

## SMALL BALL PROBABILITIES FOR THE INFINITE-DIMENSIONAL ORNSTEIN-UHLENBECK PROCESS IN SOBOLEV SPACES

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ABSTRACT. While small ball, or lower tail, asymptotic for Gaussian measures generated by solutions of stochastic ordinary differential equations is relatively well understood, a lot less is known in the case of stochastic partial differential equations. The paper presents exact logarithmic asymptotics of the small ball probabilities in a scale of Sobolev spaces when the Gaussian measure is generated by the solution of a diagonalizable stochastic parabolic equation. Compared to the finite-dimensional case, new effects appear in a certain range of the Sobolev exponents.

### 1. INTRODUCTION

A standard Gaussian random variable  $\zeta$  is very unlikely to be large:

$$\mathbb{P}(|\zeta|^2 > A) \leq e^{-A/2}, \quad A > 0,$$

(cf. [5, Lemma A.3]), but it is relatively likely to be small: by direct computation,

$$\mathbb{P}(|\zeta|^2 \leq \varepsilon) \geq \frac{\varepsilon^{1/2}}{3}, \quad \varepsilon < 1.$$

In fact, for every finite collection of iid standard Gaussian random variables  $\zeta_1, \dots, \zeta_n$ , analysis of the density of the  $\chi_n^2$  distribution shows that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-n/2} \mathbb{P} \left( \sum_{k=1}^n \zeta_k^2 \leq \varepsilon \right) = \frac{2^{(2-n)/2}}{n\Gamma(n/2)},$$

where  $\Gamma$  is the Gamma function. Similarly, for *finitely* many Gaussian random variables, the asymptotic of

$$\mathbb{P} \left( \sum_{k=1}^n a_k \zeta_k^2 \leq \varepsilon \right), \quad a_k > 0,$$

is always algebraic in  $\varepsilon$ , as  $\varepsilon \rightarrow 0$ . On the other hand, for a standard  $N$ -dimensional Brownian motion  $\mathbf{w} = \mathbf{w}(t)$ ,  $0 \leq t \leq T$ ,

$$\mathbb{P} \left( \int_0^T |\mathbf{w}(t)|^2 dt \leq \varepsilon \right) \approx e^{-N^2 T^2 / (8\varepsilon)}, \quad \varepsilon \rightarrow 0,$$

that is,

$$(1.1) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \ln \mathbb{P} \left( \int_0^T |\mathbf{w}(t)|^2 dt \leq \varepsilon \right) = -\frac{N^2 T^2}{8};$$

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cf. [8, Theorem 6.3 and Corollary 3.1] or Corollary 2.5 below.

Transition from a finite to an infinite number of Gaussian random variables typically leads to a qualitative change of behavior of small ball (or lower tail) probabilities: if  $a_k > 0$ ,  $\sum_k a_k < \infty$ , then, as  $\varepsilon \rightarrow 0$ , the decay of

$$\mathbb{P} \left( \sum_{k=1}^{\infty} a_k \zeta_k^2 \leq \varepsilon \right),$$

is usually faster than polynomial in  $\varepsilon$ .

The logarithmic asymptotic (1.1) is rather robust: if  $\mathbf{x} = \mathbf{x}(t)$  is the solution of the linear equation

$$(1.2) \quad d\mathbf{x}(t) = A\mathbf{x}(t) + Qd\mathbf{w}(t), \quad 0 < t \leq T, \quad \mathbf{x}(0) = 0,$$

with a positive-definite matrix  $Q$ , then

$$(1.3) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \ln \mathbb{P} \left( \int_0^T |\mathbf{x}(t)|^2 dt \leq \varepsilon \right) = -\frac{(T \operatorname{trace}(Q))^2}{8}$$

regardless of the matrix  $A$ ; cf. [10, Theorem 4.5]. In one-dimensional case, (1.3) continues to hold even with some time-dependent drifts [3].

A possible infinite-dimensional generalization of (1.2) is the stochastic wave equation

$$(1.4) \quad u_{tt} = u_{xx} + g(u) + \dot{W}(t, x), \quad t > 0, \quad x \in \mathbb{R},$$

where  $W = W(t, x)$  is a two-parameter Brownian sheet and  $\dot{W}(t, x) = \partial^2 W / (\partial t \partial x)$  is the corresponding space-time Gaussian white noise. Indeed, a change of variables reduces (1.4) to

$$(1.5) \quad \frac{\partial^2 v}{\partial t \partial x} = g(v) + \frac{\partial^2 \tilde{W}}{\partial t \partial x},$$

with a different Brownian sheet  $\tilde{W}$ ; cf. [16, Theorem 3.1]. Equation (1.5) can thus be considered an extension of (1.2) to two independent variables in the spirit of [4, Section 7.4.2]; according to [11], the small ball probabilities for  $u$  and  $W$  have similar asymptotics.

So far, the paper [11] appears to be the only work addressing the question of small ball probabilities for stochastic partial differential equations. The objective of the current paper is to investigate the asymptotic behavior of  $\ln \mathbb{P}(\|u\|_{L_2((0,T);H^\gamma)}^2 \leq \varepsilon)$ ,  $\varepsilon \rightarrow 0$ , where  $u$  is the solution of the stochastic parabolic equation

$$(1.6) \quad u_t(t, \mathbf{x}) + Au(t, \mathbf{x}) = \dot{W}(t, \mathbf{x}), \quad 0 < t \leq T, \quad \mathbf{x} \in G,$$

$A$  is a positive self-adjoint elliptic operator on a bounded domain  $G \subset \mathbb{R}^d$ ,  $\dot{W}$  is space-time Gaussian white noise, and  $H^\gamma$ ,  $\gamma \in \mathbb{R}$ , is the scale of Sobolev space generated by  $A$ . An expansion of the solution of (1.6) in eigenfunctions of  $A$  leads to an infinite system of ordinary differential equations, making (1.6) an infinite-dimensional version of (1.2). The

results can be summarized as follows: For both  $X = u$  and  $X = W$ , as  $\varepsilon \rightarrow 0$ ,

$$\ln \mathbb{P} \left( \int_0^T \|X(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \mathfrak{C}(\gamma) \varepsilon^{-1}, & \text{if } \gamma > d, \\ -\mathfrak{C}(T, X) \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = d, \\ -\mathfrak{C}(T, \gamma, X) \varepsilon^{-\varpi(\gamma, X)}, & \text{if } \gamma_0(X) < \gamma < d, \end{cases}$$

where  $\mathfrak{C}$ ,  $\gamma_0$ , and  $\varpi$  are suitable numbers. For example,  $\gamma_0(W) = d/2$  and  $\varpi(\gamma, W) = d/(2\gamma - d)$ . In particular, if  $\gamma > d$ , then the result is very similar to the finite-dimensional case (1.3). The details are below in Theorem 3.3 ( $X = W$ ) and Theorem 3.6 ( $X = u$ ).

Throughout the paper, for  $f(x) > 0$ ,  $g(x) > 0$ , the notation

$$f(x) \sim g(x), \quad x \rightarrow x_0,$$

means

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1,$$

$f(x) = O(g(x))$ ,  $x \rightarrow x_0$ , means  $\limsup_{x \rightarrow x_0} f(x)/g(x) < \infty$ , and  $f(x) = o(g(x))$ ,  $x \rightarrow x_0$ , means  $\lim_{x \rightarrow x_0} f(x)/g(x) = 0$ . The variable  $x$  can be discrete or continuous and the limiting value  $x_0$  finite or infinite. We also fix  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ , a stochastic basis satisfying the usual assumptions.

## 2. BACKGROUND ON SMALL BALL PROBABILITIES

Let  $\zeta_n$ ,  $n \geq 1$ , be independent identically distributed standard Gaussian random variables and let  $a_n$ ,  $n \geq 1$ , be positive real numbers such that  $\sum_n a_n < \infty$ . By direct computation,

$$\begin{aligned} \mathbb{E} \exp \left( -p \sum_{n=1}^{\infty} a_n \zeta_n^2 \right) &= \prod_{n=1}^{\infty} \mathbb{E} e^{-pa_n \zeta_n^2} = \prod_{n=1}^{\infty} (1 + 2pa_n)^{-1/2} \\ (2.1) \quad &= \exp \left( -\frac{1}{2} \sum_{n=1}^{\infty} \ln(1 + 2pa_n) \right), \quad p > 0, \end{aligned}$$

and then **Tauberian theorems** make it possible to connect the asymptotic of the right-hand side of (2.1) as  $p \rightarrow +\infty$  with the asymptotic of

$$\mathbb{P} \left( \sum_{n=1}^{\infty} a_n \zeta_n^2 \leq \varepsilon \right)$$

as  $\varepsilon \rightarrow 0$ . The most general result in this direction was obtained in [15]:

$$\begin{aligned} \mathbb{P} \left( \sum_{n=1}^{\infty} a_n \zeta_n^2 \leq \varepsilon \right) &\sim \left( 4\pi \sum_{n \geq 1} \left( \frac{a_n \mathfrak{r}(\varepsilon)}{1 + 2a_n \mathfrak{r}(\varepsilon)} \right)^2 \right)^{-1/2} \\ (2.2) \quad &\times \exp \left( \varepsilon \mathfrak{r}(\varepsilon) - \frac{1}{2} \sum_{n \geq 1} \ln(1 + 2a_n \mathfrak{r}(\varepsilon)) \right), \quad \varepsilon \rightarrow 0. \end{aligned}$$

The function  $\mathbf{r} = \mathbf{r}(\varepsilon)$  is defined implicitly by the relation

$$\varepsilon = \sum_{n \geq 1} \frac{a_n}{1 + 2a_n \mathbf{r}(\varepsilon)},$$

and this implicit dependence on  $\varepsilon$  is the main drawback of (2.2) in concrete applications.

Less precise but more explicit bounds are possible using **exponential Tauberian theorems**, such as Theorems 2.1 and 2.2 below; they are modifications of [8, Theorem 3.5] (which, in turn, is a modification of [1, Theorem 4.12.9]).

**Theorem 2.1.** *Let  $\xi$  be a non-negative random variable. Then*

$$\ln(\mathbb{E}e^{-p\xi}) \sim -\alpha p^\tau, \quad p \rightarrow +\infty, \quad \text{for some } \alpha > 0, \quad 0 < \tau < 1,$$

*holds if and only if*

$$(2.3) \quad \ln \mathbb{P}(\xi \leq \varepsilon) \sim -((1 - \tau)\alpha)^{1/(1-\tau)} \left( \frac{\tau}{1 - \tau} \right)^{\tau/(1-\tau)} \varepsilon^{-\tau/(1-\tau)}, \quad \varepsilon \rightarrow 0.$$

While (2.3) is only *logarithmic* asymptotic of the probability and is not as strong as (2.2), it is usually more explicit than (2.1) and is good enough in many applications.

When (2.3) holds, we say that the random variable  $\xi$  has the **small ball rate**

$$\varpi = \frac{\tau}{1 - \tau}$$

and the **small ball constant**

$$\mathfrak{C} = ((1 - \tau)\alpha)^{1/(1-\tau)} \left( \frac{\tau}{1 - \tau} \right)^{\tau/(1-\tau)}.$$

Occasionally, a more refined version of Theorem 2.1 is necessary.

**Theorem 2.2.** *Let  $\xi$  be a non-negative random variable. Then*

$$\ln(\mathbb{E}e^{-p\xi}) \sim -\alpha p^\tau (\ln p)^\beta, \quad p \rightarrow +\infty, \quad \text{for some } \alpha > 0, \beta > 0, \quad 0 < \tau < 1,$$

*holds if and only if*

$$\ln \mathbb{P}(\xi \leq \varepsilon) \sim -((1 - \tau)\alpha)^{1/(1-\tau)} \left( \frac{\tau}{1 - \tau} \right)^{\tau/(1-\tau)} \varepsilon^{-\tau/(1-\tau)} |\ln \varepsilon|^{\beta/(1-\tau)}.$$

**Proposition 2.3.** *Let  $x = x(t)$  be the solution of the equation*

$$dx(t) = -ax(t)dt + \sigma dw(t), \quad 0 < t < T,$$

*with  $a \in \mathbb{R}$  and  $\sigma > 0$ , and assume that the initial condition  $x(0)$  is independent of the Brownian motion  $w$  and is a Gaussian random variable with mean  $\mu_0$  and variance  $\sigma_0^2$ . Then*

$$(2.4) \quad \mathbb{E} \exp \left( -p \int_0^T x^2(t) dt \right) = \left( \frac{e^{aT}}{\cosh(\varrho T) + (a/\varrho) \sinh(\varrho T)} \right)^{1/2} \times \frac{\exp \left( -\frac{\psi \mu_0^2}{1 + 2\sigma_0^2 \psi} \right)}{\sqrt{1 + 2\sigma_0^2 \psi}},$$

where

$$\varrho = (a^2 + 2\sigma^2 p)^{1/2},$$

$$\psi = \frac{\varrho - a}{2\sigma^2} \left( 1 - \frac{e^{-\varrho T}}{\cosh(\varrho T) + (a/\varrho) \sinh(\varrho T)} \right).$$

*Proof.* This follows by direct computation using [6, Theorem 3]; see also [9, Lemma 17.3] when  $\mu_0 = \sigma_0 = 0$ .  $\square$

**Corollary 2.4.** *For the standard Brownian motion, with  $a = \mu_0 = \sigma_0 = 0$ , and  $\sigma = 1$ , equality (2.4) becomes the well-known Cameron-Martin formula:*

$$(2.5) \quad \mathbb{E} \exp \left( -p \int_0^T w^2(s) ds \right) = \left( \cosh(\sqrt{2p} T) \right)^{-1/2}.$$

As an illustration of Theorem 2.1, let us confirm (1.1).

**Corollary 2.5.** *If  $\mathbf{w} = \mathbf{w}(t)$ ,  $0 \leq t \leq T$ , is an  $N$ -dimensional standard Brownian motion, then*

$$(2.6) \quad \ln \mathbb{P} \left( \int_0^T |\mathbf{w}(t)|^2 dt \leq \varepsilon \right) \sim -\frac{N^2 T^2}{8} \varepsilon^{-1}, \quad \varepsilon \rightarrow 0.$$

*Proof.* By (2.5) and independence of the components of  $\mathbf{w}$ ,

$$\ln \mathbb{E} \exp \left( -p \int_0^T |\mathbf{w}(t)|^2 dt \right) \sim -\frac{NT}{\sqrt{2}} \sqrt{p}.$$

Then (2.6) follows from Theorem 2.1 with  $\alpha = NT/\sqrt{2}$  and  $\tau = 1/2$ .  $\square$

### 3. DIAGONALIZABLE STOCHASTIC PARABOLIC EQUATION

Let  $A$  be a positive-definite self-adjoint elliptic operator of order  $2m$  on a bounded domain  $G \subset \mathbb{R}^d$  with sufficiently smooth boundary; alternatively,  $G$  can be a smooth closed  $d$ -dimensional manifold with smooth measure  $d\mathbf{x}$ . Denote by  $\lambda_k$ ,  $k \geq 1$ , the eigenvalues of  $A$ , and by  $\varphi_k$ ,  $k \geq 1$ , the corresponding normalized eigenfunctions. Our main assumption is that the Weyl-type asymptotic holds for  $\lambda_k$ :

$$(3.1) \quad \lambda_k = \mathfrak{S} k^{2m/d} (1 + O(k^{-1/d})), \quad k \rightarrow \infty,$$

with constant  $\mathfrak{S}$  depending only on the region  $G$ ; see [14, Theorem 1.2.1]. For example, if  $A = -\Delta$  on  $G \subset \mathbb{R}^d$  with zero boundary conditions, and  $|G|$  is the Lebesgue measure of  $G$ , then  $m = 1$  and

$$\mathfrak{S} = 4\pi \left( \frac{\Gamma(1 + \frac{d}{2})}{|G|} \right)^{2/d}.$$

For  $f \in \mathcal{C}_0^\infty(G)$  [that is,  $f$  is a smooth compactly supported real-valued function on  $G$ ] and  $\gamma \in \mathbb{R}$ , define

$$f_k = \int_G f(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x} \quad \text{and} \quad \|f\|_\gamma^2 = \sum_{k=1}^{\infty} \lambda_k^{\gamma/m} f_k^2.$$

Then define the space  $H^\gamma$  as the closure of  $\mathcal{C}_0^\infty(G)$  with respect to the norm  $\|\cdot\|_\gamma$ . In particular,

$$H^0 = L_2(G), \quad H^\gamma = A^{\gamma/(2m)}(L_2(G)), \quad \|A^s f\|_\gamma = \|f\|_{\gamma+2ms}.$$

We also define

$$\begin{aligned} \langle f, g \rangle &= \int_G (A^{-\gamma} f)(\mathbf{x})(A^\gamma g)(\mathbf{x}) d\mathbf{x}, \quad f \in H^\gamma, \quad g \in H^{-\gamma}; \\ H^\infty &= \bigcap_{\gamma} H^\gamma, \quad H_\infty = \bigcup_{\gamma} H^\gamma, \end{aligned}$$

and identify  $f \in H^\gamma$  with a possibly divergent series

$$f = \sum_{k \geq 1} f_k \varphi_k, \quad f_k = \langle f, \varphi_k \rangle.$$

Note that  $\varphi_k \in H^\infty$  for all  $k \geq 1$ .

**Definition 3.1.** A cylindrical Brownian motion on  $L_2(G)$  is a Gaussian process  $\mathbf{W} = \mathbf{W}(t, f)$ , indexed by  $t \in [0, T]$  and  $f \in L_2(G)$ , such that

$$\mathbb{E} \mathbf{W}(t, f) = 0, \quad \mathbb{E}(\mathbf{W}(t, f) \mathbf{W}(s, g)) = \min(t, s) \int_G f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}.$$

**Proposition 3.2.** Define

$$w_k(t) = \mathbf{W}(t, \varphi_k)$$

and

$$(3.2) \quad W(t) = \sum_{k=1}^{\infty} w_k(t) \varphi_k.$$

Then, for  $\gamma > \frac{d}{2}$ ,

$$(3.3) \quad W \in L_2\left(\Omega \times (0, T); H^{-\gamma}\right),$$

and

$$(3.4) \quad \mathbf{W}(t, f) = \langle W(t), f \rangle, \quad f \in H^\gamma.$$

*Proof.* Definition of  $\mathbf{W}$  implies that  $w_k(t)$  are independent standard Brownian motions. Then both (3.3) and (3.4) follow by direct computation. In particular, by (3.1),

$$\mathbb{E} \|W(t)\|_{-\gamma}^2 = t \sum_{k=1}^{\infty} \lambda_k^{-\gamma/m} < \infty$$

if and only if  $\gamma > d/2$ . Moreover,

$$\mathbb{E} |\langle W(t), f \rangle|^2 = t \|f\|_0^2,$$

so that (3.4) extends to  $f \in L_2(G)$ , and, by Kolmogorov's criterion,  $W$  has a modification in  $L_2\left(\Omega; \mathcal{C}((0, T); H^{-\gamma})\right)$ ,  $\gamma > \frac{d}{2}$ .  $\square$

We then get the following analogue of (2.6).

**Theorem 3.3.** As  $\varepsilon \rightarrow 0$ ,

$$(3.5) \quad \ln \mathbb{P} \left( \int_0^T \|W(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > d, \\ -\frac{\mathfrak{G}^{-d/m} T^2}{32} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = d, \\ -\mathfrak{C}_\gamma \varepsilon^{-\varpi}, & \text{if } \frac{d}{2} < \gamma < d, \end{cases}$$

where

$$\begin{aligned} \varpi &= \frac{d}{2\gamma - d}, \quad \mathfrak{C}_\gamma = (2q - 1)^{2q} q^{-2q\varpi} 2^{(1-4q)\varpi} (T \mathfrak{G}^{-\gamma/(2m)})^{2\varpi} C_\gamma^{2q\varpi}, \\ q &= \frac{\gamma}{d}, \quad C_\gamma = \int_0^{+\infty} \frac{\ln \cosh(y)}{y^{1+(d/\gamma)}} dy. \end{aligned}$$

*Proof.* By (2.5) and (3.2),

$$(3.6) \quad \begin{aligned} \mathbb{E} \exp \left( -p \int_0^T \|W(t)\|_{-\gamma}^2 dt \right) &= \prod_{k=1}^{\infty} \mathbb{E} \exp \left( -p \lambda_k^{-\gamma/m} \int_0^T \|w_k(t)\|^2 dt \right) \\ &= \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} \ln \cosh \left( T \sqrt{2p \lambda_k^{-\gamma/m}} \right) \right), \end{aligned}$$

and by (3.1),

$$\lambda_k^{-\gamma/m} \sim \mathfrak{G}^{-\gamma/m} k^{-2\gamma/d}, \quad k \rightarrow \infty.$$

If  $\gamma > d$ , then the series  $\sum_k \lambda_k^{-\gamma/(2m)}$  converges, and the dominated convergence theorem implies

$$\lim_{p \rightarrow \infty} p^{-1/2} \sum_{k=1}^{\infty} \ln \cosh \left( T \sqrt{2p \lambda_k^{-\gamma/m}} \right) = T \sqrt{2} \left( \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right),$$

or

$$-\frac{1}{2} \sum_{k=1}^{\infty} \ln \cosh \left( T \sqrt{2p \lambda_k^{-\gamma/m}} \right) \sim -\frac{T}{\sqrt{2}} \left( \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right) \sqrt{p}, \quad p \rightarrow +\infty,$$

so that the first relation in (3.5) follows from Theorem 2.1.

If  $\frac{d}{2} < \gamma \leq d$ , then we establish the asymptotic of (3.6) by comparison with a suitable integral.

Note that

$$T \sqrt{2p \lambda_k^{-\gamma/m}} = \sqrt{p} f(k),$$

and the function  $f = f(x)$  satisfies

$$f(x) = A_\gamma x^{-\gamma/d} (1 + O(x^{-1/d})), \quad x \rightarrow +\infty,$$

with  $A_\gamma = 2^{1/2}T\mathfrak{S}^{-\gamma/(2m)}$ . Let  $q = \gamma/d$ . By direct computation, as  $p \rightarrow +\infty$ ,

$$\begin{aligned} \sum_{k=1}^{\infty} \ln \cosh \left( T \sqrt{2p\lambda_k^{-\gamma/m}} \right) &\sim \int_1^{\infty} \ln \cosh \left( \sqrt{p}f(x) \right) dx \\ &\sim q^{-1}(A_\gamma\sqrt{p})^{1/q} \int_0^{A_\gamma\sqrt{p}} \frac{\ln \cosh(y)}{y^{1+(1/q)}} dy \sim \begin{cases} 2^{-1}A_\gamma \sqrt{p} \ln p, & \text{if } q = 1; \\ q^{-1}A_\gamma^{1/q}C_q p^{1/(2q)}, & \text{if } \frac{1}{2} < q < 1. \end{cases} \end{aligned}$$

After that, Theorems 2.2 and 2.1 imply the remaining relations in (3.5). □

We now use the process  $W$  to construct an infinite-dimensional analogue of (1.2).

Given  $r > 0$ , consider the equation

$$(3.7) \quad \dot{u}(t) + A^r u(t) = \dot{W}(t), \quad 0 < t \leq T, \quad u(0) = 0.$$

For example, with

$$\begin{aligned} G &= [0, \pi], \\ A &= -\partial^2/\partial x^2, \quad \text{zero boundary conditions,} \\ \lambda_k &= k^2, \quad \varphi_k(x) = (2/\pi)^{1/2} \sin(kx). \end{aligned}$$

and  $r = 1$ , equation (3.7) becomes

$$(3.8) \quad u_t = u_{xx} + \dot{W}(t, x), \quad 0 < t \leq T, \quad 0 < x < \pi, \quad u(0, x) = u(t, 0) = u(t, \pi) = 0.$$

**Definition 3.4.** *The solution of the equation (3.7) is a mapping from  $\Omega \times [0, T]$  to  $H_\infty$  with the following properties:*

- (1) *There exists a  $\gamma \in \mathbb{R}$  such that  $u \in L_2\left(\Omega; \mathcal{C}((0, T); H^\gamma)\right)$ .*
- (2) *For every  $h \in H^{-\gamma}$ , the process  $\langle u(t), h \rangle$ ,  $0 \leq t \leq T$ , is  $\mathcal{F}_t$ -adapted.*
- (3) *For every  $h \in H^\infty$ , the equality*

$$(3.9) \quad \langle u(t), h \rangle + \int_0^t \langle u(s), A^r h \rangle ds = \langle W(t), h \rangle$$

*holds in  $L_2(\Omega \times (0, T))$ .*

**Proposition 3.5.** *Equation (3.7) has a unique solution  $u = u(t)$ . Moreover, for every  $\gamma > \frac{d}{2}$ ,*

$$(3.10) \quad u \in L_2\left(\Omega; L_2((0, T); H^{-\gamma+rm})\right) \cap L_2\left(\Omega; \mathcal{C}((0, T); H^{-\gamma})\right).$$

*Proof.* The result can be derived from general existence and uniqueness theorems for stochastic evolution equations, such as [2, Theorem 5.4] or [13, Theorem 3.1.1]; below is an outline of a direct proof.

Taking  $h = \varphi_k$  in (3.9) we find

$$(3.11) \quad \dot{u}_k(t) = -\lambda_k^r u_k(t) + \dot{w}_k(t), \quad u_k(0) = 0,$$



that is,

$$u_k(t) = \int_0^t e^{-\lambda_k^r(t-s)} dw_k(s).$$

Then

$$\mathbb{E}u_k^2(t) = \frac{1 - e^{-2\lambda_k^r T}}{2\lambda_k^r}$$

and

$$\mathbb{E} \int_0^T \|u(t)\|_{-\gamma+rm}^2 dt \leq T \sum_{k=1}^{\infty} \lambda_k^{-\gamma/m}.$$

By (3.1),

$$\lambda_k^{-\gamma/m} \sim \mathfrak{S}^{-\gamma/m} k^{-2\gamma/d}, \quad k \rightarrow \infty.$$

If  $\gamma > \frac{d}{2}$ , then  $-2\gamma/d < -1$ , and (3.10) follows.

Similarly,

$$\mathbb{E}(u_k(t) - u_k(s))^2 \leq |t - s|,$$

and then Kolmogorov's criterion implies that  $u$  has a modification in  $L_2(\Omega; \mathcal{C}((0, T); H^{-\gamma}))$ .

To establish uniqueness, note that the difference  $v$  of two solutions satisfies the deterministic equation  $\dot{v} + Av = 0$  with zero initial condition.  $\square$

**Theorem 3.6.** As  $\varepsilon \rightarrow 0$ ,

(3.12)

$$\ln \mathbb{P} \left( \int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > d, \\ -\frac{\mathfrak{S}^{-d/m} T^2}{32} \left( \frac{d}{d+2rm} \right)^2 \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = d, \\ -\mathfrak{C}_{\gamma,m} \varepsilon^{-\varpi}, & \text{if } \frac{d}{2} - rm < \gamma < d, \end{cases}$$

where

$$(3.13) \quad \begin{aligned} \mathfrak{C}_{\gamma,m} &= \frac{((1-\tau)\mathfrak{S}^{-d/(2m)}TC_{\gamma,m})^{1/(1-\tau)}}{2} (\varpi)^{\varpi}, \\ \tau &= \frac{2rm+d}{4rm+2\gamma}, \quad \varpi = \frac{\tau}{1-\tau} = \frac{2rm+d}{2\gamma+2rm-d}, \\ C_{\gamma,m} &= \int_0^{\infty} \frac{dy}{y^{2(rm+\gamma)/d} + \sqrt{y^{4(rm+\gamma)/d} + y^{2\gamma/d}}}. \end{aligned}$$

*Proof.* Define

$$(3.14) \quad A_k = \sqrt{\lambda_k^{2r} + 2p\lambda_k^{-\gamma/m}}, \quad B_k = \frac{\lambda_k^r}{A_k}.$$

By (2.4) and (3.11),

$$\begin{aligned}
(3.15) \quad & \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right) = \prod_{k=1}^{\infty} \mathbb{E} \exp \left( -p \lambda_k^{-\gamma/m} \int_0^T u_k^2(t) dt \right) \\
& = \exp \left( \frac{T}{2} \sum_{k=1}^{\infty} (\lambda_k^r - A_k) - \frac{1}{2} \sum_{k=1}^{\infty} \ln \left( \frac{1+B_k}{2} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \ln \left( 1 + \frac{1-B_k}{1+B_k} e^{-2A_k} \right) \right) \\
& := \exp \left( -S_1(p) + S_2(p) - S_3(p) \right),
\end{aligned}$$

where

$$(3.16) \quad S_1(p) = \frac{T}{2} \sum_{k=1}^{\infty} (A_k - \lambda_k^r),$$

$$(3.17) \quad S_2(p) = -\frac{1}{2} \sum_{k=1}^{\infty} \ln \left( \frac{1+B_k}{2} \right),$$

$$(3.18) \quad S_3(p) = \frac{1}{2} \sum_{k=1}^{\infty} \ln \left( 1 + \frac{1-B_k}{1+B_k} e^{-2A_k} \right).$$

The goal is to show that, as  $p \rightarrow \infty$ ,

$$(3.19) \quad -\ln \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right) \sim S_1(p)$$

and

$$(3.20) \quad S_1(p) \sim \begin{cases} \left( \frac{T}{\sqrt{2}} \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right) p^{1/2}, & \text{if } \gamma > d, \\ \frac{T \mathfrak{S}^{-d/(2m)}}{2^{3/2}} \frac{d}{d+2rm} p^{1/2} \ln p, & \text{if } \gamma = d, \\ \alpha p^{\tau} & \text{if } \frac{d}{2} - rm < \gamma < d, \end{cases}$$

where

$$\tau = \frac{2rm + d}{4rm + 2\gamma}, \quad \alpha = T \mathfrak{S}^{-d/(2m)} 2^{-(2rm+2\gamma-d)/(4rm+2\gamma)} C_{\gamma,m};$$

after that, relations (3.12) will follow from Theorems 2.1 and 2.2.

We start by establishing (3.20). We then show that  $S_2(p)$  and  $S_3(p)$  are of lower order compared to  $S_1(p)$ :

$$(3.21) \quad S_j(p) = o(S_1(p)), \quad j = 2, 3, \quad p \rightarrow \infty.$$

Note that

$$(3.22) \quad A_k - \lambda_k^r = \frac{2p \lambda_k^{-\gamma/m}}{\lambda_k^r + \sqrt{\lambda_k^{2r} + 2p \lambda_k^{-\gamma/m}}},$$

and recall that (3.1) holds. If  $\gamma > d$ , then

$$\sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} < \infty.$$

Since, for every  $k \geq 1$ ,

$$\lim_{p \rightarrow \infty} \frac{2\sqrt{p}\lambda_k^{-\gamma/m}}{\lambda_k^r + \sqrt{\lambda_k^{2r} + 2p\lambda_k^{-\gamma/m}}} = \sqrt{2}\lambda_k^{-\gamma/(2m)},$$

the dominated convergence theorem implies

$$\lim_{p \rightarrow \infty} \sum_{k=1}^{\infty} \frac{2\sqrt{p}\lambda_k^{-\gamma/m}}{\lambda_k^r + \sqrt{\lambda_k^{2r} + 2p\lambda_k^{-\gamma/m}}} = \sqrt{2} \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)}.$$

Therefore, for  $\gamma > d$ ,

$$S_1(p) \sim \left( \frac{T}{\sqrt{2}} \sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \right) \sqrt{p}, \quad p \rightarrow \infty.$$

When  $\frac{d}{2} - rm < \gamma \leq d$ , define the function

$$f(x; p) = \frac{1}{x^{2(rm+\gamma)/d} + \sqrt{x^{4(rm+\gamma)/d} + 2p\mathfrak{S}^{-2r-(\gamma/m)}x^{2\gamma/d}}}, \quad x > 0.$$

By (3.22),

$$S_1(p) = pT \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{r+(\gamma/m)} + \sqrt{\lambda_k^{2r+2(\gamma/m)} + 2p\lambda_k^{\gamma/m}}},$$

whereas (3.1) implies

$$(3.23) \quad \frac{1}{\lambda_k^{r+(\gamma/m)} + \sqrt{\lambda_k^{2r+2(\gamma/m)} + 2p\lambda_k^{\gamma/m}}} = \mathfrak{S}^{-r-(\gamma/m)} f(k + \epsilon(k); p),$$

with  $\epsilon(k) = O(k^{1-(1/d)})$ ,  $k \rightarrow \infty$ , uniformly in  $p$ .

Define

$$S_f(p) = pT\mathfrak{S}^{-r-(\gamma/m)} \int_1^{\infty} f(x; p) dx.$$

We will now show that

$$(3.24) \quad \lim_{p \rightarrow \infty} \frac{S_1(p)}{S_f(p)} = 1.$$

To begin, let us establish the asymptotic of  $S_f(p)$ . With the notations

$$(3.25) \quad \begin{aligned} \nu &= \frac{d}{2(\gamma + 2rm)}, \quad R = 2p\mathfrak{S}^{-2r-(\gamma/m)}, \quad y = xR^{-\nu}, \\ S_f(p) &= T\mathfrak{S}^{-d/(2m)} 2^{-(2rm+2\gamma-d)/(4rm+2\gamma)} p^{(2rm+d)/(4rm+2\gamma)} \\ &\times \int_{R^{-\nu}}^{\infty} \frac{dy}{y^{2(rm+\gamma)/d} + \sqrt{y^{4(rm+\gamma)/d} + y^{2\gamma/d}}}. \end{aligned}$$

If  $\gamma = d$ , then (3.25) implies

$$S_f(p) = T\mathfrak{S}^{-d/(2m)}2^{-1/2}p^{1/2} \int_{R^{-\nu}}^{+\infty} \frac{dy}{y^{2(rm+d)/d} + \sqrt{y^{4(rm+d)/d} + y^2}}.$$

By L'Hospital's rule, for every  $\kappa > 0$ ,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{|\ln \epsilon|} \int_{\epsilon}^{\infty} \frac{dy}{y^{1+\kappa} + \sqrt{y^{2+2\kappa} + y^2}} = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^{1+\kappa} + \sqrt{\epsilon^{2+2\kappa} + \epsilon^2}} = 1.$$

Therefore, as  $p \rightarrow \infty$ ,

$$S_f(p) \sim \frac{T\mathfrak{S}^{-d/(2m)}}{2^{3/2}} \frac{d}{d + 2rm} p^{1/2} \ln p.$$

If  $\frac{d}{2} - rm < \gamma < d$ , then (3.25) implies

$$S_f(p) \sim T\mathfrak{S}^{-d/(2m)}2^{-(2rm+2\gamma-d)/(4rm+2\gamma)}C_{\gamma,m}p^{(2rm+d)/(4rm+2\gamma)}, \quad p \rightarrow \infty.$$

In particular,

$$(3.26) \quad S_f(p) = \begin{cases} O(\sqrt{p} \ln p), & \text{if } \gamma = d, \\ O(p^\tau), \quad \tau = \frac{2rm + d}{4rm + 2\gamma} > \frac{1}{2}, & \text{if } \frac{d}{2} - rm < \gamma < d. \end{cases}$$

To establish (3.24), write

$$S_1(p) = S_f(p) + pT\mathfrak{S}^{-r-(\gamma/m)}S_{f,1}(p) + pTS_{f,2}(p),$$

where

$$S_{f,1}(p) = \sum_{k=1}^{\infty} f(k; p) - \int_1^{\infty} f(x; p)dx,$$

$$S_{f,2}(p) = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k^{r+(\gamma/m)} + \sqrt{\lambda_k^{2r+2(\gamma/m)} + 2p\lambda_k^{\gamma/m}}} - \mathfrak{S}^{-r-(\gamma/m)} f(k; p) \right).$$

Then (3.24) will follow from

$$(3.27) \quad pS_{f,1}(p) = o(S_f(p)), \quad p \rightarrow \infty;$$

$$(3.28) \quad pS_{f,2}(p) = o(S_f(p)), \quad p \rightarrow \infty.$$

We have

$$|S_{f,1}(p)| \leq 2 \max_{x \geq 1} f(x; p),$$

because, for fixed  $p$ ,  $0 \leq f(x; p) \rightarrow 0$ ,  $x \rightarrow +\infty$ , and the function  $f$  has at most one critical point. If  $\gamma \geq 0$ , then  $\max_{x \geq 1} f(x; p) = f(1; p) = O(p^{-1/2})$ , and (3.27) follows from (3.26).

If  $\gamma < 0$  [which is possible when  $2rm > d$ ], then

$$\arg \max_{x \geq 1} f(x; p) = O(p^{d/(4rm+2\gamma)}), \quad p \rightarrow \infty,$$

[by balancing  $x^{4(rm+\gamma)/d}$  and  $px^{2\gamma/d}$ ], so that, with  $\gamma < d$ ,

$$p \max_{x \geq 1} f(x; p) = O(p^{2rm/(4rm+2\gamma)}) = o(S_f(p)), \quad p \rightarrow \infty.$$

To get a bound on  $S_{2,f}$ , note that

$$(3.29) \quad \left| \frac{\partial f(x; p)}{\partial x} \right| \leq C_f \frac{f(x; p)}{x},$$

where  $C_f$  is a suitable constant independent of  $p$ . Together with (3.23), inequality (3.29) implies

$$|S_{f,2}(p)| \leq C_{f,2} \sum_{k=1}^{\infty} f(k; p) k^{-1/d},$$

and the constant  $C_{f,2}$  does not depend on  $p$ . By integral comparison,

$$\sum_{k=1}^{\infty} f(k; p) k^{-1/d} \sim \int_1^{\infty} f(x; p) x^{-1/d} dx, \quad p \rightarrow \infty,$$

and, similar to the derivation of (3.26),

$$p \int_1^{\infty} f(x; p) x^{-1/d} dx = o(S_f(p)), \quad p \rightarrow \infty,$$

which implies (3.28).

The asymptotic (3.20) of  $S_1(p)$  is now proved; a more compact form of (3.20) is

$$(3.30) \quad S_1(p) = \begin{cases} O(\sqrt{p}), & \text{if } \gamma > d, \\ O(\sqrt{p} \ln p), & \text{if } \gamma = d, \\ O(p^\tau), \quad \tau = \frac{2rm + d}{4rm + 2\gamma} > \frac{1}{2}, & \text{if } \frac{d}{2} - rm < \gamma < d. \end{cases}$$

It remains to establish (3.21). Recall that

$$2S_2(p) = - \sum_{k=1}^{\infty} \ln \left( \frac{1 + B_k}{2} \right);$$

cf. (3.14), (3.15), and (3.17). By definition,  $0 < B_k < 1$ , which means

$$- \ln \left( \frac{1 + B_k}{2} \right) = \ln 2 - \ln(1 + B_k) \leq 1 - B_k,$$

because the function  $h(x) = 1 + \ln(1 + x) - x$  is decreasing for  $x \in [0, 1]$  and  $h(1) = \ln 2$ . Next,

$$B_k = (1 + 2p\lambda^{-(r+(\gamma/m))})^{-1/2},$$

and therefore

$$(3.31) \quad 1 - B_k \leq \frac{2p\lambda^{-(r+(\gamma/m))}}{1 + 2p\lambda^{-(r+(\gamma/m))}},$$

because

$$1 - (1 + x)^{-1/2} \leq \frac{x}{1 + x}, \quad x > 0.$$

Using (3.1), inequality (3.31) becomes  $1 - B_k \leq g_p(k)$ , where

$$g_p(x) = \frac{C_g}{1 + (x^\mu/p)}, \quad x \geq 0,$$

$\mu = \frac{4rm + 2\gamma}{d}$ , and  $C_g$  is a suitable constant independent of  $p$ . Then, by integral comparison,

$$2S_2(p) \leq g_p(1) + \int_0^\infty g_p(x) dx.$$

Finally, by direct computation,

$$g_p(1) + \int_0^\infty g_p(x) dx = O(1) + C_g p^{1/\mu} \int_0^\infty \frac{dy}{1+y^\mu} = O(p^{1/\mu}) = O(p^{d/(4rm+2\gamma)}), \quad p \rightarrow \infty;$$

note that  $\mu > 1$  if  $\gamma > \frac{d}{2} - rm$ . Comparing with (3.30), we see that

$$S_2(p) = o(S_1(p)), \quad p \rightarrow \infty,$$

for all  $\gamma > \frac{d}{2} - rm$ .

To show that  $S_3(p) = o(S_1(p))$ ,  $p \rightarrow \infty$ , note that (3.18) and inequality  $\ln(1+x) \leq x$  imply

$$2S_3(p) \leq \sum_{k=1}^\infty e^{-2A_k} \leq \sum_{k=1}^\infty e^{-2\lambda_k^r} = O(1), \quad p \rightarrow \infty.$$

*This completes the proof of Theorem 3.6.* □

Comparing (3.5) and (3.12), we see that, for  $\gamma > d$ , the small ball behavior of both  $W$  and  $u$  is the same and, in a certain sense, similar to the finite-dimensional case (1.3). For  $\frac{d}{2} - rm < \gamma < d$ , the small ball rate for  $W$  is bigger than the small ball rate for  $u$ ; in fact, in this range of  $\gamma$ , the small ball rate is a decreasing function of  $rm$ . At  $\gamma = d$ , a kind of a phase transition takes place.

For equation (3.8), we have

$$d = r = m = \mathfrak{S} = 1, \quad \lambda_k = k^2, \quad \|f\|_\gamma^2 = \sum_{k=1}^\infty k^{2\gamma} f_k^2.$$

Then

$$\ln \mathbb{P} \left( \int_0^T \|W(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^\infty k^{-\gamma} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > 1, \\ -\frac{T^2}{32} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = 1, \\ -\mathfrak{C}_\gamma \varepsilon^{-\varpi}, & \text{if } \frac{1}{2} < \gamma < 1, \end{cases}$$

where

$$\varpi = \frac{1}{2\gamma - 1}, \quad \mathfrak{C}_\gamma = (2\gamma - 1)^{2\gamma} \gamma^{-2\gamma\varpi} 2^{(1-4\gamma)\varpi} T^{2\varpi} C_\gamma^{2\gamma\varpi}, \quad C_\gamma = \int_0^{+\infty} \frac{\ln \cosh(y)}{y^{1+(1/\gamma)}} dy.$$

Similarly,

$$\ln \mathbb{P} \left( \int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} k^{-\gamma} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > 1, \\ -\frac{T^2}{288} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = 1, \\ -\mathfrak{C}_\gamma \varepsilon^{-\varpi}, & \text{if } -\frac{1}{2} < \gamma < 1, \end{cases}$$

where

$$\mathfrak{C}_\gamma = \frac{((1-\tau)TC_\gamma)^{1/(1-\tau)}}{2} (\varpi)^\varpi, \quad \tau = \frac{3}{4+2\gamma}, \quad \varpi = \frac{\tau}{1-\tau} = \frac{3}{2\gamma+1},$$

$$C_\gamma = \int_0^\infty \frac{dy}{y^{2(1+\gamma)} + \sqrt{y^{4(1+\gamma)} + y^{2\gamma}}}.$$

In particular, taking  $\gamma = 0$ , we get a rather explicit logarithmic asymptotic

$$(3.32) \quad \ln \mathbb{P} \left( \int_0^T \int_0^\pi u^2(t, x) dx dt \leq \varepsilon \right) \sim -\frac{81}{512} C_0^4 T^4 \varepsilon^{-3},$$

where

$$(3.33) \quad C_0 = \int_0^\infty \frac{dy}{y^2 + \sqrt{y^4 + 1}} \approx 1.236.$$

Figure 1 presents the small ball rates  $\varpi$  when  $\gamma < 1$  for the solution of (3.8) (bold curve) and for the underlying noise  $W$ .

#### 4. FURTHER DIRECTIONS

4.1. **Equivalent norms on  $H^\gamma$ .** If  $\mathbf{a} = \{a_k, k \geq 1\}$ , is a sequence of real numbers such that

$$(4.1) \quad c_1 k \leq a_k \leq c_2 k$$

for some  $0 < c_1 \leq c_2$  and all  $k$ , then, by (3.1),

$$\|f\|_{\gamma; \mathbf{a}}^2 = \sum_{k=1}^{\infty} a_k^{2\gamma/d} f_k^2$$

defines an equivalent norm on  $H^\gamma$ . When  $\gamma > d$ , we get immediate analogues of Theorems 3.3 and 3.6.

**Proposition 4.1.** *Let  $X$  denote either  $u$  or  $W$ . For  $\gamma > d$ ,*

$$(4.2) \quad \ln \mathbb{P} \left( \int_0^T \|X(t)\|_{\gamma; \mathbf{a}}^2 dt \leq \varepsilon \right) \sim -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} a_k^{-\gamma/d} \right)^2 \varepsilon^{-1}, \quad \varepsilon \rightarrow 0.$$

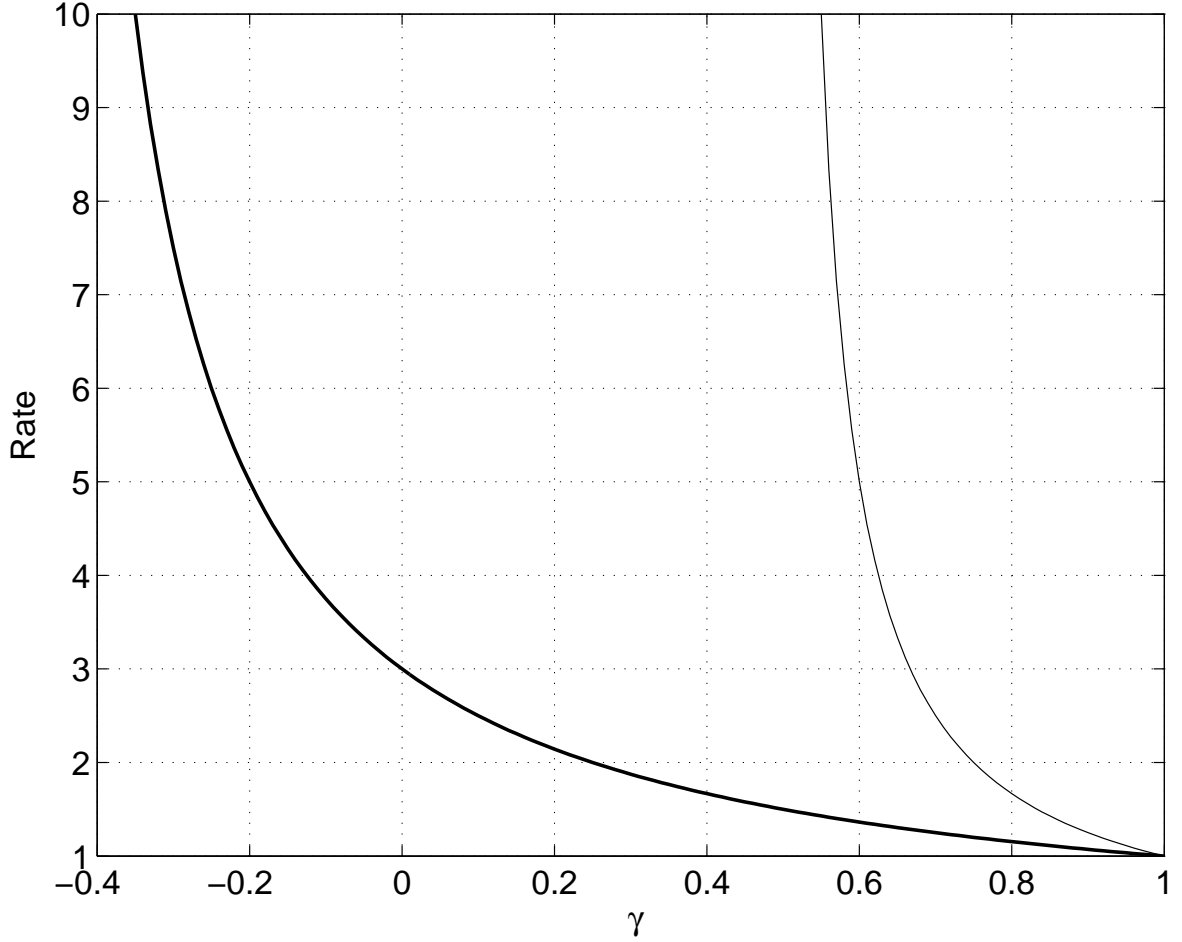


FIGURE 1. Stochastic heat equation on  $[0, \pi]$ : Small ball rate in  $H^{-\gamma}$ ,  $\gamma < 1$ , for the solution (bold curve) and the noise.

*Proof.* Direct computations show that

$$\ln \mathbb{E} \exp \left( -p \int_0^T \|X(t)\|_{\gamma; \mathbf{a}}^2 dt \right) \sim -2^{-3/2} T \left( \sum_{k=1}^{\infty} a_k^{-\gamma/d} \right) \sqrt{p}, \quad p \rightarrow \infty,$$

either by (3.19) [when  $X = u$ ] or by (3.6) [when  $X = W$ ], and then (4.2) follows from Theorem 2.1.  $\square$

When  $\gamma \leq d$ , analogues of Theorems 3.3 and 3.6 exist under an additional assumption about the numbers  $a_k$ .

**Proposition 4.2.** *Assume that*

$$(4.3) \quad \lim_{k \rightarrow \infty} \frac{a_k}{k} = c_a > 0.$$



Then, as  $\varepsilon \rightarrow 0$ ,

$$(4.4) \quad \ln \mathbb{P} \left( \int_0^T \|W(t)\|_{-\gamma; \mathbf{a}}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{32c_a^2} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = d, \\ -\mathfrak{C}_\gamma \varepsilon^{-\varpi}, & \text{if } \frac{d}{2} < \gamma < d, \end{cases}$$

where

$$\begin{aligned} \varpi &= \frac{d}{2\gamma - d}, \quad \mathfrak{C}_\gamma = (2q - 1)^{2q} q^{-2q\varpi} 2^{(1-4q)\varpi} (T c_a^{-\gamma/d})^{2\varpi} C_\gamma^{2q\varpi}, \\ q &= \frac{\gamma}{d}, \quad C_\gamma = \int_0^{+\infty} \frac{\ln \cosh(y)}{y^{1+(d/\gamma)}} dy. \end{aligned}$$

For the solution  $u = u(t)$  of equation (3.7),

$$(4.5) \quad \ln \mathbb{P} \left( \int_0^T \|u(t)\|_{-\gamma; \mathbf{a}}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{32c_a^2} \left( \frac{d}{d + 2rm} \right)^2 \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = d, \\ -\mathfrak{C}_{\gamma, m} \varepsilon^{-\varpi}, & \text{if } \frac{d}{2} - rm < \gamma < d, \end{cases}$$

where

$$\begin{aligned} \mathfrak{C}_{\gamma, m} &= \frac{((1 - \tau) \mathfrak{S}^{r(\gamma-d)/(\gamma+2rm)} c_a^{-2\gamma\tau/d} T C_{\gamma, m})^{1/(1-\tau)}}{2} (\varpi)^\varpi, \\ \tau &= \frac{2rm + d}{4rm + 2\gamma}, \quad \varpi = \frac{\tau}{1 - \tau} = \frac{2rm + d}{2\gamma + 2rm - d}, \\ C_{\gamma, m} &= \int_0^\infty \frac{dy}{y^{2(rm+\gamma)/d} + \sqrt{y^{4(rm+\gamma)/d} + y^{2\gamma/d}}}. \end{aligned}$$

*Proof.* By (3.1) and (4.3),

$$\lambda_k^{\gamma/m} \sim \mathfrak{S}^{\gamma/m} k^{2\gamma/d} \sim c_a^{2\gamma/d} k^{2\gamma/d}, \quad k \rightarrow \infty.$$

To derive (4.4), it remains to replace  $\mathfrak{S}$  in (3.5) with  $c_a^{2m/d}$ . After a slightly more detailed analysis, (4.5) follows from (3.12) in a similar way. Note that if  $c_a = \mathfrak{S}^{d/(2m)}$ , then (4.5) becomes (3.12).  $\square$

In the special case  $a_k = k$ , an alternative proof of Proposition 4.2 is possible using the results from [7, Example 2]; for technical reasons, such a proof is usually not possible under the general assumption (4.3). Without (4.3) (that is, assuming only (4.1)), a precise logarithmic asymptotic of the small ball probabilities may not exist when  $\gamma \leq d$ , but the corresponding upper and lower bounds can still be derived.

**4.2. Noise with correlation in space.** A generalization of  $\mathbf{W}$  is Q-cylindrical Brownian motion  $\mathbf{W}^Q$  defined by

$$\mathbb{E} \left( \mathbf{W}^Q(t, f) \mathbf{W}^Q(s, g) \right) = \min(t, s) \int_G (Qf)(x) g(x) dx,$$

where  $Q$  is a non-negative symmetric operator on  $L_2(G)$ . If

$$Q\varphi_k = q_k^2\varphi_k, \quad q_k > 0,$$

then, similar to (3.2), we set

$$W^Q(t) = \sum_{k=1}^{\infty} \varphi_k \mathbf{W}^Q(t, \varphi_k)$$

and consider the equation

$$\dot{u}(t) = A^r u + \dot{W}^Q(t), \quad u(0) = 0.$$

If furthermore

$$q_k \sim c_q k^\sigma, \quad k \rightarrow \infty,$$

then the question about the asymptotic of  $\mathbb{P}\left(\int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon\right)$ ,  $\varepsilon \rightarrow 0$ , is reduced to the corresponding question for the solution of the equation

$$\dot{v}(t) = A^r v(t) + \dot{W}(t),$$

where  $v = Q^{-1/2}u$ . For example, if  $q_k = \lambda_k^s$ , that is,  $Q = A^{2s}$ ,  $s \in \mathbb{R}$ , then

$$\mathbb{P}\left(\int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon\right) = \mathbb{P}\left(\int_0^T \|v(t)\|_{-\gamma+2ms}^2 dt \leq \varepsilon\right).$$

**4.3. Non-zero initial condition.** Assume that the initial condition  $u(0)$  in (3.7) is independent of  $W$  and has the form

$$u(0) = \sum_{k=1}^{\infty} u_k(0)\varphi_k,$$

where  $u_k(0)$  are independent Gaussian random variables with mean  $\mu_k$  and variance  $\sigma_k^2$ . To ensure (3.10), condition

$$(4.6) \quad \sum_{k=1}^{\infty} k^{-2\gamma/d} (\mu_k^2 + \sigma_k^2) < \infty$$

must hold for all  $\gamma > d/2$ .

In the finite-dimensional case (1.2), it is known [10, Theorem 4.5] that the initial condition may affect the small ball constant but not the small ball rate: if  $\mathbf{x}(0)$  is a Gaussian random vector independent of  $\mathbf{w}$ , then

$$\ln \mathbb{P}\left(\int_0^T |\mathbf{x}(t)|^2 dt \leq \varepsilon\right) \sim -\mathfrak{C} \varepsilon^{-1}, \quad \varepsilon \rightarrow 0,$$

where  $\mathfrak{C}$  may depend on the mean and covariance of  $\mathbf{x}(0)$ . In particular, if the covariance matrix of  $\mathbf{x}(0)$  is non-singular, then  $\mathfrak{C} = T^2/8$ , that is, the initial condition does not change the small ball asymptotic at the logarithmic level. The corresponding results in the infinite-dimensional case are as follows.

**Proposition 4.3.** *Assume that (4.6) holds.*

(1) If

$$(4.7) \quad \inf_k \sigma_k = \sigma_0 > 0,$$

then (3.12) holds.

(2) If  $\gamma > d$  and  $\sigma_k = 0$  for all  $k \geq 1$ , then

$$(4.8) \quad \ln \mathbb{P} \left( \int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim -\frac{1}{8} \left( \sum_{k=1}^{\infty} (T + \mu_k^2) \lambda_k^{-\gamma/(2m)} \right)^2 \varepsilon^{-1}, \quad \varepsilon \rightarrow 0.$$

*Proof.* The idea is to trace the contributions of the initial condition throughout the proof of Theorem 3.6. In particular, our objective is the asymptotic of  $\ln \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right)$  as  $p \rightarrow \infty$ .

By (2.4), the initial condition contributes an extra multiplicative term

$$\exp(-S_{0,1}(p) - S_{0,2}(p)),$$

where

$$S_{0,1}(p) = \sum_{k=1}^{\infty} \frac{\mu_k^2 \psi_{0,k}(p)}{1 + 2\sigma_k^2 \psi_{0,k}(p)}, \quad 2S_{0,2}(p) = \sum_{k=1}^{\infty} \ln(1 + 2\sigma_k^2 \psi_{0,k}(p)),$$

$$\psi_{0,k}(p) = \frac{p \lambda_k^{-\gamma/m}}{\lambda_k^r + \sqrt{\lambda_k^{2r} + 2p \lambda_k^{-\gamma/m}}}.$$

Also recall that, with zero initial condition, the dominant term is

$$S_1(p) = pT \sum_{k=1}^{\infty} \frac{\lambda_k^{-\gamma/m}}{\lambda_k^r + \sqrt{\lambda_k^{2r} + 2p \lambda_k^{-\gamma/m}}}.$$

Under (4.7),

$$S_{0,1}(p) \leq \sum_{k=1}^{\infty} \frac{p \mu_k^2 \lambda_k^{-\gamma/m}}{\lambda_k^r + p \sigma_0^2 \lambda_k^{-\gamma/m}} = o(S_1(p)), \quad p \rightarrow \infty.$$

Similarly,  $S_{0,2}(p) = o(S_1(p))$ ,  $p \rightarrow \infty$ . In other words, if the variance of the initial condition is strictly non-degenerate, then the initial condition does not affect the asymptotic of  $\ln \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right)$  as  $p \rightarrow \infty$ .

If  $\gamma > d$  and  $\sigma_k = 0$ , then (4.6) implies

$$\sum_{k=1}^{\infty} \lambda_k^{-\gamma/(2m)} \mu_k^2 < \infty,$$

and, similar to the proof of (3.19),

$$\ln \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right) \sim -\frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (T + \mu_k^2) \lambda_k^{-\gamma/(2m)}.$$

After that, Theorem 2.1 implies (4.8). □

The stationary case requires special consideration; cf. [3] for one-dimensional OU process.

**Proposition 4.4.** *Assume that  $\mu_k = 0$  and  $\sigma_k^2 = (2\lambda_k^r)^{-1}$ . Then (3.12) holds.*

*Proof.* Even though direct application of Proposition 4.3(1) is not possible because now  $\inf_k \sigma_k = 0$  and (4.7) fails, very little changes in the actual proof: with only the  $S_{0,2}(p)$  term present, we see that  $S_{0,2}(p) = o(S_1(p))$ ,  $p \rightarrow \infty$ , still holds, that is, the initial condition does not affect the asymptotic of  $\ln \mathbb{E} \exp \left( -p \int_0^T \|u(t)\|_{-\gamma}^2 dt \right)$ .  $\square$

If  $\gamma \leq d$ , then initial condition can affect the small ball rate. For example, assume that the initial condition in (3.8) is non-random [ $\sigma_k = 0$ ] and  $\mu_k = \sqrt{\ln k}$ , so that (4.6) holds. Using (2.4), (3.19), and (3.22),

$$\ln \mathbb{E} \exp \left( -p \int_0^T \int_0^\pi u^2(t, x) dx dt \right) \sim -p \sum_{k=1}^{\infty} \frac{T + \ln k}{k^2 + \sqrt{k^4 + 2p}}, \quad p \rightarrow \infty.$$

Similar to derivation of (3.20),

$$\sum_{k=1}^{\infty} \frac{T + \ln k}{k^2 + \sqrt{k^4 + 2p}} \sim \frac{\ln p}{4(2p)^{1/4}} \int_0^\infty \frac{dy}{y^2 + \sqrt{y^4 + 1}},$$

that is,

$$\ln \mathbb{E} \exp \left( -p \int_0^T \int_0^\pi u^2(t, x) dx dt \right) \sim -2^{-9/4} C_0 p^{3/4} \ln p,$$

with  $C_0$  from (3.33). By Theorem 2.2 with  $\alpha = 2^{-9/4} C_0$ ,  $\beta = 1$ ,  $\tau = 3/4$ ,

$$\ln \mathbb{P} \left( \int_0^T \int_0^\pi u^2(t, x) dx dt \leq \varepsilon \right) \sim -\frac{27C_0^4}{2^{17}} \varepsilon^{-3} |\ln \varepsilon|^4,$$

which is very different from (3.32): the rate has an additional logarithmic term and the constant does not depend on  $T$ .

**4.4. Other types of parabolic equations.** Consider a linear operator  $A$  on a separable Hilbert space  $H$ . If  $A$  is symmetric and has a pure point spectrum  $0 < \lambda_1 \leq \lambda_2 \leq \dots$ , and the corresponding eigenfunctions  $\varphi_k$  form an orthonormal basis in  $H$ , then all the constructions from Section 3 can be repeated, and an analog of Theorem 3.6 can be stated and proved for the evolution equation

$$(4.9) \quad u_t(t) + Au(t) = \dot{W}(t), \quad u(0) = 0,$$

where  $\dot{W}$  is a cylindrical Brownian motion on  $H$ .

The details depend on the asymptotic behavior of  $\lambda_k$  as  $k \rightarrow \infty$ . For example, consider the equation

$$u_t(t, x) = \frac{1}{2} (u_{xx}(t, x) - x^2 u(t, x) - u(t, x)) + \dot{W}(t, x), \quad t > 0, \quad x \in \mathbb{R}, \quad u(0, x) = 0.$$

Then

$$H = L_2(\mathbb{R}), \quad A = \frac{1}{2} \left( -\frac{\partial^2}{\partial x^2} + x^2 + 1 \right),$$

and

$$\lambda_k = k, \quad \varphi_k(x) = (-1)^k \frac{1}{\sqrt{2^k k!}} \pi^{-1/4} e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2};$$

cf. [12, Section 1.4]. Since operator  $A$  has order 2, we define

$$H^\gamma = A^{\gamma/2} L_2(\mathbb{R}), \quad \|f\|_\gamma^2 = \sum_{k=1}^{\infty} k^\gamma f_k^2.$$

Recall that the norm in the traditional Sobolev space on  $\mathbb{R}$  is

$$\|f\|_\gamma^2 = \int_{-\infty}^{+\infty} |\hat{f}(y)|^2 (1+y^2)^\gamma dy;$$

$\hat{f}$  is the Fourier transform of  $f$ . In particular, it follows that

$$\mathbb{E}\|W(t)\|_\gamma^2 = \infty$$

for every  $\gamma \in \mathbb{R}$  and every  $t > 0$  [roughly speaking, because  $\sum_k \varphi_k^2(x) = \delta(x)$  and each  $\varphi_k$  is an eigenfunction of the Fourier transform], and consequently the solution of  $v_t = v_{xx} + \dot{W}(t, x)$ ,  $x \in \mathbb{R}$ , does not belong to any traditional Sobolev space on  $\mathbb{R}$ . On the other hand, similar to Propositions 3.2 and 3.5,

$$W \in L_2(\Omega \times (0, T); H^{-\gamma}), \quad u \in L_2(\Omega \times (0, T); H^{-\gamma+1}), \quad \gamma > 1;$$

$u$  is the solution of (4.9). The corresponding small ball asymptotics can also be derived.

**Proposition 4.5.** *The following relations hold as  $\varepsilon \rightarrow 0$ :*

$$\ln \mathbb{P} \left( \int_0^T \|W(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} k^{-\gamma/2} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > 2, \\ -\frac{T^2}{32} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = 2, \\ -\mathfrak{C}_\gamma \varepsilon^{-1/(\gamma-1)}, & \text{if } 1 < \gamma < 2, \end{cases}$$

where

$$\mathfrak{C}_\gamma = (\gamma - 1)^\gamma \gamma^{-2\gamma\varpi} 2^{(1-2\gamma)\varpi} T^{2\varpi} C_\gamma^{\gamma\varpi}, \quad \varpi = \frac{1}{\gamma - 1}, \quad C_\gamma = \int_0^{+\infty} \frac{\ln \cosh(y)}{y^{1+(2/\gamma)}} dy,$$

and

$$\ln \mathbb{P} \left( \int_0^T \|u(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim \begin{cases} -\frac{T^2}{8} \left( \sum_{k=1}^{\infty} k^{-\gamma} \right)^2 \varepsilon^{-1}, & \text{if } \gamma > 2, \\ -\frac{T^2}{128} \varepsilon^{-1} |\ln \varepsilon|^2, & \text{if } \gamma = 2, \\ -\mathfrak{C}_\gamma \varepsilon^{-2/\gamma}, & \text{if } 0 < \gamma < 2, \end{cases}$$

where

$$\mathfrak{C}_\gamma = \frac{((1-\tau)TC_\gamma)^{1/(1-\tau)}}{2} (\varpi)^\varpi, \quad \tau = \frac{2}{2+\gamma}, \quad \varpi = \frac{\tau}{1-\tau} = \frac{2}{\gamma},$$

$$C_\gamma = \int_0^\infty \frac{dy}{y^{1+\gamma} + \sqrt{y^{2+2\gamma} + y^\gamma}}.$$

*Proof.* The case of  $W$  follows from the asymptotic of

$$\sum_{k=1}^{\infty} \ln \cosh \left( T \sqrt{2pk^{-\gamma}} \right), \quad p \rightarrow \infty,$$

similar to the proof of Theorem 3.3.

The case of  $u$  follows the same steps as the proof of Theorem 3.6.  $\square$

## 5. SUMMARY

Consider the equation

$$\dot{u}(t) + A^r u = \dot{W}^Q(t), \quad u(0) = 0,$$

with  $r > 0$ , and assume that the positive-definite operators  $A$  and  $Q$  commute, have purely point spectrum and act in a scale  $H^\gamma$  of Hilbert spaces, and

$$\gamma_0 = \inf \{s > 0 \mid \mathfrak{i} \circ Q : H^0 \rightarrow H^{-s} \text{ is trace class}\} < \infty,$$

where  $\mathfrak{i}$  is the embedding operator. If  $\gamma > \gamma_0$ , then the logarithmic asymptotic of the small ball probabilities is similar to finite-dimensional case (1.3):

$$\ln \mathbb{P} \left( \int_0^T \|X(t)\|_{-\gamma}^2 dt \leq \varepsilon \right) \sim - \frac{(T \operatorname{trace}(\mathfrak{i} \circ Q))^2}{8} \varepsilon^{-1}, \quad \varepsilon \rightarrow 0,$$

for both  $X = u$  and  $X = W^Q$ . Infinite-dimensional effects appear when  $\gamma \leq \gamma_0$ : the small ball rate now depends on  $\gamma$  and can be arbitrarily large, whereas the small ball constant depends on the operator  $A$ .

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