$\mathbb{E}\big(\bar{W}(x)\bar{W}(y)\big)=\min(x,y)-xy;$$

cf. [16, Chapter VI].

What would a multi-dimensional version of these results be? In other words, what are the scaling limits of discrete random objects in the plane or in the space, or in higher dimensions?

In many models [13, 20, etc.] this limiting object is a Gaussian free field [25]. While well-known in theoretical physics, for example, as a starting point in the construction of certain quantum field theories [27], the Gaussian free field is a relatively new area of research in mathematics.

Let $\mathcal{O}\subseteq\mathbb{R}^{\mathrm{d}}$ be a domain and let $\Phi_{\mathcal{O}}=\Phi_{\mathcal{O}}(x,y),\ x,y\in\mathcal{O}$, be Green's function of the Laplacian $\boldsymbol{\Delta}$ in $\mathcal{O}$ with suitable homogeneous boundary conditions. A *Gaussian free field* on $\mathcal{O}$ is usually defined as a (generalized) Gaussian process $\bar{W}=\bar{W}(x),\ x\in\mathcal{O}$, such that

$$\mathbb{E}\bar{W}(x)=0,\ \mathbb{E}\Big(\bar{W}(x)\bar{W}(y)\Big)=\Phi_{\mathcal{O}}(x,y),\quad x,y\in\mathcal{O}. \tag{1.1}$$

If $\mathrm{d}>1$, then the function $\Phi_{\mathcal{O}}$ has a singularity on the diagonal $x=y$. By (1.1), $\mathbb{E}|\bar{W}(x)|^2=+\infty$ for all $x\in\mathcal{O}$, meaning that $\bar{W}$ must indeed be a generalized process,

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or a random generalized function (distribution), indexed by test functions on $\mathcal{O}$ rather than points in $\mathcal{O}$.

Let us assume that the equation

$$\boldsymbol{\Delta} v = -f \tag{1.2}$$

is well-posed in a sufficiently rich class $\mathcal{G}$ of functions f on $\mathcal{O}$ and the solution of (1.2) can be written as

$$v(x) = \int_{\mathcal{O}} \Phi_{\mathcal{O}}(x,y) f(y)\, dy.$$

Then the Gaussian free field $\bar{W}$ on $\mathcal{O}$ is defined as a collection of zero-mean Gaussian random variables $\bar{W}[f],\ f \in \mathcal{G}$, such that

$$\mathbb{E}\Big(\bar{W}[f]\bar{W}[g]\Big) = \iint\limits_{\mathcal{O}\times\mathcal{O}} \Phi_{\mathcal{O}}(x,y) f(x) g(y)\, dxdy, \quad f, g \in \mathcal{G}. \tag{1.3}$$

If $\bar{W} = \bar{W}(x),\ x \in \mathcal{O}$, is a collection of Gaussian random variables satisfying (1.1), then $\bar{W}$ defines a random distribution on $\mathcal{G}$ by

$$\bar{W}[f] = \int_{\mathcal{O}} \bar{W}(x) f(x)\, dx, \quad f \in \mathcal{G},$$

which is a collection of zero-mean Gaussian random variables satisfying (1.3).

Let $\mathbb{F} = \big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ be a stochastic basis with the usual assumptions [12, Definition I.1.3], on which countably many independent standard Brownian motions $w_k = w_k(t),\ t \geq 0,\ k = 1, 2, \ldots$ are defined. The stochastic basis $\mathbb{F}$ will be fixed throughout the rest of the paper.

The *space-time Gaussian white noise* $\dot{W} = \dot{W}(t,x)$ on $\mathcal{O}$ is a collection of zero-mean Gaussian random variables $\dot{W}[f]$, $f \in L_2\big((0,+\infty) \times \mathcal{O}\big)$, such that

$$\mathbb{E}\Big(\dot{W}[f]\dot{W}[g]\Big) = \int_0^{+\infty}\int_{\mathcal{O}} f(t,x) g(t,x)\, dxdt.$$

Given an orthonormal basis $\{\mathfrak{h}_k = \mathfrak{h}_k(x),\ k \geq 1\}$ in $L_2(\mathcal{O})$, the process $\dot{W}$ can be written as a (formal) sum

$$\dot{W}(t,x) = \sum_{k=1}^{\infty} \mathfrak{h}_k(x) \dot{w}_k(t). \tag{1.4}$$

Similarly,

$$W(t,x) = \sum_{k=1}^{\infty} \mathfrak{h}_k(x) w_k(t) \tag{1.5}$$

is called *cylindrical Brownian motion* on $L_2(\mathcal{O})$. For a square integrable function $f = f(t,x)$,

$$\int_0^t \int_{\mathcal{O}} f(s,y) W(ds,dy) = \sum_{k=1}^{\infty} \int_0^t \left(\int_{\mathcal{O}} f(s,y)\mathfrak{h}_k(y)\, dy\right) dw_k(s);$$

cf. [28, Chapter 2].

The objective of this paper is to characterize the Gaussian free field as the stationary solution of a heat equation driven by space-time Gaussian white noise.

Theorem 1.1. *Let $u = u(t, x)$ be a solution of*

$$u_t(t, x) = \nu\, \mathbf{\Delta} u(t, x) + \sigma \dot{W}(t, x),\ t > 0,\ x \in \mathcal{O} \subseteq \mathbb{R}^{\mathrm{d}}, \tag{1.6}$$

with initial condition $u(0, x) = \varphi(x)$ independent of W and with constant $\nu > 0$, $\sigma > 0$; suitable boundary conditions are imposed if $\mathcal{O} \subset \mathbb{R}^{\mathrm{d}}$.

Then, as $t \to +\infty$, u converges weakly to a scalar multiple of the Gaussian free field on $\mathcal{O}$.

In other words, as $t \to +\infty$, the solution of the stochastic parabolic equation (1.1) converges in distribution to the solution of the stochastic elliptic equation

$$(-\nu\mathbf{\Delta})^{1/2} v(x) = \sigma V(x),$$

where V is Gaussian white noise (or an isonormal Gaussian process) on $L_2(\mathcal{O})$. By comparison, direct computations show that, as $t \to +\infty$, the solution of the deterministic heat equation $u_t = \nu\mathbf{\Delta} u + f(x)$ in a bounded domain or in $\mathbb{R}^{\mathrm{d}}$, $d \geq 3$, with a smooth compactly supported f, converges to the solution of the elliptic equation $\nu\mathbf{\Delta} v = -f$, but this convergence does not in general hold in $\mathbb{R}$ and $\mathbb{R}^2$.

The starting point in the proof of Theorem 1.1 is well-posedness of equation (1.6): a suitably defined solution exits, is unique, and depends continuously on the initial condition. The argument is relatively straightforward in a bounded domain: the Fourier method shows that (1.6) is well posed as long as the deterministic heat equation is well posed; see Theorem 3.5 below. The analysis is more complicated in the whole space, where the stochastic term does not allow a direct application of deterministic theory and makes it necessary to consider the solution in special spaces of generalized functions.

Once the solution of (1.6) is constructed, the rest of the proof of Theorem 1.1 can be summarized as follows. Denote by $G_{\mathcal{O}} = G_{\mathcal{O}}(t, x, y)$ the heat kernel for equation (1.6) so that, for $f \in \mathcal{G}$, the solution $u^{\mathrm{H},f} = u^{\mathrm{H},f}(t, x)$ of the deterministic heat equation with initial condition f is

$$u^{\mathrm{H},f}(t, x) = \int_{\mathcal{O}} G_{\mathcal{O}}(t, x, y) f(y) dy. \tag{1.7}$$

The main consequence of well-posedness of (1.6) is that the solution can be written as

$$u(t, x) = u^{\mathrm{H},\varphi}(t, x) + \sigma \int_0^t \int_{\mathcal{O}} G(t - s, x, y)\, W(ds, dy) \tag{1.8}$$

and, because $G_{\mathcal{O}}(t, x, y) = G_{\mathcal{O}}(t, y, x)$,

$$\begin{aligned} u[t, f] &:= \int_{\mathcal{O}} u(t, x) f(x) dx = u^{\mathrm{H},\varphi}[t, f] + \sigma \int_0^t \int_{\mathcal{O}} \left(\int_{\mathcal{O}} G_{\mathcal{O}}(t - s, x, y) f(x) dx \right) W(ds, dy) \\ &= u^{\mathrm{H},\varphi}[t, f] + \sigma \int_0^t \int_{\mathcal{O}} u^{\mathrm{H},f}(t - s, y)\, W(ds, dy); \end{aligned} \tag{1.9}$$

cf. [28, Chapter 9] in the case $\mathcal{O} = \mathbb{R}^{\mathrm{d}}$, $\mathrm{d} \geq 3$. As a result,

$$\mathbb{E}\Big(u[t,f]u[t,g]\Big) = \mathbb{E}\Big(u^{\mathrm{H},\varphi}[t,f]u^{\mathrm{H},\varphi}[t,g]\Big) + \sigma^2 \int_0^t \int_{\mathcal{O}} u^{\mathrm{H},f}(t-s,y)u^{\mathrm{H},g}(t-s,y)\,dy\,ds,$$

and if

$$\lim_{t\to+\infty} u^{\mathrm{H},f}(t,x) = 0 \tag{1.10}$$

in an appropriate way, then

$$\lim_{t\to+\infty} \mathbb{E}\Big(u[t,f]u[t,g]\Big) = \sigma^2 \int_0^{+\infty} \int_{\mathcal{O}} u^{\mathrm{H},f}(s,y)u^{\mathrm{H},g}(s,y)\,dy\,ds.$$

Moreover, by (1.7) and the semigroup property of $G_{\mathcal{O}}$,

$$\int_{\mathcal{O}} u^{\mathrm{H},f}(s,y)u^{\mathrm{H},g}(s,y)\,dy = \iint_{\mathcal{O}\times\mathcal{O}} G_{\mathcal{O}}(2s,x,y)f(x)g(y)\,dxdy.$$

If we also have

$$\int_0^{+\infty} G_{\mathcal{O}}(s,x,y)\,ds = \frac{1}{\nu}\Phi_{\mathcal{O}}(x,y), \tag{1.11}$$

then, combining the above computations with (1.3), we get the convergence

$$\lim_{t\to+\infty} \mathbb{E}\Big(u[t,f]u[t,g]\Big) = \frac{\sigma^2}{2\nu}\mathbb{E}\Big(\bar{W}[f]\bar{W}[g]\Big). \tag{1.12}$$

A major part of the paper consists in providing the details in the above arguments, in particular,

(1) Constructing the solution of (1.6) and interpreting (1.8);
(2) Identifying a suitable function class $\mathcal{G}$ and verifying (1.9), (1.10);
(3) Working around (1.11): this step turns out to be a major technical difference between a bounded domain and the whole space;
(4) Interpreting both u and $\bar{W}$ as Gaussian measures on a suitable Hilbert space so that (1.12) will indeed imply the required convergence.

We will also see that, similar to the deterministic problem, the cases $\mathcal{O} = \mathbb{R}$ and $\mathcal{O} = \mathbb{R}^2$ require special considerations, partly because of the failure of (1.11) and partly because of unexpected difficulties interpreting (1.3).

Section 2 summarizes the construction and general properties of Gaussian processes indexed by elements of a separable Hilbert space, which, in particular, provides an interpretation of the diverging series (1.4) and (1.5). Sections 3 and 4 present the precise statement and proof of Theorem 1.1 in a bounded domain and in the whole space, respectively. Section 5 discusses the special features of the one-dimensional case, and Section 6 summarizes the results and puts them in a broader context.

The symbol $\sim$ has the same meaning as in [19, Formula 2.1.1]:

$$f(x) \sim g(x),\ x \to x_0 \ \Leftrightarrow\ \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

A long bar $\overline{z}$ over a symbol denoted complex conjugations; it should not be confused with a short bar $\bar{W}$ in the notation of the Gaussian free field.

2. Gaussian Processes and Measures on Hilbert Spaces

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)_0$ and norm $\|\cdot\|_0$, and let Λ be a linear operator on H with the following properties:

[O1] $(\Lambda f, g)_0 = (f, \Lambda g)$ for all f, g in the domain of Λ;
[O2] $(\Lambda f, f)_0 > 0,\ f \neq 0,\ f$ in the domain of Λ;
[O3] There is an orthonormal basis $\{\mathfrak{h}_k,\ k \geq 1\}$ in H such that

$$\Lambda \mathfrak{h}_k = \lambda_k \mathfrak{h}_k,\ k \geq 1;\ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots ;\ \lim_{k\to\infty} k^{-\alpha}\lambda_k = c_\Lambda \tag{2.1}$$

for some $\alpha > 0$, $c_\Lambda > 0$.

For $f \in H$, write

$$f_k = (f, \mathfrak{h}_k)_0.$$

Definition 2.1. *The* `Hilbert scale` $\mathbb{H}_\Lambda$ *generated by the operator* Λ *is the collection of the Hilbert spaces* $\{H^\gamma,\ \gamma \in \mathbb{R}\}$, *where*

- $H^0 = H$;
- $H^\gamma = \{f \in H : \sum_{k\geq 1} \lambda_k^{2\gamma} f_k^2 < \infty\}$ *if* $\gamma > 0$;
- H^γ *is the closure of* H *with respect to the norm* $\|f\|_\gamma$, *where*

$$\|f\|_\gamma^2 = \sum_{k\geq 1} \lambda_k^{2\gamma} f_k^2, \tag{2.2}$$

if $\gamma < 0$.

Equality (2.2) defines the norm in every H^γ, $\gamma \in \mathbb{R}$,

$$H^\gamma = \Lambda^{-\gamma} H,\ (f,g)_\gamma = (\Lambda^\gamma f, \Lambda^\gamma g)_0 = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} f_k g_k,$$

and

$$f = \sum_{k=1}^{\infty} f_k \mathfrak{h}_k \in H^\gamma \iff \sum_{k=1}^{\infty} k^{2\alpha\gamma} f_k^2 < \infty.$$

Proposition 2.2. *If* $\mathbb{H}_\Lambda = \{H^\gamma,\ \gamma \in \mathbb{R}\}$ *is the Hilbert scale from Definition 2.1, then, for every* $\gamma_1 > \gamma_2$, *the space* H^{γ_1} *is densely and compactly embedded into* H^{γ_2}; *the embedding is Hilbert-Schmidt if* $\gamma_1 - \gamma_2 > 1/(2\alpha)$.

Proof. The construction of $\mathbb{H}_\Lambda$ implies density of the embedding, whereas assumption (2.1) about the eigenvalues of Λ implies that the embedding is compact and, as long as $\sum_k \lambda_k^{2(\gamma_2-\gamma_1)} < \infty$, it is Hilbert-Schmidt. □

Definition 2.3. *Let* U *be a separable Hilbert space with inner product* $(\cdot,\cdot)_U$.

(1) *A Q-*`Brownian motion` *$W = W(t)$ on U is a collection of zero-mean Gaussian processes $\{W[t,h],\ h \in H,\ t \geq 0\}$ such that $\mathbb{E}\Big(W[t,h]W[s,g]\Big) = \min(t,s)\,(Qh,g)_U$ for some linear operator Q on U. In the case Q is the identity operator, W is called a* `cylindrical Brownian motion` *on U.*

(2) *A Q-*`Brownian motion` *$W = W(t)$ on U is called U-valued if*

$$W[t,h] = \big(W(t),h\big)_U \tag{2.3}$$

and the process W on the right-hand side of (2.3) satisfies

$$W \in L_2\Big(\Omega; \mathcal{C}\big((0,T);U\big)\Big)$$

for all $T > 0$.

A Q-Brownian motion on U is U-valued if and only if the operator Q is trace class on U; cf. [6, Propositions 4.3 and 4.4]. It is convenient to re-state [6, Proposition 4.7] in the setting of the Hilbert scale $\mathbb{H}_\Lambda$.

Proposition 2.4. *A cylindrical Brownian motion on H has a representation*

$$W(t) = \sum_{k\geq 1} \mathfrak{h}_k w_k(t), \tag{2.4}$$

where $w_k(t) = W[t,\mathfrak{h}_k]$, $k \geq 1$, are independent standard Brownian motions, and $W \in L_2\Big(\Omega; \mathcal{C}((0,T);H^{-\gamma})\Big)$ for all $T > 0$, $\gamma > 1/(2\alpha)$. Equivalently, a cylindrical Brownian motion H is an $H^{-\gamma}$-valued Q-Gaussian process, $\gamma > 1/(2\alpha)$, with $Q = \mathfrak{j}\mathfrak{j}'$, where $\mathfrak{j}$ is the embedding operator $H \to H^{-\gamma}$ and $\mathfrak{j}' : H^{-\gamma} \to H$ is the adjoint of $\mathfrak{j}$.

We will also need a stationary version of Definition 2.3.

Definition 2.5. *Let U be a separable Hilbert space with inner product $(\cdot,\cdot)_U$.*

(1) *A Q-*`Gaussian process` *W on U is a collection of zero-mean Gaussian random variables $\{W[h],\ h \in H\}$ such that $\mathbb{E}\Big(W[h]W[g]\Big) = (Qh,g)_U$ for some linear operator Q on U. In the case Q is the identity operator, W is called an* `isonormal Gaussian process`*;* cf. [18, Definition 1.1.1].

(2) *A Q-*`Gaussian process` *W on U is called U-valued if*

$$W[h] = \big(W,h\big)_U \tag{2.5}$$

and the random variable W on the right-hand side of (2.5) satisfies $W \in L_2\big(\Omega;U\big)$.

A Q-Gaussian process on U is U-valued if and only if the operator Q is trace class on U; cf. [17, Theorem 3.2.39]. In the Hilbert scale $\mathbb{H}_\Lambda$, we have a version of Proposition 2.4.

Proposition 2.6. *Given an $r \in \mathbb{R}$, an isonormal Gaussian process on H^r has a representation*

$$W = \sum_{k\geq 1} \lambda_k^{-r} \mathfrak{h}_k \zeta_k,$$

where $\zeta_k = W[\mathfrak{h}_k]$, $k \geq 1$, *are iid Gaussian random variables, and* $W \in L_2(\Omega; H^{r-\gamma})$ *for all* $\gamma > 1/(2\alpha)$. *Equivalently, an isonormal Gaussian process on* H^r *is an* $H^{r-\gamma}$*-valued* Q*-Gaussian process for every* $\gamma > 1/(2\alpha)$, *with* $Q = \mathfrak{j}\mathfrak{j}'$, *where* $\mathfrak{j}$ *is the embedding operator* $H^r \to H^{r-\gamma}$ *and* $\mathfrak{j}' : H^{r-\gamma} \to H^r$ *is its adjoint.*

Proof. This follows by direct computation after observing that the collection

$$\{\lambda_k^{-r}\mathfrak{h}_k, \ k \geq 1\}$$

is an orthonormal basis in H^r. In particular,

$$Q\mathfrak{h}_k = \lambda^{-2\gamma}\mathfrak{h}_k.$$

□

Remark 2.7. While every Hilbert space is self-dual, there is an alternative notion of duality in a Hilbert scale $\mathbb{H}_\Lambda$: for every $\gamma_0 \in \mathbb{R}$ and every $\gamma > 0$, the spaces $H^{\gamma_0+\gamma}$ and $H^{\gamma_0-\gamma}$ are dual relative to the inner product in H^{γ_0}; the duality $\langle\cdot,\cdot\rangle_{\gamma_0,\gamma}$ is given by

$$f \in H^{\gamma_0+\gamma}, g \in H^{\gamma_0-\gamma} \ \mapsto \langle f, g\rangle_{\gamma_0,\gamma} = \lim_{n\to\infty}(f, g_n)_{\gamma_0}, \tag{2.6}$$

where $g_n \in H^{\gamma_0}$ and $\lim_{n\to\infty}\|g - g_n\|_{\gamma_0-\gamma} = 0$. With respect to $\langle\cdot,\cdot\rangle_{0,|r|}$ duality, an isonormal Gaussian process on H^r from Proposition 2.6 becomes an isonormal Gaussian process on H^{-r}. Indeed, if $r > 0$, then, for $f \in H^{-r}$, we *define*

$$\langle W, f\rangle_{0,r} = \sum_{k=1}^{\infty}\frac{f_k}{\lambda_k^r}\zeta_k$$

so that

$$\mathbb{E}\Big(\langle W, f\rangle_{0,r}\langle W, g\rangle_{0,r}\Big) = \sum_{k=1}^{\infty}\frac{f_k g_k}{\lambda_k^{2r}} = (f, g)_{-r}.$$

The case $r < 0$ is similar.

Remark 2.8. Let V be an isonormal Gaussian process on H. By direct computation, an isonormal Gaussian process W on H^r is the unique solution of the stochastic elliptic equation

$$\Lambda^{r/2}W = V; \tag{2.7}$$

cf. [17, Theorem 4.2.2].

By the Bochner-Minlos theorem [5, Theorem 2.27], a U-valued Q-Gaussian process W defines a centered Gaussian measure μ_W on U by

$$\mu_W(A) = \mathbb{P}(W \in A),$$

where A is a Borel sub-set of U, and, for every $f \in U$,

$$\int_U e^{\mathrm{i}(f,g)_U}\, d\mu_W(g) = \mathbb{E}e^{\mathrm{i}(W,f)_U} = \exp\left(-\frac{1}{2}(Qf, f)_U\right).$$

The *Cameron-Martin space* of the measure μ_W is the collection of all $h \in U$ such that the measure μ_W^h defined by $\mu_W^h(A) = \mu_W(A + h)$ is equivalent to μ_W [3, Section 2.4].

Proposition 2.9. *Let $\mathbb{H}_\Lambda$ be the Hilbert scale from Definition 2.1. If W is an isonormal Gaussian process on H^r, then W generates a Gaussian measure μ_W on every $H^{r-\gamma}$ with $\gamma > 1/(2\alpha)$, and the Cameron-Martin space of this measure is H^r.*

Proof. This is a combination of two results, [3, Lemma 2.1.4 and Theorem 3.5.1], in the Hilbert space setting. □

The following result is a re-statement of [3, Example 3.8.13(iii)].

Proposition 2.10. *Let $W_n, n \geq 1$, be a collection of U-valued Q_n-Gaussian processes. Then the convergence in distribution $\lim_{n\to\infty} W_n \stackrel{d}{=} W$, for a U-valued Q-Gaussian process W, takes place if and only if*

$$\lim_{n\to\infty} \sqrt{Q_n} = \sqrt{Q}$$

in the Hilbert-Schmidt norm.

In particular, if all the operators Q_n and Q have a common system of eigenfunctions $\{h_k, k \geq 1\}$ that is an orthonormal basis in U, so that $Q_n h_k = \alpha_{k,n},\ Qh_k = \alpha_k h_k$, then $\lim_{n\to\infty} W_n \stackrel{d}{=} W$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\sqrt{\alpha_{k,n}} - \sqrt{\alpha_k}\right)^2 = 0. \tag{2.8}$$

3. Bounded Domain in $\mathbb{R}^{\mathrm{d}}$

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^{\mathrm{d}}$ and let $\boldsymbol{\Delta}$ be the Laplacian on $\mathcal{O}$ with some homogeneous boundary conditions so that

[A1] The eigenfunction $\mathfrak{h}_k,\ k \geq 1$, of $\boldsymbol{\Delta}$ form an orthonormal basis in $L_2(\mathcal{O})$;
[A2] The eigenvalues $-\lambda_k^2,\ k \geq 1$, of $\boldsymbol{\Delta}$ satisfy $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and there exists a number $c_{\mathcal{O}} > 0$ such that

$$\lambda_k \sim c_{\mathcal{O}} k^{1/\mathrm{d}}. \tag{3.1}$$

There are various sufficient conditions ensuring [A1] and [A2]: see, for example, [23, Section 1.1.7].

Taking $H = L_2(\mathcal{O})$ and $\Lambda = (-\boldsymbol{\Delta})^{1/2}$, we see that conditions [O1]–[O3] hold, with $\alpha = 1/\mathrm{d}$, and we construct the Hilbert scale $\mathbb{H}_\Lambda$ as in Definition 2.1. In particular,

$$f = \sum_{k=1}^{\infty} f_k \mathfrak{h}_k \in H^\gamma \iff \sum_{k=1}^{\infty} k^{2\gamma/\mathrm{d}} f_k^2 < \infty.$$

3.1. Green's Functions and Gaussian Free Fields.

For $\nu > 0$, consider the heat equation

$$u_t(t,x) = f(x) + \nu \int_0^t \boldsymbol{\Delta} u(s,x)\, ds,\ t \geq 0,\ x \in \mathcal{O}, \tag{3.2}$$

and the Poisson equation

$$\nu\boldsymbol{\Delta}v(x) = -g(x), \quad x \in \mathcal{O}. \tag{3.3}$$

Writing

$$f(x) = \sum_{k=1}^{\infty} f_k\mathfrak{h}_k(x),\ g = \sum_{k=1}^{\infty} g_k\mathfrak{h}_k(x),\ u(t,x) = \sum_{k=1}^{\infty} u_k(t)\mathfrak{h}_k(x),\ v(x) = \sum_{k=1}^{\infty} v_k\mathfrak{h}_k(x),$$

we can solve equations (3.2) and (3.3).

Proposition 3.1. (1) *For every $f \in H^\gamma$, the unique solution of* (3.2) *is*

$$u(t,x) = \sum_{k=1}^{\infty} e^{-\lambda_k^2\nu t} f_k\mathfrak{h}_k(x) = \int_{\mathcal{O}} G_{\mathcal{O}}(t,x,y)f(y)\,dy,$$

where

$$G_{\mathcal{O}}(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k^2\nu t}\mathfrak{h}_k(x)\mathfrak{h}_k(y).$$

The operator norm of the heat semigroup

$$S_t : f \mapsto \int_{\mathcal{O}} G_{\mathcal{O}}(t,x,y)f(y)\,dy \tag{3.4}$$

is decaying exponentially in time on every H^γ:

$$\|S_tf\|_\gamma \le e^{-\lambda_1 t}\|f\|_\gamma. \tag{3.5}$$

(2) *For every $g \in H^\gamma$, the unique solution of* (3.3) *is*

$$v(x) = \sum_{k=1}^{\infty} \frac{g_k}{\lambda_k^2\nu}\,\mathfrak{h}_k(x) = \int_{\mathbb{R}^{\mathrm{d}}} \Phi_{\mathcal{O}}(x,y)g(y)\,dy,$$

where

$$\Phi_{\mathcal{O}}(x,y) = \sum_{k=1}^{\infty} \frac{\mathfrak{h}_k(x)\mathfrak{h}_k(y)}{\lambda_k^2\nu} = \int_0^{+\infty} G_{\mathcal{O}}(t,x,y)\,dt.$$

In particular, equality (1.11) *holds.*

Definition 3.2. *The $(\boldsymbol{\Delta},\mathcal{O})$-Gaussian free field $\bar{W}$ is an isonormal Gaussian process on H^1.*

The point is that, in a bounded domain $\mathcal{O}$, there are many different Gaussian free fields, depending on the boundary conditions of the operator $\boldsymbol{\Delta}$. For example, with *zero boundary conditions*, we take $f, g \in H^1$ and integrate by parts to find

$$\mathbb{E}\big(\bar{W}[f]\bar{W}[g]\big) = (f,g)_1 = (\Lambda f, \Lambda g)_0 = -(f, \boldsymbol{\Delta} g)_0 = -\int_{\mathcal{O}} f(x)\boldsymbol{\Delta} g(x)\,dx = (\nabla f, \nabla g)_0,$$

which, for $\mathrm{d} = 2$, is the same as [25, Definition 2.12]. More generally, by (2.7),

$$(-\boldsymbol{\Delta})^{1/2}\bar{W} = V,$$

where V is an isonormal Gaussian process on $L_2(\mathcal{O})$.

Proposition 3.3. *Under the assumptions* [A1], [A2], *the* $(\mathbf{\Delta}, \mathcal{O})$*-Gaussian free field* $\bar{W}$ *has a representation*

$$\bar{W}(x) = \sum_{k=1}^{\infty} \frac{\zeta_k}{\lambda_k}\, \mathfrak{h}_k(x), \tag{3.6}$$

with iid standard Gaussian random variables ζ_k*, and defines a centered Gaussian measure on* $H^{1-(\mathrm{d}/2)-\varepsilon}$ *for every* $\varepsilon > 0$*; the Cameron-Martin space of this measure is* H^1*.*

Proof. This follows from Proposition 2.6 with $r = 1$ and $\alpha = 1/\mathrm{d}$. □

3.2. **Main Result.** Given $\nu > 0$, $\sigma > 0$, and a cylindrical Brownian motion W on $L_2(\mathcal{O})$, consider the evolution equation

$$u(t) = \varphi + \nu \int_0^t \mathbf{\Delta} u(s)\, ds + \sigma W(t),\ t > 0, \tag{3.7}$$

with initial condition φ independent of W.

Definition 3.4. *Given* $\varphi \in L_2(\Omega; H^r)$*, a solution of* (3.7) *is an adapted process with values in* $L_2\big(\Omega \times [0,T]; H^{r+1}\big) \bigcap L_2\big(\Omega; \mathcal{C}((0,T); H^r\big)$*, such that equality* (3.7) *holds in* H^{r-1} *for all* $t \geq 0$ *with probability one.*

Theorem 3.5. *If* $\varphi \in L_2(\Omega; H^{-\gamma})$ *and* $\gamma > \mathrm{d}/2$*, then, under assumptions* [A1], [A2]*, equation* (3.7) *has a unique solution and, for every* $T > 0$*,*

$$\mathbb{E} \sup_{0<t<T} \|u(t)\|_{-\gamma}^2 + \mathbb{E} \int_0^T \|u(t)\|_{1-\gamma}^2\, dt \leq C(\gamma, T)(1 + \mathbb{E}\|\varphi\|_{-\gamma}^2); \tag{3.8}$$

$C = C(\gamma, T)$ *is a number depending only on* T *and* γ*. Moreover,*

(1) *For every* $t > 0$*,* $u(t) \in L_2\big(\Omega; H^{1-\gamma}\big)$ *and*

$$u(t) = S_t\varphi + \sum_{k=1}^{\infty} \bar{u}_k(t)\mathfrak{h}_k, \tag{3.9}$$

where S_t *is the heat semigroup* (3.4) *and* $\bar{u}_k(t),\ k \geq 1$*, are independent Gaussian random variables with mean zero and variance*

$$\mathbb{E}\bar{u}_k^2(t) = \frac{\sigma^2}{2\nu\lambda_k^2}\Big(1 - e^{-2\nu\lambda_k^2 t}\Big). \tag{3.10}$$

(2) *As* $t \to +\infty$*, the* $H^{1-\gamma}$*-valued random variables* $u(t)$ *converge weakly to* $\sigma(2\nu)^{-1/2}\bar{W}$*, where* $\bar{W}$ *is the* $(\mathbf{\Delta}, \mathcal{O})$*-Gaussian free field.*

Proof. The first part of the theorem follows directly from [22, Theorem 3.1] after the identifications

$$A = \nu\mathbf{\Delta},\ \mathbb{X} = H^{1-\gamma},\ \mathbb{H} = H^{-\gamma},\ \mathbb{X}' = H^{-\gamma-1},\ M(t) = W(t),$$

because, by Proposition 2.4,

$$W \in L_2\big(\Omega; \mathcal{C}((0,T); H^{-\gamma}\big),\ \gamma > \frac{\mathrm{d}}{2}.$$

To establish (3.9), we write

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\mathfrak{h}_k$$

and combine (3.7) with (2.4) to get

$$u_k(t) = \varphi_k - \nu\lambda_k^2 \int_0^t u_k(s)\,ds + \sigma w_k(t);$$

recall that $\boldsymbol{\Delta}\mathfrak{h}_k = -\lambda_k^2\mathfrak{h}_k$. Then

$$u_k(t) = \varphi_k e^{-\nu\lambda_k^2 t} + \bar{u}_k(t),$$

where

$$\bar{u}_k(t) = \sigma \int_0^t e^{-\nu\lambda_k^2(t-s)} dw_k(s).$$

Next,

$$\mathbb{E}\bar{u}_k^2(t) = \sigma^2 \int_0^t e^{-2\nu\lambda_k^2(t-s)}\,ds,$$

and (3.9) follows. In particular,

$$\mathbb{E}\|u(t)\|_{1-\gamma}^2 \leq \frac{\mathbb{E}\|\varphi\|_{-\gamma}^2}{\nu t} + \frac{\sigma^2}{2\nu}\sum_{k=1}^{\infty} \lambda_k^{-2\gamma}, \tag{3.11}$$

so that

$$u(t) \in L_2(\Omega; H^{1-\gamma}), \quad t > 0. \tag{3.12}$$

Note that (3.11) cannot be used to establish (3.8), whereas (3.8) does not necessarily imply (3.12).

Finally, consider the $H^{1-\gamma}$-valued Q-Gaussian process

$$W = \frac{\sigma}{\sqrt{2\nu}} \sum_{k=1}^{\infty} \frac{\zeta_k}{\lambda_k}\,\mathfrak{h}_k.$$

By Proposition 3.3, $\bar{W}$ is a multiple of a $(\boldsymbol{\Delta}, \mathcal{O})$-Gaussian free field. On the other hand, by Proposition 2.6,

$$Q\mathfrak{h}_k = \frac{\sigma^2}{2\nu^2\lambda_k^{2\gamma}}\,\mathfrak{h}_k.$$

Similarly, for each $t > 0$, equality (3.10) implies that $u(t)$ is the $H^{1-\gamma}$-valued $Q(t)$-Gaussian process and

$$Q(t)\mathfrak{h}_k = \frac{\sigma^2}{2\nu\lambda_k^{2\gamma}}\left(1 - e^{-2\nu\lambda_k^2 t}\right)\mathfrak{h}_k.$$

Note that, for $x \in [0, 1]$,

$$0 \leq 1 - (1-x)^{1/2} \leq x$$

and so

$$0 \leq 1 - \left(1 - e^{-2\nu\lambda_k^2 t}\right)^{1/2} \leq e^{-2\nu\lambda_k^2 t} \leq e^{-2\nu\lambda_1^2 t}. \tag{3.13}$$

The required convergence now follows by combining Proposition 2.10 and (2.8) with (3.5). $\square$

Corollary 3.6. (1) *Equation* (3.7) *is ergodic and the unique invariant measure is the distribution of* $\sigma(2\nu)^{-1/2}\bar{W}$ *on* $H^{1-\gamma}$.

(2) *If* $\varphi \overset{d}{=} \sigma(2\nu)^{-1/2}\bar{W}$, *then* $u(t) \overset{d}{=} \sigma(2\nu)^{-1/2}\bar{W}$ *for all* $t > 0$.

(3) *If* $\mathbb{E}\varphi_k = 0$ *for all* k, *then, for each* $t > 0$, *the measure generated by* $u(t)$ *on* $H^{1-\gamma}$ *is absolutely continuous with respect to the measure generated by* $\sigma(2\nu)^{-1/2}\bar{W}$.

Proof. The first two statements are an immediate consequence of (3.9). The third statement follows from a theorem of Kakutani [3, Example 2.7.6]: two zero-mean Gaussian product measures are equivalent if and only if the corresponding *standard deviations* m_k, n_k satisfy

$$\sum_{k=1}^{\infty}\left(\frac{m_k}{n_k}-1\right)^2 < \infty :$$

in our case,

$$m_k = n_k\big(1-e^{-2\nu\lambda_k^2 t}\big)^{1/2},$$

and we have (3.13). □

4. The whole space $\mathbb{R}^{\mathrm{d}}$

There are two special features of the bounded domain that are absent in the whole space:

- The operator Λ generating the scale $\mathbb{H}_\Lambda$ commutes with the operator $\boldsymbol{\Delta}$ in the equations (3.2) and (3.3) we want to solve, and has the property that $\Lambda^{-\gamma}$ is Hilbert-Schmidt on H for sufficiently large $\gamma > 0$;
- The assumption $\lambda_1 > 0$ ensures (3.5), that is, the operator norm of the heat semigroup decays exponentially in time.

As a result, despite its simple form, equation (1.6) in $\mathbb{R}^{\mathrm{d}}$ is not covered by such standard references as [15] (because of the structure of the noise) and [4] (because of the particular form of the evolution operator). Accordingly, we study (1.6) in $\mathbb{R}^{\mathrm{d}}$ by combining very general results from [6] and [22] with very specific computations using (1.8).

4.1. **Function Spaces.** There are three families of spaces that appear in the analysis of partial differential equations on $\mathbb{R}^{\mathrm{d}}$:

(1) `Homogeneous Sobolev spaces` $\dot{H}^\gamma$, $\gamma \in \mathbb{R}$, the collection of generalized functions $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ such that the Fourier transform $\hat{f} = \hat{f}(\xi)$ of f is locally integrable and

$$\|f\|^2_{\dot{H}^\gamma} := \int_{\mathbb{R}^{\mathrm{d}}} |\xi|^{2\gamma}|\hat{f}(\xi)|^2\, d\xi < \infty; \tag{4.1}$$

when $\gamma < \mathrm{d}/2$, $\dot{H}^\gamma$ is also known as the *Riesz potential* space [24];

(2) `Nonhomogenous Sobolev, or Bessel potential, spaces` H^γ, $\gamma \in \mathbb{R}$, the collection of generalized functions $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ such that the Fourier transform $\hat{f} = \hat{f}(\xi)$ of f is locally square integrable and

$$\|f\|_{H^\gamma}^2 := \int_{\mathbb{R}^{\mathrm{d}}} (\varepsilon + |\xi|^2)^\gamma |\hat{f}(\xi)|^2 \, d\xi < \infty; \tag{4.2}$$

(3) The Hilbert scale $\mathbb{H}_{\tilde{\Lambda}} = \{\tilde{H}^\gamma, \ \gamma \in \mathbb{R}\}$, constructed according to Definition 2.1 with $H = L_2(\mathbb{R}^{\mathrm{d}})$ and $\tilde{\Lambda}$ defined by

$$\tilde{\Lambda}^2 : f(x) \mapsto -\boldsymbol{\Delta} f(x) + |x|^2 f(x), \ \ f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}}). \tag{4.3}$$

The operator $\tilde{\Lambda}^2$ has pure point spectrum so that (2.1) holds with $\alpha = 1/(2\mathrm{d})$, and the eigenfunctions, known as the Hermite functions, form an orthonormal basis in $L_2(\mathbb{R}^{\mathrm{d}})$; cf. [10, Section 1.5] or [28, Example 4.2].

Recall that the *normalized Hermite polynomials* are

$$\mathrm{H}_n(x) = \frac{(-1)^n}{\pi^{1/4} 2^{n/2} (n!)^{1/2}} \, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots;$$

the *Hermite functions*

$$h_n(x) = e^{-x^2/2} \mathrm{H}_n(x)$$

form an orthonormal basis in $L_2(\mathbb{R})$ and satisfy

$$-h_n''(x) + x^2 h_n(x) = (2n+1) h_n(x).$$

The orthonormal basis in $L_2(\mathbb{R}^{\mathrm{d}})$,

$$h_{\mathbf{n}}(x_1, \ldots, x_{\mathrm{d}}) = \prod_{j=1}^{\mathrm{d}} h_{n_j}(x_j),$$

is indexed by $\mathbf{n} = (n_1, \ldots, n_{\mathrm{d}})$, $n_j = 0, 1, 2, \ldots$ so that

$$\tilde{\Lambda}^2 h_{\mathbf{n}} = \lambda_{\mathbf{n}}^2 h_{\mathbf{n}} = \big(2(n_1 + \cdots + n_{\mathrm{d}}) + \mathrm{d}\big) h_{\mathbf{n}}.$$

A non-decreasing ordering of $\lambda_{\mathbf{n}}^2$ brings us to the setting of Definition 2.1. In particular,

$$\lambda_n^2 \sim (2\mathrm{d}!)^{1/\mathrm{d}} \, n^{1/\mathrm{d}},$$

cf. [26, Theorem 30.1], and

$$f = \sum_{k=1}^{\infty} f_k \mathfrak{h}_k \in \tilde{H}^\gamma \iff \sum_{k=1}^{\infty} k^{\gamma/\mathrm{d}} f_k^2 < \infty.$$

The norms (4.2) are equivalent for different $\varepsilon > 0$ and (4.1) is a formal limit of (4.2) as $\varepsilon \to 0$. We could interpret (4.1) and (4.2) as

$$\dot{H}^\gamma = \dot{\Lambda}^{-\gamma} L_2(\mathbb{R}^{\mathrm{d}}), \quad H^\gamma = \Lambda^{-\gamma} L_2(\mathbb{R}^{\mathrm{d}}),$$

with

$$\dot{\Lambda} = (-\boldsymbol{\Delta})^{1/2} : \hat{f}(\xi) \ \mapsto |\xi| \hat{f}(\xi), \quad \Lambda = (\varepsilon - \boldsymbol{\Delta})^{1/2} : \hat{f}(\xi) \ \mapsto (\varepsilon + |\xi|^2)^{1/2} \hat{f}(\xi),$$

but it is still not possible to construct the scales as in Definition 2.1: the operators $\dot{\Lambda}$ and Λ do not have a pure point spectrum, and, in addition, the spaces $\dot{H}^\gamma$ are complete with respect to the norm $\| \bullet \|_{\dot{H}^\gamma}$ if and only if $\gamma < \mathrm{d}/2$ [1, Proposition 1.3.4]. In particular, $\dot{H}^1$ is not a Hilbert space when $\mathrm{d} = 1, 2$.

It follows from the definitions that $H^\gamma \subset \dot{H}^\gamma$ and $\tilde{H}^\gamma \subset H^\gamma$ for $\gamma > 0$, and $\dot{H}^\gamma \subset H^\gamma$ for $\gamma < 0$. Also, by duality, $H^\gamma \subset \tilde{H}^\gamma$ for $\gamma < 0$. To summarize,

$$\begin{cases} \tilde{H}^\gamma \subset H^\gamma \subset \dot{H}^\gamma, & \gamma > 0, \\ \tilde{H}^0 = H^0 = \dot{H}^0 = L_2(\mathbb{R}^{\mathrm{d}}), & \gamma = 0, \\ \dot{H}^\gamma \subset H^\gamma \subset \tilde{H}^\gamma, & \gamma < 0. \end{cases} \tag{4.4}$$

One of the technical difficulties in studying equation (1.6) on $\mathbb{R}^{\mathrm{d}}$ is that, while the spaces $\dot{H}^\gamma$ and H^γ are "custom-made" for the operator $\boldsymbol{\Delta}$, the cylindrical Brownian motion $W = W(t)$ on $L_2(\mathbb{R}^{\mathrm{d}})$ does not belong to any of those space, even though we do have $\psi W(t) \in H^{-\gamma}$, $\gamma > \mathrm{d}/2$ for every $t > 0$ and every smooth function ψ with compact support [28, Proposition 9.5]. On the other hand, by Proposition 2.4, we have

$$W \in L_2\big(\Omega; \mathcal{C}((0,T); \tilde{H}^{-\gamma}\big),\ T > 0, \tag{4.5}$$

for every $\gamma > \mathrm{d}$, meaning that the basic existence/uniqueness result for (1.6) must be established in $\tilde{H}^\gamma$. Another useful feature of the spaces $\tilde{H}^\gamma$ is the equalities

$$\mathcal{S}(\mathbb{R}^{\mathrm{d}}) = \bigcap_\gamma \tilde{H}^\gamma, \quad \mathcal{S}'(\mathbb{R}^{\mathrm{d}}) = \bigcup_\gamma \tilde{H}^\gamma;$$

cf. [2].

Definition 4.1. *The* `Gaussian free field` $\bar{W}$ *on* $\mathbb{R}^{\mathrm{d}}$, $\mathrm{d} \geq 3$, *is an isonormal Gaussian process on* $\dot{H}^1$. *The* `Euclidean free field of mass` $\sqrt{\varepsilon}$ *is an isonormal Gaussian process* $\bar{W}_\varepsilon$ *on* H^1.

We also denote by $\tilde{W}$ an isonormal Gaussian process on $\tilde{H}^1$.

To state a definition of $\bar{W}$ that works for all d, denote by $\mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$ the collection of functions from $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$ for which the Fourier transform is equal to zero near the origin.

Definition 4.2. *The* `Gaussian free field` $\bar{W}$ *on* $\mathbb{R}^{\mathrm{d}}$, $\mathrm{d} \geq 1$, *is a collection of zero-mean Gaussian random variables* $\bar{W}[f]$, $f \in \mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$ *such that*

$$\mathbb{E}\Big(\bar{W}[f]\bar{W}[g]\Big) = \int_{\mathbb{R}^{\mathrm{d}}} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2}\, d\xi. \tag{4.6}$$

In the language of quantum field theory [10, p. 103], construction of a zero-mass free field ($\varepsilon = 0$) in dimensions one and two requires different sets of test functions.

For $\mathrm{d} \geq 3$, Definitions 4.1 and 4.2 are equivalent. Indeed, the space $\mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$ is dense in $\dot{H}^\gamma$ for $\gamma < \mathrm{d}/2$ [1, Proposition 1.35] and, for $|\gamma| < \mathrm{d}/2$, the spaces $\dot{H}^\gamma$ and $\dot{H}^{-\gamma}$ are dual relative to the inner product of $L_2(\mathbb{R}^{\mathrm{d}})$ [1, Proposition 1.36]. Thus, if $\mathrm{d} \geq 3$, then the isonormal Gaussian process on $\dot{H}^1$ satisfies (4.6) with an interpretation of $\bar{W}[f]$ as duality relative to $L_2(\mathbb{R}^{\mathrm{d}})$ (as opposed to inner product in $\dot{H}^1$; cf. Remark 2.7).

Definitions 4.2 is also consistent with (1.1). Indeed, the function $\xi \mapsto |\xi|^{-2}$ is a homogeneous distribution in $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ and, for $\mathrm{d} \neq 2$, the Fourier transform of this distribution is the fundamental solution of the Poisson equation on $\mathbb{R}^{\mathrm{d}}$; cf. [8, Chapter 32]. When $\mathrm{d} = 2$, there are some issues with uniqueness, which can be resolved, for example, by restricting the set of test functions to $\mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$.

Finally, by (2.7), if V is an isonormal Gaussian process on $L_2(\mathbb{R}^{\mathrm{d}})$, then

$$(-\boldsymbol{\Delta})^{1/2}\bar{W} = V, \quad (\varepsilon - \boldsymbol{\Delta})^{1/2}\bar{W}_\varepsilon = V, \quad \tilde{\Lambda}\tilde{W} = V.$$

4.2. **Deterministic Equations and Fundamental Solutions.** For $\nu > 0, \varepsilon \geq 0$ and $f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$, consider the heat equation

$$u_t(t,x) = \nu\boldsymbol{\Delta}u(t,x) - \varepsilon u(t,x), \ t > 0, \ x \in \mathbb{R}^{\mathrm{d}}, \tag{4.7}$$

with initial condition $u(0,x) = f(x)$, and the Poisson equation

$$\nu\boldsymbol{\Delta}v(x) - \varepsilon v(x) = -g(x), \quad x \in \mathbb{R}^{\mathrm{d}}. \tag{4.8}$$

The number $\varepsilon > 0$ in $\mathbb{R}^{\mathrm{d}}$ is the analog of $\lambda_1 > 0$ in the bounded domain.

Below is a summary of the well-known results.

- The unique solution of (4.7) in $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$ is
$$u(t,x) = \int_{\mathbb{R}^{\mathrm{d}}} G_{\varepsilon,\mathrm{d}}(t,x)f(y)\,dy,$$
where
$$G_{\varepsilon,\mathrm{d}}(t,x) = \frac{1}{(4\pi\nu t)^{\mathrm{d}/2}} \exp\left(-\varepsilon t - \frac{|x|^2}{4\nu t}\right); \tag{4.9}$$
cf. [14, Theorem 8.4.2].
- The unique solution of (4.8) in $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$ is
$$v(x) = \int_{\mathbb{R}^{\mathrm{d}}} \Phi_{\varepsilon,\mathrm{d}}(x-y)g(y)\,dy, \tag{4.10}$$
where
$$\Phi_{\varepsilon,\mathrm{d}}(x) = \int_0^{+\infty} G_{\varepsilon,\mathrm{d}}(t,x)\,dt; \tag{4.11}$$
cf. [14, Theorems 1.2.1 and 1.6.2, and Exercise 1.6.5].
- If $\varepsilon = 0$ and $d \geq 2$, then the unique solution of (4.8) in $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$ is
$$v(x) = \int_{\mathbb{R}^{\mathrm{d}}} \Phi_{0,\mathrm{d}}(x-y)g(y)\,dy,$$
where
$$\Phi_{0,\mathrm{d}}(x) = \begin{cases} -\dfrac{1}{2\pi\nu} \ln \dfrac{|x|}{\sqrt{\nu}}, & \mathrm{d} = 2, \\[2ex] \dfrac{\Gamma\left(\frac{\mathrm{d}}{2} - 1\right)}{4\pi^{\mathrm{d}/2}\nu|x|^{\mathrm{d}-2}}, & \mathrm{d} \geq 3, \end{cases}$$
and
$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt;$$

cf. [9, Section 2.2.1].

Denote by $K_p = K_p(x)$, $p, x \in \mathbb{R}$, the modified Bessel function of the second kind [19, Section 10.25].

Proposition 4.3 (cf. [10, Proposition 7.2.1])**.** *The following equalities hold:*

$$\Phi_{\varepsilon,\mathrm{d}}(x) = (2\pi\nu)^{-\mathrm{d}/2}\left(\frac{\varepsilon\nu}{x^2}\right)^{(\mathrm{d}-2)/4} K_{(\mathrm{d}-2)/2}\big(\sqrt{\varepsilon/\nu}\,|x|\big), \tag{4.12}$$

$$\lim_{\varepsilon\to 0}\Phi_{\varepsilon,\mathrm{d}}(x) = \Phi_{0,\mathrm{d}}(x),\ \ x \in \mathbb{R}^{\mathrm{d}} \setminus \{0\}, \quad \mathrm{d} \geq 3. \tag{4.13}$$

In particular,

$$\Phi_{\varepsilon,\mathrm{d}}(x) = \begin{cases} \dfrac{1}{2\sqrt{\varepsilon\nu}}\, e^{-\sqrt{\varepsilon/\nu}\,|x|}, & \mathrm{d} = 1,\\[2ex] \dfrac{1}{2\pi\nu}\, K_0\big(\sqrt{\varepsilon/\nu}\,|x|\big), & \mathrm{d} = 2,\\[2ex] \dfrac{1}{4\pi\nu|x|}\, e^{-\sqrt{\varepsilon/\nu}\,|x|}, & \mathrm{d} = 3. \end{cases} \tag{4.14}$$

Proof. Equality (4.12): combine (4.11) with [11, Formula 3.471.9]. Equality (4.13): use the properties of the function K_p, in particular, $K_p(x) = K_{-p}(x),\ \nu > 0$ [19, Formula 10.27.3] and $K_p(x) \sim 2^{p-1}\Gamma(p)x^{-p},\ x \to 0,\ p > 0$ [19, Formula 10.30.2]. Of course, one can get (4.13) directly by passing to the limit in (4.11). Equality (4.14) is a particular case of (4.12) because

$$K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z};$$

cf. [19, Formula 10.39.2]. □

Combining (4.14) with $K_0(x) \sim -\ln x,\ x \to 0$ [19, Formula 10.30.3], we see that, in the case $\mathrm{d} = 2$, equality (4.13) is missed by a logarithmic term: for fixed $x \neq 0$,

$$\Phi_{\varepsilon,2}(x) \sim \Phi_{0,2}(x) - \frac{1}{4\pi\nu}\ln\varepsilon,\ \ \varepsilon \to 0.$$

Representation (4.10) has a version in the Fourier domain, with no explicit dependence on d:

$$\hat{v}(\xi) = \frac{\hat{f}(\xi)}{\varepsilon + \nu|\xi|^2}.$$

Passing to the limit $\varepsilon \to 0$, we get

$$\hat{v}(\xi) = \frac{\hat{f}(\xi)}{\nu|\xi|^2}.$$

Event though the function $\xi \mapsto |\xi|^{-2}$ is not integrable at zero for $\mathrm{d} = 1, 2$, it defines a homogenous distribution on $\mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$, and its inverse Fourier transform is equal to $\Phi_{0,\mathrm{d}}$ [8, Chapter 32].

To conclude, we summarize how the main operators act in the spaces $\tilde{H}^\gamma$.

Proposition 4.4. *For every $\gamma \in \mathbb{R}$,*

(C1) *the operator $\boldsymbol{\Delta}$ extends to a bounded linear operator from $\tilde{H}^{\gamma+2}$ to $\tilde{H}^{\gamma}$;*
(C2) *the heat semigroup*

$$S_t : f(x) \mapsto \int_{\mathbb{R}^{\mathrm{d}}} G_{\nu\varepsilon,\mathrm{d}}(t, x-y) f(y)\, dy,\ \varepsilon \geq 0,\ t > 0,\ f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}}), \tag{4.15}$$

extends to a bounded linear operator on $\tilde{H}^{\gamma}$, and, if $\varepsilon > 0$, then

$$\|S_t f\|_{\tilde{H}^{\gamma}} \leq C e^{-\delta t} \|f\|_{\tilde{H}^{\gamma}} \tag{4.16}$$

for some $C > 0$, $\delta > 0$ and all $f \in \tilde{H}^{\gamma}$.

Proof. (C1) Direct computations show that $\boldsymbol{\Delta}$ is bounded from $\tilde{H}^{2k+2}$ to $\tilde{H}^{2k}$ for every $k = 0, 1, 2, \ldots$. The case of $\gamma > 0$ then follows by interpolation and $\gamma < 0$, by duality.

(C2) This follows by [21, Theorem 2.4]. □

4.3. **Main Results.** Given $\nu > 0$, $\sigma > 0$, $\varepsilon > 0$, a cylindrical Brownian motion W on $L_2(\mathbb{R}^{\mathrm{d}})$, and $\varphi \in L_2(\Omega; \tilde{H}^r)$ independent of W, consider stochastic evolution equations

$$u(t) = \varphi + \nu \int_0^t (\boldsymbol{\Delta} - \varepsilon) u(s)\, ds + \sigma W(t), \tag{4.17}$$

$$u(t) = \varphi + \nu \int_0^t \boldsymbol{\Delta} u(s)\, ds + \sigma W(t), \tag{4.18}$$

$$u(t) = \varphi + \nu \int_0^t \tilde{\Lambda}^2 u(s)\, ds + \sigma W(t), \tag{4.19}$$

with $\tilde{\Lambda}^2$ from (4.3). In physics literature, the deterministic version of (4.19) is known as the Hermite heat equation [7].

Definition 4.5. *For each of the three equations, given the initial condition $\varphi \in L_2(\Omega; \tilde{H}^r)$, a solution $u = u(t)$ on $[0, T]$ is an adapted process with values in $L_2\big(\Omega \times (0,T); \tilde{H}^{r+1}\big) \bigcap L_2\big(\Omega; \mathcal{C}((0,T); \tilde{H}^{\gamma}\big)$, such that the corresponding equality holds in $\tilde{H}^{r-1}$ for all $t \in [0,T]$ with probability one.*

Theorem 4.6. *Assume that $\varphi \in \tilde{H}^{-\gamma}$ and $\gamma > \mathrm{d}$. Then*

(1) *Equation (4.17) has a unique solution for every $T > 0$;*
(2) *The solution has a representation*

$$u(t) = S_t \varphi + \int_0^t S_{t-s} dW(s), \tag{4.20}$$

with S_t from (4.15);
(3) *as $t \to +\infty$, the solution converges in distribution to $(2\nu)^{-1/2}\sigma W_{\varepsilon}$, that is, the Gaussian measure generated on $\tilde{H}^{-\gamma}$ by the solution converges weakly to the Gaussian measure generated by $(2\nu)^{-1/2}\sigma W_{\varepsilon}$.*

Proof. The general theory of SPDEs in the Sobolev spaces H^γ, such as [15], is not applicable because the process $W = W(t)$ does not take values in any of H^γ. Similarly, the results from [4] do not apply because the operator $(-\boldsymbol{\Delta})^{-1}$ is not Hilbert-Schmidt on $L_2(\mathbb{R}^{\mathrm{d}})$.

Fortunately, for existence and uniqueness of solution, relation (4.5) and first part of Proposition 4.4 make it possible to apply [22, Theorem 3.1] with

$$A = \nu(\boldsymbol{\Delta} - \varepsilon),\ \mathbb{X} = \tilde{H}^{1-\gamma},\ \mathbb{H} = \tilde{H}^{-\gamma},\ \mathbb{X}' = \tilde{H}^{-\gamma-1},\ M(t) = W(t).$$

Similarly, the second part of Proposition 4.4 makes it possible to apply [6, Theorem 5.4], from which (4.20) follows.

To prove convergence, note that, by (4.11), the general argument outlined in Introduction works, with $\mathcal{O} = \mathbb{R}^{\mathrm{d}}$ and $\mathcal{G} = \mathcal{S}(\mathbb{R}^{\mathrm{d}})$. Keeping in mind that the fundamental solution for (4.17) is $G_{\nu\varepsilon,\mathrm{d}}$, which, by (4.9), acts as the multiplier

$$\hat{f}(\xi) \ \mapsto e^{-\nu(|\xi|^2+\varepsilon)t}\hat{f}(\xi)$$

in the Fourier domain, we easily complete the proof. □

Theorem 4.7. *If $\varphi \in \tilde{H}^{-\gamma}$ and $\gamma > \mathrm{d}$, then equation (4.18) has a unique solution for every $T > 0$ and the solution has a representation*

$$u(t) = S_t\varphi + \int_0^t S_{t-s}dW(s), \tag{4.21}$$

where S_t is from (4.15) with $\varepsilon = 0$.

If $\varphi \in H^{-\gamma}$ and $\gamma > \mathrm{d}$, then, as $t \to +\infty$, the solution converges in distribution to $(2\nu)^{-1/2}\sigma\bar{W}$, that is, the Gaussian measure generated on $\tilde{H}^{-\gamma}$ by the solution converges weakly to the Gaussian measure generated by $(2\nu)^{-1/2}\sigma\bar{W}$.

Proof. Existence, uniqueness, and representation (4.21) of the solution follow in the same way as in the proof of Theorem 4.6. To prove the convergence as $t \to +\infty$, we streamline the notations by setting $G = G(t, x)$ to be the heat kernel for equation (4.18):

$$G(t, x) = \frac{1}{(4\nu t)^{\mathrm{d}/2}}\, e^{-|x|^2/(4\nu t)}.$$

Given a function $f = f(x)$ from $\mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$, denote by $u^{\mathrm{H},f} = u^{\mathrm{H},f}(t, x)$ the solution of the deterministic heat equation with initial condition f:

$$u^{\mathrm{H},f}(t, x) = \int_{\mathbb{R}^{\mathrm{d}}} G(t, x - y)f(y)dy.$$

Then

$$u(t, x) = u^{\mathrm{H},\varphi}(t, x) + \sigma \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} G(t - s, x - y)\, W(ds, dy)$$

and, using (2.6) and [6, Theorem 5.4],

$$
\begin{aligned}
u[t,f] &:= \langle f, u(t)\rangle_{0,\gamma} \\
&= u^{\mathrm{H},\varphi}[t,f] + \sigma \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} \left(\int_{\mathbb{R}^{\mathrm{d}}} G(t-s, x-y) f(x)dx \right) W(ds,dy) \\
&= u^{\mathrm{H},\varphi}[t,f] + \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} u^{\mathrm{H},f}(t-s,y)\, W(ds,dy).
\end{aligned}
$$

By independence of φ and W,

$$
\begin{aligned}
\mathbb{E}\Big(u[t,f]u[t,g]\Big) &= \mathbb{E}\Big(u^{\mathrm{H},\varphi}[t,f]u^{\mathrm{H},\varphi}[t,g]\Big) + \sigma^2 \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} u^{\mathrm{H},f}(t-s,y) u^{\mathrm{H},g}(t-s,y)\, dy\, ds \\
&= \mathbb{E}\Big(u^{\mathrm{H},\varphi}[t,f]u^{\mathrm{H},\varphi}[t,g]\Big) + \sigma^2 \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} u^{\mathrm{H},f}(s,y) u^{\mathrm{H},g}(s,y)\, dy\, ds.
\end{aligned}
$$

Next,

$$
\hat{u}^{\mathrm{H},f}(s,\xi) = \hat{f}(\xi) e^{-s\nu|\xi|^2},
$$

and then the Fourier isometry implies

$$
\begin{aligned}
\mathbb{E}\Big(u[t,f]u[t,g]\Big) = &\iint\limits_{\mathbb{R}^{\mathrm{d}}\times\mathbb{R}^{\mathrm{d}}} e^{-t\nu(|\xi|^2+|\eta|^2)} \mathbb{E}\big(\hat{\varphi}(\xi)\hat{\varphi}(\eta)\big) \overline{\hat{f}(\xi)\hat{g}(\eta)}\, d\xi\, d\eta \\
&+ \sigma^2 \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} \hat{f}(\xi)\overline{\hat{g}(\xi)} e^{-2s\nu|\xi|^2}\, d\xi\, ds.
\end{aligned}
\tag{4.22}
$$

The first term on the right-hand side of (4.22) goes to zero as $t \to \infty$ by the dominated convergence theorem, because, by assumption,

$$
\int_{\mathbb{R}^{\mathrm{d}}} (1+|\xi|^2)^{-\gamma} \mathbb{E}|\hat{\varphi}(\xi)|^2 d\xi < \infty
$$

for some $\gamma > \mathrm{d}$, and, for $f, g \in \mathcal{S}_0(\mathbb{R}^{\mathrm{d}})$,

$$
\sup_{\xi} (1+|\xi|^2)^{\gamma} |\hat{f}(\xi)| < \infty, \quad \sup_{\eta} (1+|\eta|^2)^{\gamma} |\hat{g}(\eta)| < \infty.
$$

With $\varepsilon = 0$, we no longer have (4.16) and therefore have to make additional assumptions about the initial condition to achieve the desired convergence.

As a result,

$$
\lim_{t\to+\infty} \mathbb{E}\Big(u[t,f]u[t,g]\Big) = \sigma^2 \int_{\mathbb{R}^{\mathrm{d}}} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{2\nu|\xi|^2}\, d\xi.
$$

Together with (4.6), the last equality completes the proof. □

Analysis of equation (4.19) in the scale $\mathbb{H}_{\tilde{\Lambda}}$ is equivalent to analysis of equation (3.7) in the scale $\mathbb{H}_{\Lambda}$: similar to Theorem 3.5 and Corollary 3.6, the distribution of $\sigma(2\nu)^{-1/2}\tilde{W}$ is the unique invariant measure for equation (4.19). The only difference is that now we have λ_k of order $k^{1/(2\mathrm{d})}$ rather than $k^{1/\mathrm{d}}$.

Let $\mathfrak{h}_k,\ k \geq 1$, be the Hermite functions and let $\lambda_k,\ k \geq 1$, be the corresponding eigenvalues of the operator $\tilde{\Lambda}^2$.

Theorem 4.8. *Assume that* $\varphi \in \tilde{H}^{-\gamma}$ *and* $\gamma > \mathrm{d}$. *Then*

(1) *Equation* (4.19) *has a unique solution for every* $T > 0$*;*
(2) *For every* $t > 0$, $u(t) \in L_2\big(\Omega; \tilde{H}^{1-\gamma}\big)$ *and*

$$u(t) = \sum_{k=1}^{\infty} e^{-\nu\lambda_k^2 t}\varphi_k \mathfrak{h}_k + \sum_{k=1}^{\infty} \tilde{u}_k(t)\mathfrak{h}_k,$$

where and $\tilde{u}_k(t)$, $k \geq 1$, *are independent Gaussian random variables with mean zero and variance*

$$\mathbb{E}\tilde{u}_k^2(t) = \frac{\sigma^2}{2\nu\lambda_k^2}\Big(1 - e^{-2\nu\lambda_k^2 t}\Big);$$

(3) *As* $t \to +\infty$, *the* $\tilde{H}^{1-\gamma}$*-valued random variables* $u(t)$ *converge weakly to* $\sigma(2\nu)^{-1/2}\tilde{W}$*;*
(4) *Equation* (4.19) *is ergodic and the unique invariant measure is the distribution of* $\sigma(2\nu)^{-1/2}\tilde{W}$ *on* $\tilde{H}^{1-\gamma}$*;*
(5) *If* $\varphi \overset{d}{=} \sigma(2\nu)^{-1/2}\tilde{W}$, *then* $u(t) \overset{d}{=} \sigma(2\nu)^{-1/2}\tilde{W}$ *for all* $t > 0$*;*
(6) *If* $\mathbb{E}\varphi = 0$, *then, for each* $t > 0$, *the measure generated by* $u(t)$ *on* $\tilde{H}^{1-\gamma}$ *is absolutely continuous with respect to the measure generated by* $\sigma(2\nu)^{-1/2}\tilde{W}$.

5. Some Comments on the One-Dimensional Case

In one space dimension, the Gaussian free field $\bar{W} = \bar{W}(x)$, $x \in \mathcal{O} \subseteq \mathbb{R}$, is a regular, as opposed to a generalized, process: $\bar{W}(x)$ is a zero-mean Gaussian random variable for each $x \in \mathcal{O}$,

If $\mathcal{O} = (a, b)$ is a bounded interval, then

$$\bar{W}(x) = \sum_{k=1}^{\infty} \frac{\zeta_k}{\lambda_k}\, \mathfrak{h}_k(x). \tag{5.1}$$

In (5.1),

- ζ_k, $k \geq 1$, are idd standard Gaussian random variables;
- $\mathfrak{h}_k = \mathfrak{h}_k(x)$ and $-\lambda_k^2 < 0$, $k \geq 1$, are the normalized eigenfunctions and the eigenvalues of the Laplacian on (a, b) with suitable boundary conditions:

$$\mathfrak{h}_k''(x) = -\lambda_k^2 \mathfrak{h}_k(x),\ \int_a^b \mathfrak{h}_k^2(x)\, dx = 1,\ \int_a^b \mathfrak{h}_k(x)\mathfrak{h}_m(x)\, dx = 0,\ k \neq m.$$

The series in (5.1) converges with probability one because, by (3.1), $\lambda_k \sim ck$. Moreover,

$$\mathbb{E}\bar{W}(x) = 0,\ \mathbb{E}\Big(\bar{W}(x)\bar{W}(y)\Big) = \sum_{k=1}^{\infty} \frac{\mathfrak{h}_k(x)\mathfrak{h}_k(y)}{\lambda_k^2} = \Phi_{\mathcal{O}}(x, y),$$

where $\Phi_{\mathcal{O}}$ is Green's function of the Laplacian on (a, b) with appropriate boundary conditions, which also shows that (5.1) is the *Karhunen-Loève decomposition* of $\bar{W}$ [17, Example 3.2.18]. Zero boundary conditions $\mathfrak{h}_k(a) = \mathfrak{h}_k(b) = 0$ imply $\bar{W}$ is a (multiple of a) Brownian bridge on $[a, b]$, whereas $\mathfrak{h}_k(a) = \mathfrak{h}_k'(b) = 0$ imply $\bar{W}$ is a (multiple of a) standard Brownian motion. Of course, (5.1) is a particular case of (3.6) and is consistent with the general definition (1.1) of the Gaussian free field.

For unbounded intervals, the convention is somewhat different.

If $\mathcal{O} = (0, +\infty)$, then $\bar{W}$ is defined as the standard Brownian motion; this convention, in particular, means the boundary condition at $x = 0$ is fixed and is equal to zero.

If $\mathcal{O} = \mathbb{R}$, then the Gaussian free field on $\mathcal{O}$ is defined by

$$\breve{W}(x) = \begin{cases} W(x), & x > 0 \\ V(-x), & x < 0. \end{cases} \tag{5.2}$$

In (5.2), W and V are independent standard Brownian motions. In particular, $\breve{W}$ is a zero-mean Gaussian process with covariance $\rho(x, y) = \mathbb{E}\big(\breve{W}(x)\breve{W}(y)\big)$ given by

$$\rho(x, y) = \begin{cases} \min(|x|, |y|), & xy > 0, \\ 0, & xy < 0. \end{cases} \tag{5.3}$$

Equality (5.3) means that $\breve{W}$ is not a Gaussian free field in the sense of the general definition (1.1) but rather the two-sided standard Brownian motion, which also happens to be the *Lévy Brownian motion* on $\mathbb{R}$; cf. [16, Chapter VIII]. Indeed, (5.3) implies

$$\mathbb{E}\big(\breve{W}(x) - \breve{W}(y)\big)^2 = |x - y|, \ x, y \in \mathbb{R}.$$

An alternative description of $\breve{W}$ on $\mathbb{R}$ is a random generalized function acting on $f \in \mathcal{S}(\mathbb{R})$ by

$$\breve{W}[f] = \int_{-\infty}^{+\infty} f(x)\breve{W}(x)\, dx. \tag{5.4}$$

Then

$$\mathbb{E}\Big(\breve{W}[f]\breve{W}[g]\Big) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)\rho(x, y)\, dxdy. \tag{5.5}$$

Given a function $f \in \mathcal{S}(\mathbb{R})$, define the function $F = F(x)$ by

$$F(x) = \begin{cases} \displaystyle\int_{-\infty}^{x} f(t)\, dt, & x < 0; \\ \\ \displaystyle -\int_{x}^{+\infty} f(t)\, dt, & x > 0. \end{cases} \tag{5.6}$$

By direct computation, the function F is continuous except possibly at $x = 0$, and so

$$F'(x) = f(x) - \left(\int_{-\infty}^{+\infty} f(t)\, dt\right)\delta(x),$$

where $\delta(x)$ is the point mass (Dirac delta function) at zero. Moreover, if $f \in \mathcal{S}(\mathbb{R})$, then, for all $p > 0$,

$$\lim_{|x|\to+\infty} |x|^p \; |F(x)| = 0. \tag{5.7}$$

We can integrate by parts in (5.4) using (5.2):

$$\breve{W}[f] = \int_0^{+\infty} F(-x)\, dV(x) - \int_0^{+\infty} F(x)\, dW(x). \tag{5.8}$$

The transition from (5.4) to (5.8) essentially relies on (5.7) and the equality $\breve{W}(0) = 0$. By (5.8), we get an alternative form of (5.5):

$$\mathbb{E}\Big(\breve{W}[f]\breve{W}[g]\Big) = \int_{-\infty}^{+\infty} F(x)G(x)\, dx; \tag{5.9}$$

the function G is constructed from the function g according to (5.6).

If

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\mathfrak{i}x\xi} f(x)\, dx$$

is the Fourier transform of f, then, by direct computation,

$$\hat{F}(\xi) = \frac{\hat{f}(\xi) - \hat{f}(0)}{\mathfrak{i}\xi}; \tag{5.10}$$

recall that

$$\hat{f}(0) = \int_{-\infty}^{+\infty} f(x)\, dx.$$

By (5.10) and the L_2 isometry of the Fourier transform, (5.9) becomes

$$\mathbb{E}\Big(\breve{W}[f]\breve{W}[g]\Big) = \int_{-\infty}^{+\infty} \frac{\big(\hat{f}(\xi) - \hat{f}(0)\big)\overline{\big(\hat{g}(\xi) - \hat{g}(0)\big)}}{|\xi|^2}\, d\xi; \tag{5.11}$$

as usual, $\overline{z}$ denotes the complex conjugate of z. The integral on the right-hand side of (5.11) converges as long as the functions $\hat{f}$ and $\hat{g}$ are differentiable at zero, which is the case, for example, if

$$\int_{-\infty}^{+\infty} |xf(x)|\, dx < \infty, \quad \int_{-\infty}^{+\infty} |xg(x)|\, dx < \infty, \tag{5.12}$$

and certainly holds if $f, g \in \mathcal{S}(\mathbb{R})$. With (5.3) in mind, condition (5.12) is also sufficient for convergence of the integral on the right-hand side of (5.5).

Equality (5.11) confirms that $\breve{W}$ from (5.2) is not a Gaussian free field in the sense of Definition 4.2.

6. Summary and Further Directions

Let $\mathcal{L}$ be a self-adjoint elliptic operator on a separable Hilbert space H. Under suitable conditions, we expect that, as $t \to +\infty$, the solution of the parabolic equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f$$

to converge to the solution of the elliptic equation

$$\mathcal{L}v = -f.$$

The results of this paper show that, under some conditions, the solution of the stochastic evolution equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u + \dot{W}, \tag{6.1}$$

driven by a cylindrical Brownian motion on H, converges in distribution to the solution of

$$(-\mathcal{L})^{1/2}v = V, \tag{6.2}$$

where V is an isonormal Gaussian process on H. In particular, we establish this convergence when $\mathcal{L}$ is the Laplace operator and the solution of (6.2) is the Gaussian free field. One could study equation (6.1) with other operators $\mathcal{L}$ and driving processes $\dot{W}$, resulting in different limits coming out of equation (6.2). Beside purely mathematical interest, another motivation for this study is scaling limits of (mostly yet to be discovered) discrete models.

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