A Summary of Counting Methods[1]

Multiplication principle. With n_k choices for option k, $k = 1, \dots, N$, the total number of choices is

$$\prod_{k=1}^{N} n_k = n_1 \cdot n_2 \cdots n_N.$$

Examples

- CA Lic plate of the type 4MLG145 (with $1-8$ for the first number, no other restrictions) gives $8 \cdot 26^3 \cdot 10^3 = 140{,}608{,}000$ possibilities.
- all 4-digit numbers: from $0 = 0000$ to 9999, that is, 10^4 total.
- Given a *factorization* $N = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ of an integer N into k distinct prime factors, the total number of the divisors of N (including 1 and N) is

  $$d(N) = (n_1+1)(n_2+1)\cdots(n_k+1).$$

  For example, if $N = 45 = 3^2 \cdot 5$, then, with $p_1 = 3, n_1 = 2; p_2 = 5, n_2 = 1$, we get $d(N) = (2+1)\cdot(1+1) = 3 \cdot 2 = 6$. Indeed, the divisors of 45 are $1, 3, 5, 9, 15, 45$. The number

  $$278914005382139703576000 = 2^6 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$$

  has $7 \cdot 6 \cdot 4 \cdot 3 \cdot 2^{11} = 1032192$ divisors; it is the smallest number with at least one million divisors; for more, see `https://oeis.org/A002182`
- `occupancy problem` with m (different) balls and n (different) boxes, gives

  $$n^m = (\#\text{of boxes})^{\#\text{of balls}}$$

  as the total number of configurations. Counting 4-digit numbers is an example, with boxes representing digits and balls selecting those digits, so that the number 155=0155 corresponds to one ball in box 0, one ball in box 1 and two balls in box 5.

Permutations. The number of ways to order n objects is, by multiplication principle, $n \cdot (n-1) \cdots 2 \cdot 1 = \prod_{k=1}^{n} k = n!$ The convention is $0! = 1$. The *basic Stirling formula* is

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

in the sense that the ratio of the left and right goes to 1 as $n \to \infty$.

Examples

- The number of different "anagrams" (not all of them are actual words) of the word representation is $\frac{14!}{3!(2!)^3}$, where 14 is the total number of letters, of which there are three "e" and three more letters (r,t,n) come in pairs.

Combinations n choose k, or *binomial coefficients*, for non-negative integers n, k:

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & n \ge k \\ 0, & n < k. \end{cases}$$

One of the main relations is the *binomial formula*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Pascal's triangle is one way to compute $\binom{n}{k}$ for fixed n and all k.

Partitions.

(1) By default, `partition` refers to an *unordered* partition of a positive integer; $p(n)$ denotes the total number of such partitions. For example, $p(3) = 3$: $3, 2+1, 1+1+1$. Similarly, $p(5) = 7$. There is no simple formula for $p(n)$, but we have a generating function and an asymptotic relation:

$$\sum_{n=0}^{\infty} p(n) z^n = 1 + z + 2z^2 + \ldots = \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \quad p(n) \sim \frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{2n/3}}.$$

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(2) An *ordered partition* of a positive integer n is a positive integer solution of the equation $x_1+\ldots+x_k=n$ with $1\le k\le n$. The number of such solutions is $\binom{n-1}{k-1}$, meaning that the total number of ordered partitions of n is $\sum_{k=1}^{n}\binom{n-1}{k-1}=2^{n-1}$. For example, the four ordered partitions of the number 3 are $3, 2+1, 1+2, 1+1+1$.

More generally, for two positive integers k, N, the number of *non-negative* integer solutions of the equation $x_1+\ldots+x_k=N$ is $\binom{N+k-1}{k-1}$. Equivalently, the number of strictly positive integer solutions is $\binom{N-1}{k-1}$.

(3) *Ordered partitions of a set* with n elements are counted by the `ordered Bell numbers` $a(n)$, also known as the `Fubini numbers`, with convention $a(0)=1$. Each partition represents a possible outcome in a race with n participants when ties are allowed, or a way to group n iterated integrals [whence the Fubini connection]. For example, for a set $\{1,2,3\}$ with three elements, $a(3)=13$, as there are 6 ways to order $\{\{1\},\{2\},\{3\}\}$ [corresponding to no ties in a race], another 6 ways coming from $\{\{1,2\},\{3\}\}$ [corresponding to a two-way tie in the race], and one more, $\{\{1,2,3\}\}$, corresponding to a tree-way tie. There is no simple formula for $a(n)$, but we have an *exponential* generating function and an asymptotic relation:

$$\sum_{n=0}^{\infty}\frac{a(n)}{n!}z^n=\frac{1}{2-e^z},\quad a(n)\sim\frac{n!}{2(\ln 2)^{n+1}}.$$

(4) *Unordered partitions of a set* with n elements are counted by the (usual) `Bell numbers` B_n, with convention $B_0=1$. For example, for a set $\{1,2,3\}$ with three elements, $B_3=5$, as there is one partition into single elements $\{\{1\},\{2\},\{3\}\}$, three partitions of the form $\{\{1,2\},\{3\}\}$, and the "trivial" partition $\{\{1,2,3\}\}$. We have an*exponential* generating function, a recursion (by partitioning sub-sets of size k), and an asymptotic expansion:

$$\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n=e^{e^z-1},\quad B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_k,$$
$$\frac{\ln B_n}{n}=\ln n-\ln(\ln n)-1+\frac{\ln(\ln n)}{\ln n}+\frac{1}{\ln n}+o(1/\ln n),\ n\to\infty.$$

In particular, $B_n=e^{-1}\sum_{k=0}^{\infty}k^n/k!$, and, as $n\to+\infty$,

$$B(n)\sim n^{-1/2}\big(\lambda(n)\big)^{n+(1/2)}e^{\lambda(n)-n-1},$$

where $\lambda=\lambda(x)$ is the function inverse to $f(x)=x\ln x, x>1$: $\lambda(n)\ln\lambda(n)=n$.

Because $\ln 2<1$ and $\frac{\ln n!}{n}=\ln n-1+\frac{\ln n}{n}+O(1/n)$, we have, for all sufficiently large positive integer n, $p(n)<2^n<B_n<n!<a(n)<n^n$.

Bijections. A bijection is a function that is both one-to-one and onto. Given two *finite* sets A and B, a bijection from A to B exists if and only if the sets A and B have the same number of elements.

EXAMPLES

- Given a set $\Omega=\{\omega_1,\ldots,\omega_n\}$ with n elements, there is bijection from the set of all subsets of Ω to binary strings of length n: given a sub-set $A\subseteq\Omega$, a string $(x_1,\ldots,x_n)$, with $x_k=1$ if $\omega_k\in A$ and $x_k=0$ if $\omega_k\notin A$, represents A. With this representation, $\emptyset$ corresponds to $(0,\ldots,0)$ and Ω corresponds to $(1,\ldots,1)$, and we conclude that there are 2^n subsets of Ω.

Inclusion-Exclusion Principle: If $A_1,A_2,\ldots,A_n$ are finite sets, then, with $|B|$ denoting the number of elements in the set B,

$$|A_1\cup A_2\cup\ldots\cup A_n|=\sum_{k=1}^{n}|A_k|-\sum_{k_1<k_2}|A_{k_1}\cap A_{k_2}|+\sum_{k_1<k_2<k_3}|A_{k_1}\cap A_{k_2}\cap A_{k_3}|-\ldots$$
$$+(-1)^{n+1}|A_1\cap\ldots\cap A_n|.$$

EXAMPLES

- The number of ways to distribute 20 different gifts among 7 (different) children so that each child gets at least one gift is the total number of ways to distribute 20 gifts among 7 children minus the number of cases in which somebody get no gifts:

$$7^{20}-\left(\binom{7}{1}6^{20}-\binom{7}{2}5^{20}+\binom{7}{3}4^{20}-\binom{7}{4}3^{20}+\binom{7}{5}2^{20}-\binom{7}{6}\right)\approx 5.6\cdot 10^{16}.$$