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ON GENERALIZED MALLIAVIN CALCULUS

S. V. LOTOTSKY, B. L. ROZOVSKII, AND D. SELEŠI

Abstract. The Malliavin derivative, divergence operator (Skorokhod integral), and the Ornstein-Uhlenbeck operator are extended from the traditional Gaussian setting to nonlinear generalized functionals of white noise. These extensions are related to the new developments in the theory of stochastic PDEs, in particular elliptic PDEs driven by spatial white noise and quantized nonlinear equations.

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1. Introduction

Currently, the predominant *driving random source* in Malliavin calculus is the isonormal Gaussian process (white noise) $\dot{W}$ on a separable Hilbert space $\mathcal{U}$ [15, 18]. This process is in effect a *linear* combination of a countable collection $\boldsymbol{\xi} := \{\xi_i\}_{i\geq 1}$ of independent standard Gaussian random variables.

In the first part of this paper (Sections 2–4) we extend Malliavin calculus to the driving random source given by a nonlinear functional $u := u(\boldsymbol{\xi})$ of white noise. More specifically, we study `the main operators of` Malliavin calculus: Malliavin derivative $\mathbf{D}_u(f)$; divergence operator $\boldsymbol{\delta}_u(f)$, and Ornstein-Uhlenbeck operator $\mathcal{L}_u(f)$ with respect to a generalized random element $u = \sum_{|\boldsymbol{\alpha}|<\infty} u_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}$, where $\{\xi_{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| < \infty\}$ is the Cameron-Martin basis in the Wiener Chaos space, $\boldsymbol{\alpha}$ is a multiindex and $u_{\boldsymbol{\alpha}}$ belongs to a certain Hilbert space U. The term "generalized" indicates that

$$\|u\|_X^2 = \sum_{|\boldsymbol{\alpha}|<\infty} \|u_{\boldsymbol{\alpha}}\|_X^2 = \infty.$$

Looking for driving random sources that are generalized random elements is quite reasonable: after all, the Gaussian white noise that often drives the equation of interest is itself a generalized random element.

Our interest in this subject was prompted by some open problems in the theory and applications stochastic partial differential equations (SPDEs). In particular:

`(A) Non-adapted SPDEs, including elliptic and parabolic equations with time-independent random forcing;`

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(B) SPDEs driven by random sources more general then Brownian motion, for example, nonlinear functionals of Gaussian white noise;

(C) Stochastic quantization of non-linear SPDEs.

These issues are discussed in the second part of the paper (Sections 5 and 6).

Skorokhod integral (Malliavin divergence operator) is a standard tool in the L_2 theory of non-adapted stochastic differential equations. However, simple examples of SPDEs from classes (A) and (B) indicate that their solutions have infinite energy (L_2 norm). To address this issue, one must allow the "argument" $f = \sum_{|\boldsymbol{\alpha}|<\infty} f_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}$ also to be a generalized random elements taking values in an appropriate Hilbert space.

Stochastic SPDEs with infinite energy are not a rarity. One simple example is the heat equation driven by multiplicative space-time white noise $\dot{W}(t,x)$ with dimension of x two or higher:

$$u_t = \boldsymbol{\Delta} u + u\dot{W}. \tag{1.1}$$

Examples in one space dimension also exist:

$$du = u_{xx}dt + \sigma u_x dw(t),\ \ \sigma^2 > 2, \tag{1.2}$$

or

$$du = u_{xx}dt + u_{xx}dw(t). \tag{1.3}$$

Elliptic equations with random inputs (including random coefficients) is another large class of SPDEs that generate solutions with infinite variance. A classic example is equation

$$\Delta u = \dot{W},\ \text{for } x \in D = (0,1)^d,\ u(x) = 0 \text{ for } x \in \partial D$$

when $d \geq 4$.

Another important example of an elliptic SPDE with an infinite energy solution is

$$\big(a(x) \diamond u_x(t,x)\big)_x = f(x), \quad x \in \mathbb{R}, \tag{1.4}$$

where

$$a(x) = e^{\diamond \dot{W}(x)} := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\mathfrak{e}^{\boldsymbol{\alpha}}(x)}{\boldsymbol{\alpha}!}\,\mathrm{H}_{\boldsymbol{\alpha}} \tag{1.5}$$

is the *positive* noise process ([4, Section 2.6]), $\{\mathfrak{e}_k(x),\ k \geq 1\}$ are the Hermite functions, and $\mathrm{H}_{\boldsymbol{\alpha}} = \xi_{\boldsymbol{\alpha}}\sqrt{\boldsymbol{\alpha}!}$. The function $a = a(x)$ defined by (1.5) models permeability of a random medium.

A somewhat similar example with coefficient $a(x)$ taking negative values with positive probability was considered in [25].

Beside the study of equation of the type (1.1)–(1.4), an important impetus for development of generalized Malliavin calculus is the problem of unbiased stochastic perturbations for nonlinear deterministic PDE. Unbiased stochastic perturbations can be produced by way of *stochastic quantization*.[1] Roughly speaking, stochastic quantization procedure consists in approximating the product uv by $\boldsymbol{\delta}_v(u)$ where

[1] "Stochastic quantization" is not a standard term. For background see [20, 6, 17] and [16, Section 6].

$\boldsymbol{\delta}_v$ is the generalized Malliavin divergence operator, where u and v are functions of Gaussian white noise $\dot{W}$ or, equivalently, the sequence $\boldsymbol{\xi} := \{\xi_i\}_{i\geq 1}$. The nature of this approximation could be explained by the following formula:

$$v \cdot u = \boldsymbol{\delta}_u(v) + \sum_{n=1}^{\infty} \frac{\boldsymbol{\delta}_{\mathbf{D}^n_{\dot{W}}(u)}\left(\mathbf{D}^n_{\dot{W}}(v)\right)}{n!} \tag{1.6}$$

(see [17]). Formula (1.6) implies that $\boldsymbol{\delta}_u(v)$ is the highest stochastic order approximation for the product $v \cdot u$. For example, if $u = \xi_{\boldsymbol{\alpha}}$ and $v = \xi_{\boldsymbol{\beta}}$, then (1.6) yields $\xi_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\beta}} = \boldsymbol{\delta}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}}) + \sum_{\boldsymbol{\gamma}<\boldsymbol{\alpha}+\boldsymbol{\beta}} c_{\boldsymbol{\gamma}}\xi_{\boldsymbol{\gamma}}$. Since $\boldsymbol{\delta}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}}) = \xi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ it is indeed the *highest stochastic order* component of the Wiener chaos expansion of $\xi_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\beta}}$. Note that, if v and u are real valued square integrable random variables, then $\boldsymbol{\delta}_u(v) = u \diamond v$ where $\diamond$ stands for Wick product [2, 4, 26]. Equations subjected to stochastic quantization procedure are usually referred to as "quantized".

Interesting examples of quantized stochastic PDEs include randomly forced equations of Burgers [7] and Navier-Stokes [17]. For example, let us consider Burgers equation with deterministic initial condition and simple Gaussian forcing given by

$$u_t = u_{xx} + uu_x + e^{-x^2}\xi, \tag{1.7}$$

where ξ is a standard Gaussian random variable. The quantized version of this equation is given by

$$v_t = v_{xx} + \boldsymbol{\delta}_v(v_x) + e^{-x^2}\xi\,, \tag{1.8}$$

where $\boldsymbol{\delta}_v(v_x)$ is Malliavin divergence operator of v_x with respect to the solution v of (1.8). It can be shown that $\bar{v}(t,x) := \mathbb{E}v(t,x)$ solves the deterministic Burgers equation $\bar{v}_t(t,x) = \bar{v}_{xx}(t,x) + \bar{v}(t,x)\bar{v}_x(t,x)$. In other words, in contrast to the standard stochastic Burgers equation, its quantized version provides an unbiased random perturbation of the solution of the deterministic Burgers equation $u_t = u_{xx} + uu_x$. Thus, the quantized version (1.8) of stochastic Burgers equation is an *unbiased* perturbation of the standard Burgers equation (1.7).

Note that, in all examples we have discussed, the variance of a generalized random element u was given by the diverging sum $\sum_{\boldsymbol{\alpha}} \|u_{\boldsymbol{\alpha}}\|_X^2 = \infty$. However, the rate of divergence could differ substantially from case to case. To study this rate, we introduce a rescaling operator $\mathcal{R}$ defined by $\mathcal{R}\xi_{\boldsymbol{\alpha}} = r_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}$, where *weights* $r_{\boldsymbol{\alpha}}$ are positive numbers selected in such a way that the weighted sum $\sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}^2 \|u_{\boldsymbol{\alpha}}\|_X^2$ becomes finite. Of course, it could be done in many ways. A particular choice of weights (weighted spaces) depends on the specifics of the problem, for example on the type of the stochastic PDE in question. A special case of the aforementioned rescaling procedure was originally introduced in quantum physics and referred to as "*second quantization*" [24]. It was limited to a special but very important class of weights. In this paper they are referred to as *sequence weights.*

Quantum physics has brought about a number of important precursors to Malliavin calculus. For example, *creation* and *annihilation* operators correspond to Malliavin divergence and derivative operators, respectively, for a single Gaussian random variable. The original definition of Wick product [26] is not related to the Malliavin divergence operator or Skorokhod integral but remarkably these notions coincide in some situations. In fact, standard Wick product could be interpreted as Skorokhod integral with respect to square integrable processes generated by Gaussian white noise, while the classic Malliavin divergence operator integrates only with respect to isonormal Gaussian process. In Section 3, we demonstrate that Malliavin divergence operator could be extended to the setting where both the integrand and the integrator are generalized random elements in a Hilbert space; however, we were unable to extend Wick product to the same extent.

To summarize, in this paper, we construct and investigate the three main operators of Malliavin calculus: the derivative operator $\mathbf{D}_u(v)$, the divergence operator $\boldsymbol{\delta}_u(f)$, and the Ornstein-Uhlenbeck operator $\mathcal{L}_u(v) = \boldsymbol{\delta}_u \circ \mathbf{D}_u(v)$ when u, v, and f are Hilbert space-valued generalized random elements. Section 2 reviews the main constructions of the Malliavin calculus in the form suitable for generalizations. Section 3 presents the definitions of the Malliavin derivative, Skorokhod integral, and Ornstein-Uhlenbeck operator in the most general setting of weighted chaos spaces. Section 4 presents a more detailed analysis of the operators on some special classes of spaces. In particular, in Section 4 we study the strong continuity of these operators with respect to the "argument". We show that

$$\|\mathcal{A}_u(v)\|_a \leq C\big(\|u\|_b\big)\, \|v\|_c,$$

where $\|\cdot\|_i,\ i = a, b, c$ are norms in the suitable spaces, the function C is independent of v, and $\mathcal{A}$ is one of the operators $\mathbf{D}$, $\boldsymbol{\delta}$, $\mathcal{L}$. In Section 5 we generalize the isonormal Gaussian process $\dot{W}$ to $\dot{Z}_N = \sum_{0<|\boldsymbol{\alpha}|\leq N} \mathfrak{u}^{\otimes \boldsymbol{\alpha}}\, \mathrm{H}_{\boldsymbol{\alpha}}$, $1 \leq N \leq \infty$, where $\{\mathfrak{u}_k,\ k \geq 1\}$ is an orthonormal basis in $\mathcal{U}$. Parameter N is the *stochastic order* of $\dot{Z}_N$. In particular, the stochastic order of $\dot{W}$ is 1, while the stochastic order of the positive noise (1.5) is infinity.

We investigate the properties of the corresponding operators $\mathcal{D}_{\dot{Z}_N}$, $\boldsymbol{\delta}_{\dot{Z}_N}$, $\mathcal{L}_{\dot{Z}_N}$, and use the results to characterize certain spaces of generalized random elements via the action on the stochastic exponential. In Section 6 we introduce two new classes of stochastic partial differential equations driven by $\dot{Z}_N$, prove the main existence and uniqueness theorems, and consider some particular examples.

More specifically, in Section 6 we study general parabolic and elliptic SPDEs of the form

$$\dot{u}(t) = \mathbf{A}(t)u\,(t) + f(t) + \boldsymbol{\delta}_{\dot{Z}_\infty}\big(\mathbf{M}(t)u(t)\big),\ 0 \leq t \leq T,$$

and

$$\mathbf{A}u + \boldsymbol{\delta}_{\dot{Z}_\infty}\big(\mathbf{M}u\big) = f,$$

where $\mathbf{A} : V \to V'$ and $\mathbf{M} : V \to V' \otimes \bigoplus_{k\geq 1} \mathcal{U}^{\otimes k}$.

Roughly speaking, the only assumptions on the operators $\mathbf{A}$ and $\mathbf{M}$ are that $\mathbf{M}$ and $\mathbf{A}^{-1}\mathbf{M}$ are appropriately bounded. The initial conditions and the free forces are assumed to be from the spaces based on Kondratiev space $\mathcal{S}_{-1}$. Appropriate counter examples indicate that our results are sharp.

2. Review of the traditional Malliavin Calculus

The starting point in the development of Malliavin calculus is the *isonormal Gaussian process* (also known as Gaussian white noise) $\dot{W}$: a Gaussian system $\{\dot{W}(u),\ u \in \mathcal{U}\}$ indexed by a separable Hilbert space $\mathcal{U}$ and such that $\mathbb{E}\dot{W}(u) = 0$, $\mathbb{E}\big(\dot{W}(u)\,\dot{W}(v)\big) = (u,v)_{\mathcal{U}}$. The objective of this section is to outline a different but equivalent construction.

Let $\mathbb{F} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is the σ-algebra generated by a collection of independent standard Gaussian random variables $\{\xi_i\}_{i\geq 1}$. Given a real separable Hilbert space X, we denote by $L_2(\mathbb{F}; X)$ the Hilbert space of square-integrable $\mathcal{F}$-measurable X-valued random elements f. When $X = \mathbb{R}$, we often write $L_2(\mathbb{F})$ instead of $L_2(\mathbb{F};\mathbb{R})$. Finally, we fix a real separable Hilbert space $\mathcal{U}$ with an orthonormal basis $\mathfrak{U} = \{\mathfrak{u}_k,\ k \geq 1\}$.

Definition 2.1. *A* `Gaussian white noise` $\dot{W}$ *on* $\mathcal{U}$ *is a formal series*

$$\dot{W} = \sum_{k\geq 1} \xi_k \mathfrak{u}_k. \tag{2.1}$$

Given an isonormal Gaussian process $\dot{W}$ and an orthonormal basis $\mathfrak{U}$ in $\mathcal{U}$, representation (2.1) follows with $\xi_k = \dot{W}(\mathfrak{u}_k)$. Conversely, (2.1) defines an isonormal Gaussian process on $\mathcal{U}$ by $\dot{W}(u) = \sum_{k\geq 1}(u, \mathfrak{u}_k)_{\mathcal{U}}\,\xi_k$. To proceed, we need to review several definitions related to *multi-indices*. Let $\mathcal{J}$ be the collection of multi-indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ such that $\alpha_k \in \{0, 1, 2, \ldots\}$ and $\sum_{k\geq 1}\alpha_k < \infty$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{J}$, we define

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad |\boldsymbol{\alpha}| = \sum_{k\geq 1} \alpha_k, \quad \boldsymbol{\alpha}! = \prod_{k\geq 1} \alpha_k!.$$

By definition, $\boldsymbol{\alpha} > 0$ if $|\boldsymbol{\alpha}| > 0$, and $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ if $\beta_k \leq \alpha_k$ for all $k \geq 1$. If $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$, then

$$\boldsymbol{\alpha} - \boldsymbol{\beta} = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots).$$

Similar to the convention for the usual binomial coefficients,

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \begin{cases} \dfrac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha} - \boldsymbol{\beta})!\boldsymbol{\beta}!}, & \text{if } \boldsymbol{\beta} \leq \boldsymbol{\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

We use the following notation for the special multi-indices:

(1) $\mathbf{(0)}$ is the multi-index with all zero entries: $\mathbf{(0)}_k = 0$ for all k;
(2) $\boldsymbol{\varepsilon}(i)$ is the multi-index of length 1 and with the single non-zero entry at position i: i.e. $\boldsymbol{\varepsilon}(i)_k = 1$ if $k = i$ and $\boldsymbol{\varepsilon}(i)_k = 0$ if $k \neq i$. We also use convention $\boldsymbol{\varepsilon}(0) = \mathbf{(0)}$.

Given a sequence of positive numbers $\mathfrak{q} = (q_1, q_2, \ldots)$ and a real number ℓ, we define the sequence $\mathfrak{q}^{\ell\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{J}$, by $\mathfrak{q}^{\ell\boldsymbol{\alpha}} = \prod_k q_k^{\ell\alpha_k}$. In particular, $(2\mathbb{N})^{\ell\boldsymbol{\alpha}} = \prod_{k\geq 1}(2k)^{\ell\alpha_k}$.

Next, we recall the construction of an orthonormal basis in $L_2(\mathbb{F}; X)$. Define the collection of random variables $\Xi = \{\xi_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{J}\}$ as follows:

$$\xi_{\boldsymbol{\alpha}} = \prod_{k \geq 1} \frac{\mathrm{H}_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}},$$

where H_n is the Hermite polynomial of order n: $\mathrm{H}_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}$. Sometimes it is convenient to work with unnormalized basis elements $\mathrm{H}_{\boldsymbol{\alpha}}$, defined by

$$\mathrm{H}_{\boldsymbol{\alpha}} = \sqrt{\boldsymbol{\alpha}!}\, \xi_{\boldsymbol{\alpha}} = \prod_{k \geq 1} \mathrm{H}_{\alpha_k}(\xi_k). \tag{2.2}$$

Theorem 2.2 (Cameron-Martin [1]). *The set Ξ is an orthonormal basis in $L_2(\mathbb{F}; X)$: if $v \in L_2(\mathbb{F}; X)$ and $v_{\boldsymbol{\alpha}} = \mathbb{E}(v\, \xi_{\boldsymbol{\alpha}})$, then $v = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} v_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}$ and $\mathbb{E}\|v\|_X^2 = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \|v_{\boldsymbol{\alpha}}\|_X^2$.*

If the space $\mathcal{U}$ is n-dimensional, then the multi-indices are restricted to the the set

$$\mathcal{J}_n = \{\boldsymbol{\alpha} \in \mathcal{J} : \alpha_k = 0,\ k > n\}.$$

The three main operators of the Malliavin calculus are

(1) The `(Malliavin) derivative` $\mathbf{D}_{\dot{W}}$;
(2) The `divergence operator` $\boldsymbol{\delta}_{\dot{W}}$, also known as the `Skorokhod integral`;
(3) The `Ornstein-Uhlenbeck operator` $\mathcal{L}_{\dot{W}} = \boldsymbol{\delta}_{\dot{W}}\, \mathbf{D}_{\dot{W}}$.

For reader's convenience, we summarize the main properties of $\mathbf{D}_{\dot{W}}$ and $\boldsymbol{\delta}_{\dot{W}}$; all the details are in [18, Chapter 1].

(1) $\mathbf{D}_{\dot{W}}$ is a closed unbounded linear operator from $L_2(\mathbb{F}; X)$ to $L_2(\mathbb{F}; X \otimes \mathcal{U})$; the domain of $\mathbf{D}_{\dot{W}}$ is denoted by $\mathbb{D}^{1,2}(\mathbb{F}; X)$;
(2) If $v = F\big(\dot{W}(h_1), \ldots, \dot{W}(h_n)\big)$ for a polynomial $F = F(x_1, \ldots, x_n)$ and $h_1, \ldots, h_n \in X$, then

$$\mathbf{D}_{\dot{W}}(v) = \sum_{k=1}^{n} \frac{\partial F}{\partial x_k} \big(\dot{W}(h_1), \ldots, \dot{W}(h_n)\big)\, h_k. \tag{2.3}$$

(3) $\boldsymbol{\delta}_{\dot{W}}$ is the adjoint of $\mathbf{D}_{\dot{W}}$ and is a closed unbounded linear operator from $L_2(\mathbb{F}; X \otimes \mathcal{U})$ to $L_2(\mathbb{F}; X)$ such that

$$\mathbb{E}\big(\varphi \boldsymbol{\delta}_{\dot{W}}(f)\big) = \mathbb{E}\big(f, \mathbf{D}_{\dot{W}}(\varphi)\big)_{\mathcal{U}} \tag{2.4}$$

for all $\varphi \in \mathbb{D}^{1,2}(\mathbb{F}; \mathbb{R})$ and $f \in \mathbb{D}^{1,2}(\mathbb{F}; X \otimes \mathcal{U})$. Equivalently,

$$\big(v, \boldsymbol{\delta}_{\dot{W}}(f)\big)_{L_2(\mathbb{F};X)} = \big(f, \mathbf{D}_{\dot{W}}(v)\big)_{L_2(\mathbb{F};X\otimes\mathcal{U})} \tag{2.5}$$

for all $v \in \mathbb{D}^{1,2}(\mathbb{F}; X)$ and $f \in \mathbb{D}^{1,2}(\mathbb{F}; X \otimes \mathcal{U})$.

The following theorem provides representations of the operators $\mathbf{D}_{\dot{W}}$, $\boldsymbol{\delta}_{\dot{W}}$, and $\mathcal{L}_{\dot{W}}$ in the basis Ξ. These representations are well known (e.g. [18, Chapter 1]).

Theorem 2.3. *(1) If $v \in L_2(\mathbb{F}; X)$ and*

$$\sum_{\boldsymbol{\alpha} \in \mathcal{J}} |\boldsymbol{\alpha}|\, \|v_{\boldsymbol{\alpha}}\|_X^2 < \infty, \tag{2.6}$$

then $\mathbf{D}_{\dot W}(v) \in L_2(\mathbb{F}; X\otimes \mathcal{U})$ *and*

$$\mathbf{D}_{\dot W}(v) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\sum_{k\geq 1}\sqrt{\alpha_k}\,\xi_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}(k)}\,v_{\boldsymbol{\alpha}}\otimes \mathfrak{u}_k. \tag{2.7}$$

(2) If $f=\sum_{\boldsymbol{\alpha}\in\mathcal{J},\,k\geq 1} f_{k,\boldsymbol{\alpha}}\otimes \mathfrak{u}_k\,\xi_{\boldsymbol{\alpha}}$, *and*

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J},\,k\geq 1} |\boldsymbol{\alpha}|\|f_{k,\boldsymbol{\alpha}}\|_X^2<\infty, \tag{2.8}$$

then $\boldsymbol{\delta}_{\dot W}(f)\in L_2(\mathbb{F};X)$ *and*

$$\boldsymbol{\delta}_{\dot W}(f) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{k\geq 1}\sqrt{\alpha_k} f_{k,\boldsymbol{\alpha}-\boldsymbol{\varepsilon}(k)}\right)\xi_{\boldsymbol{\alpha}}. \tag{2.9}$$

(3) If $v\in L_2(\mathbb{F};X)$ *and*

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} |\boldsymbol{\alpha}|^2\,\|v_{\boldsymbol{\alpha}}\|_X^2<\infty, \tag{2.10}$$

then $\mathcal{L}_{\dot W}(v)\in L_2(\mathbb{F};X)$ *and*

$$\mathcal{L}_{\dot W}(v) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} |\boldsymbol{\alpha}| v_{\boldsymbol{\alpha}}\,\xi_{\boldsymbol{\alpha}}. \tag{2.11}$$

Remark 2.4. There is an important technical difference between the derivative and the divergence operators:

- For the operator $\mathbf{D}_{\dot W}$, $\big(\mathbf{D}_{\dot W}(v)\big)_{\boldsymbol{\alpha}} = \sum_{k\geq 1}\sqrt{\alpha_k+1}\,v_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}(k)}\otimes\mathfrak{u}_k$; in general, the sum on the right-hand side contains infinitely many terms and will diverge without additional conditions on v, such as (2.6).
- For the operator $\boldsymbol{\delta}_{\dot W}$, $\big(\boldsymbol{\delta}_{\dot W}(f)\big)_{\boldsymbol{\alpha}} = \sum_{k\geq 1}\sqrt{\alpha_k} f_{k,\boldsymbol{\alpha}-\boldsymbol{\varepsilon}(k)}$; the sum on the right-hand side always contains finitely many terms, because only finitely many of α_k are not equal to zero. Thus, for *fixed* $\boldsymbol{\alpha}$, $\big(\boldsymbol{\delta}_{\dot W}(f)\big)_{\boldsymbol{\alpha}}$ is defined without any additional conditions on f.

3. Generalizations to weighted chaos spaces

Recall that $\dot W$, as defined by (2.1), is not a $\mathcal{U}$-valued random element, but a *generalized* random element on $\mathcal{U}$: $\dot W(h) = \sum_{k\geq 1}(h,\mathfrak{u}_k)_{\mathcal{U}}\,\xi_k$, where the series on the right-hand side converges with probability one for every $h\in\mathcal{U}$. The objective of this section is to find similar interpretations of the series in (2.7), (2.9), and (2.11) if the corresponding conditions (2.6), (2.8), (2.10) fail. Along the way, it also becomes natural to allow other generalized random elements to replace $\dot W$.

We start with the construction of *weighted chaos spaces*. Let $\mathcal{R}$ be a bounded linear operator on $L_2(\mathbb{F})$ defined by $\mathcal{R}\xi_{\boldsymbol{\alpha}} = r_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}$ for every $\boldsymbol{\alpha}\in\mathcal{J}$, where the *weights* $\{r_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha}\in\mathcal{J}\}$ are positive numbers.

Given a Hilbert space X, we extend $\mathcal{R}$ to an operator on $L_2(\mathbb{F};X)$ by defining $\mathcal{R}f$ as the unique element of $L_2(\mathbb{F};X)$ such that, for all $g\in X$,

$$\mathbb{E}(\mathcal{R}f,g)_X = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} r_{\boldsymbol{\alpha}}\mathbb{E}\big((f,g)_X\xi_{\boldsymbol{\alpha}}\big).$$

Denote by $\mathcal{R}L_2(\mathbb{F};X)$ the closure of $L_2(\mathbb{F};X)$ with respect to the norm

$$\|f\|^2_{\mathcal{R}L_2(\mathbb{F};X)} := \|\mathcal{R}f\|^2_{L_2(\mathbb{F};X)} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} r^2_{\boldsymbol{\alpha}}\|f_{\boldsymbol{\alpha}}\|^2_X.$$

In what follows, we will identify the operator $\mathcal{R}$ with the corresponding collection $(r_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha}\in\mathcal{J})$. Note that if $u\in\mathcal{R}_1L_2(\mathbb{F};X)$ and $v\in\mathcal{R}_2L_2(\mathbb{F};X)$, then both u and v belong to $\mathcal{R}L_2(\mathbb{F};X)$, where $r_{\boldsymbol{\alpha}}=\min(r_{1,\boldsymbol{\alpha}},r_{2,\boldsymbol{\alpha}})$. As usual, the argument X will be omitted if $X=\mathbb{R}$.

Important particular cases of $\mathcal{R}L_2(\mathbb{F};X)$ are

(1) The `sequence spaces` $L_{2,\mathfrak{q}}(\mathbb{F};X)$, corresponding to the weights $r_{\boldsymbol{\alpha}}=\mathfrak{q}^{\boldsymbol{\alpha}}$, where $\mathfrak{q}=\{q_k,\ k\geq 1\}$ is a sequence of positive numbers; see [11, 9, 19]. Given a real number p, one can also consider the spaces

$$L^p_{2,\mathfrak{q}}(\mathbb{F};X)=L_{2,\mathfrak{q}^p}(\mathbb{F};X), \tag{3.1}$$

where $\mathfrak{q}^p=\{q_k^p,\ k\geq 1\}$. In particular, $L^1_{2,\mathfrak{q}}=L_{2,\mathfrak{q}}$; $L^{-1}_{2,\mathfrak{q}}=L_{2,1/\mathfrak{q}}$. Under the additional assumption $q_k\geq 1$ we have an embedding similar to the usual Sobolev spaces: $L^p_{2,\mathfrak{q}}(\mathbb{F};X)\subset L^r_{2,\mathfrak{q}}(\mathbb{F};X),\ p>r$.

(2) The `Kondratiev spaces` $(\mathcal{S})_{\rho,\ell}(X)$, corresponding to the weights $r_{\boldsymbol{\alpha}}=(\boldsymbol{\alpha}!)^{\rho/2}(2\mathbb{N})^{\ell\boldsymbol{\alpha}},\ \rho\in[-1,1],\ \ell\in\mathbb{R}$, see [4].

(3) The `sequence Kondratiev spaces` $(\mathcal{S})_{\rho,\mathfrak{q}}(X)$, corresponding to the weights $r_{\boldsymbol{\alpha}}=(\boldsymbol{\alpha}!)^{\rho/2}\mathfrak{q}^{\boldsymbol{\alpha}}$, $\rho\in[-1,1]$. We will see below in Section 6 that the sequence Kondratiev spaces $(\mathcal{S})_{\rho,\mathfrak{q}}(X)$, which include both $L_{2,\mathfrak{q}}(X)$ and $(\mathcal{S})_{\rho,\ell}(X)$ as particular cases, are of interest in the study of stochastic evolution equations. We will write $\|\cdot\|_{\rho,\mathfrak{q},X}$ to denote the norm in $(\mathcal{S})_{\rho,\mathfrak{q}}(X)$.

The Cauchy-Schwartz inequality leads to two natural definitions of duality between spaces of generalized random elements, which we denote by $\langle\cdot,\cdot\rangle$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, respectively. If $u\in(\mathcal{S})_{\rho,\mathfrak{q}}(X)$ and $v\in(\mathcal{S})_{-\rho,\mathfrak{q}^{-1}}(X)$, then

$$\langle u,v\rangle=\sum_{\boldsymbol{\alpha}}(u_{\boldsymbol{\alpha}},v_{\boldsymbol{\alpha}})_X. \tag{3.2}$$

The result of this duality is a number extending the notion of $\mathbb{E}(u,v)_X$ to generalized X-valued random elements.

If $u\in(\mathcal{S})_{\rho,\mathfrak{q}}(X\otimes V)$ and $v\in(\mathcal{S})_{-\rho,\mathfrak{q}^{-1}}(V)$, then

$$\langle\!\langle u,v\rangle\!\rangle=\sum_{\boldsymbol{\alpha}}(u_{\boldsymbol{\alpha}},v_{\boldsymbol{\alpha}})_V. \tag{3.3}$$

This duality produces an element of the Hilbert space X.

Taking projective and injective limits of weighted spaces leads to constructions similar to the Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. Of special interest are

(1) The `power sequence` spaces $L^+_{2,\mathfrak{q}}(\mathbb{F};X)=\bigcap_{p\in\mathbb{R}}L^p_{2,\mathfrak{q}}(\mathbb{F};X)$, $L^-_{2,\mathfrak{q}}(\mathbb{F};X)=\bigcup_{p\in\mathbb{R}}L^p_{2,\mathfrak{q}}(\mathbb{F};X)$, where $\mathfrak{q}=\{q_k,\ k\geq 1\}$ is a sequence with $q_k\geq 1$ (see (3.1)).

(2) The spaces $\mathcal{S}^{\rho}(X)$ and $\mathcal{S}_{-\rho}(X)$, $0 \le \rho \le 1$ of `Kondratiev test functions and distributions`: $\mathcal{S}^{\rho}(X) = \bigcap_{\ell\in\mathbb{R}}(\mathcal{S})_{\rho,\ell}(X)$, $\mathcal{S}_{-\rho}(X) = \bigcup_{\ell\in\mathbb{R}}(\mathcal{S})_{-\rho,\ell}(X)$. In the traditional white noise setting, $X = \mathbb{R}^d$, $\rho = 0$ corresponds to the Hida spaces, and the term *Kondratiev spaces* is usually reserved for $\mathcal{S}^1(\mathbb{R}^d)$ and $\mathcal{S}_{-1}(\mathbb{R}^d)$. The similar constructions of the test functions and distributions are possible for the sequence Kondratiev spaces.

If the space $\mathcal{U}$ is finite-dimensional, then the sequence $\mathfrak{q}$ can be taken finite, with as many elements as the dimension of $\mathcal{U}$. In this case, certain Kondratiev spaces are bigger than any sequence space. The precise result is as follows.

Proposition 3.1. *If $\mathcal{U}$ is finite-dimensional, then*

$$L_{2,\mathfrak{q}}(X) \subset (\mathcal{S})_{-\rho,-\ell}(X) \tag{3.4}$$

for every $\rho > 0$, $\ell \ge \rho$ and every $\mathfrak{q}$.

Proof. Let n be the dimension of $\mathcal{U}$ and $r = \min\{q_1, \ldots, q_n\}$. Define $\mathfrak{r} = \{r, \ldots, r\}$. Then $L_{2,\mathfrak{q}}(X) \subset L_{2,\mathfrak{r}}(X)$. On the other hand, for all $\boldsymbol{\alpha} \in \mathcal{J}$, $(2\mathbb{N})^2\boldsymbol{\alpha}! \ge |\boldsymbol{\alpha}|!$, so that $(r^{2|\boldsymbol{\alpha}|}(\boldsymbol{\alpha}!)^{\rho}(2\mathbb{N})^{2\rho})^{-1} \le (r^{2|\boldsymbol{\alpha}|}(|\boldsymbol{\alpha}|!)^{\rho})^{-1} \le C(r)$, which means $L_{2,\mathfrak{r}}(X) \subset (\mathcal{S})_{-\rho,-\ell}(X)$.

□

Analysis of the proof shows that, in general, an inclusion of the type (3.4) is possible if and only if there is a uniform in $\boldsymbol{\alpha}$ bound of the type $(\mathfrak{q}^{2\boldsymbol{\alpha}}(|\boldsymbol{\alpha}|!)^{\rho})^{-1} \le C(2\mathbb{N})^{p\boldsymbol{\alpha}}$; the constants C and p can depend on the sequence $\mathfrak{q}$. If the space $\mathcal{U}$ is infinite-dimensional, then such a bound may exist for certain sequences $\mathfrak{q}$ (such as $\mathfrak{q} = \mathbb{N}$), and may fail to exist for other sequences (such as $\mathfrak{q} = \exp(\mathbb{N})$).

Definition 3.2. *A* `generalized` X`-valued random element` *is an element of the set $\bigcup \mathcal{R}L_2(\mathbb{F}; X)$, with the union taken over all weight sequences $\mathcal{R}$.*

To complete the discussion of weighted spaces, we need the following results about multi-indexed series.

Proposition 3.3. *Let $\mathfrak{r} = \{r_k,\ k \ge 1\}$ be a sequence of positive numbers.*

(1) If $\sum_{k\ge1} r_k < \infty$, then

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\mathfrak{r}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = \exp\left(\sum_{k\ge1} r_k\right). \tag{3.5}$$

(2) If $\sum_{k\ge1} r_k < \infty$ and $r_k < 1$ for all k, then, for every $\boldsymbol{\alpha} \in \mathcal{J}$,

$$\sum_{\boldsymbol{\beta}\in\mathcal{J}} \binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}} \mathfrak{r}^{\boldsymbol{\beta}} = \left(\prod_{k\ge1} \frac{1}{1-r_k}\right)(1-\mathfrak{r})^{-\boldsymbol{\alpha}}, \tag{3.6}$$

where $1-\mathfrak{r}$ is the sequence $\{1-r_k,\ k\ge1\}$. In particular,

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} (2\mathbb{N})^{-\ell\boldsymbol{\alpha}} < \infty \tag{3.7}$$

for all $\ell > 1$; cf. [4, Proposition 2.3.3].

(3) For every $\boldsymbol{\alpha} \in \mathcal{J}$,

$$\sum_{\boldsymbol{\beta} \in \mathcal{J}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathfrak{r}^{\boldsymbol{\beta}} = (1 + \mathfrak{r})^{\boldsymbol{\alpha}}, \tag{3.8}$$

where $1 + \mathfrak{r}$ is the sequence $\{1 + r_k,\ k \geq 1\}$.

Proof. Note that $\exp\left(\sum_{k \geq 1} r_k\right) = \prod_{k \geq 1} \sum_{n \geq 1} r_k^n/n!$, $\prod_{k \geq 1} 1 - r_k{}^{-1} = \prod_{k \geq 1} \sum_{n \geq 1} r_k^n$. By assumption, $\lim_{k \to \infty} r_k = 0$, and therefore $\prod_{k \geq 1} r_k^{n_k} = 0$ unless only finitely many of n_k are not equal to zero. Then both (3.5) and (3.6) with $\boldsymbol{\alpha} = (\mathbf{0})$ follow. For general $\boldsymbol{\alpha}$, (3.6) follows from $\sum_{k \geq 0} \binom{n+k}{k} x^k = (1-x)^{-n-1}$, $|x| < 1$, which, in turn, follows by differentiating n times the equality $\sum_k x^k = (1-x)^{-1}$. Recall that

$$\sum_k r_k < \infty,\ 0 < r_k < 1 \quad \Rightarrow \quad 0 < \prod_k \frac{1}{1 - r_k} < \infty.$$

Equality (3.8) follows from the usual binomial formula. □

Corollary 3.4. *(a) For every collection $f_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{J}$ of elements from X there exists a weight sequence $r_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{J}$ such that $\sum_{\boldsymbol{\alpha} \in \mathcal{J}} \|f_{\boldsymbol{\alpha}}\|_X^2 r_{\boldsymbol{\alpha}}^2 < \infty$.*

(b) If $q_k > 1$ and $\sum_{k \geq 1} 1/q_k < \infty$, then the space $L^+_{2,\mathfrak{q}}(X)$ is nuclear.

(c) The space $\mathcal{S}^{\rho}(X)$ is nuclear for every $\rho \in [0, 1]$.

Proof. (a) In view of (3.7), one can take, for example, $r_{\boldsymbol{\alpha}} = (2\mathbb{N})^{-\boldsymbol{\alpha}}(1 + \|f_{\boldsymbol{\alpha}}\|_X)^{-1}$.

(b) By (3.6), the embedding $L^{p+1}_{2,\mathfrak{q}}(X) \subset L^{p}_{2,\mathfrak{q}}(X)$ is Hilbert-Schmidt for every $p \in \mathbb{R}$.

(c) Note that $\sum_{k \geq 1} (2k)^{-2} < \infty$. Therefore, by (3.6), the embedding $(\mathcal{S})_{\rho,\ell+1}(X) \subset (\mathcal{S})_{\rho,\ell}(X)$ is Hilbert-Schmidt for every $\ell \in \mathbb{R}$. □

To summarize, an element f of $\mathcal{R}L_2(\mathbb{F}; X)$ can be identified with a formal series $\sum_{\boldsymbol{\alpha} \in \mathcal{J}} f_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}$, where $f_{\boldsymbol{\alpha}} \in X$ and $\sum_{\boldsymbol{\alpha} \in \mathcal{J}} \|f_{\boldsymbol{\alpha}}\|_X^2 r_{\boldsymbol{\alpha}}^2 < \infty$. Conversely, every formal series

$$f = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} f_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}, \tag{3.9}$$

$f_{\boldsymbol{\alpha}} \in X$, is a generalized X-valued random element. Using (2.2), we get an alternative representation of the generalized X-valued random element (3.9):

$$f = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \bar{f}_{\boldsymbol{\alpha}}\, \mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.10}$$

with $\bar{f}_{\boldsymbol{\alpha}} \in X$. By (2.2), $\bar{f}_{\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}/\sqrt{\boldsymbol{\alpha}!}$.

The following definition extends the three operators of the Malliavin calculus to generalized random elements.

Definition 3.5. *Let $u = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} u_{\boldsymbol{\alpha}}\, \xi_{\boldsymbol{\alpha}}$ be a generalized $\mathcal{U}$-valued random element, $v = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} v_{\boldsymbol{\alpha}}\, \xi_{\boldsymbol{\alpha}}$, a generalized X-valued random element, and $f = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} f_{\boldsymbol{\alpha}}\, \xi_{\boldsymbol{\alpha}}$, a generalized $X \otimes \mathcal{U}$-valued random element.*

(1) The `Malliavin derivative` *of v with respect to u is the generalized $X\otimes\mathcal{U}$-valued random element*

$$\mathbf{D}_u(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\;v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\otimes u_{\boldsymbol{\beta}}\right)\xi_{\boldsymbol{\alpha}} \tag{3.11}$$

provided the inner sum is well-defined.

(2) The `Skorokhod integral` *of f with respect to u is a generalized X-valued random element*

$$\boldsymbol{\delta}_u(f)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}\sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}\;(f_{\boldsymbol{\beta}},u_{\boldsymbol{\alpha}-\boldsymbol{\beta}})_{\mathcal{U}}\right)\xi_{\boldsymbol{\alpha}}. \tag{3.12}$$

(3) The `Ornstein-Uhlenbeck` *operator with respect to u, when applied to v, is a generalized X-valued random element*

$$\mathcal{L}_u(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{J}}\sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\beta}}}\;v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}(u_{\boldsymbol{\gamma}},u_{\boldsymbol{\alpha}-\boldsymbol{\beta}})_{\mathcal{U}}\right)\xi_{\boldsymbol{\alpha}}, \tag{3.13}$$

provided the inner sum is well-defined.

For future reference, here are the equivalent forms of (3.11), (3.12), and (3.13) using the un-normalized expansion (3.10): if $u=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\bar{u}_{\boldsymbol{\alpha}}\,\mathrm{H}_{\boldsymbol{\alpha}}$, $v=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\bar{v}_{\boldsymbol{\alpha}}\,\mathrm{H}_{\boldsymbol{\alpha}}$, $f=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\bar{f}_{\boldsymbol{\alpha}}\,\mathrm{H}_{\boldsymbol{\alpha}}$, then

$$\mathbf{D}_u(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!}\bar{v}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\otimes\bar{u}_{\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.14}$$

$$\boldsymbol{\delta}_u(f)=\sum_{\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}(\bar{f}_{\boldsymbol{\beta}},\bar{u}_{\boldsymbol{\alpha}-\boldsymbol{\beta}})_{\mathcal{U}}\mathrm{H}_{\boldsymbol{\alpha}}=\sum_{\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}(\bar{f}_{\boldsymbol{\alpha}-\boldsymbol{\beta}},\bar{u}_{\boldsymbol{\beta}})_{\mathcal{U}}\mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.15}$$

$$\mathcal{L}_u(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{J}}\frac{(\boldsymbol{\beta}+\boldsymbol{\gamma})!}{\boldsymbol{\beta}!}\;\bar{v}_{\boldsymbol{\beta}+\boldsymbol{\gamma}}(\bar{u}_{\boldsymbol{\gamma}},\bar{u}_{\boldsymbol{\alpha}-\boldsymbol{\beta}})_{\mathcal{U}}\right)\mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.16}$$

The definitions imply that both $\mathbf{D}$ and $\boldsymbol{\delta}$ are *bi-linear* operators: $\mathcal{A}_{au+bv}(w)=a\mathcal{A}_u(w)+b\mathcal{A}_v(w)$, $\mathcal{A}_u(av+bw)=a\mathcal{A}_u(v)+b\mathcal{A}_u(w)$, $a,b\in\mathbb{R}$, for all suitable u,v,w; $\mathcal{A}$ is either $\mathbf{D}$ or $\boldsymbol{\delta}$. The operation $\mathcal{L}_u(v)$ is linear in v for fixed u, but is not linear in u. The equality $\mathbf{D}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}})=\sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}\xi_{\boldsymbol{\alpha}-\boldsymbol{\beta}}$ shows that, in general $\mathbf{D}_u(v)\neq\mathbf{D}_v(u)$. The definition of the Skorokhod integral $\boldsymbol{\delta}_u(f)$ has a built-in non-symmetry between the integrator u and the integrand f: they have to belong to different spaces. This is necessary to keep the definition consistent with (2.9). Similar non-symmetry holds for the Ornstein-Uhlenbeck operator $\mathcal{L}_u(v)$. Still, we will see later that $\boldsymbol{\delta}_u(f)=\boldsymbol{\delta}_f(u)$ if both f and u are real-valued. If $\mathbf{D}_u(v)$ is defined, then $\mathcal{L}_u(v)=\boldsymbol{\delta}_u(\mathbf{D}_u(v))$, but $\mathcal{L}_u(v)$ can exist even when $\mathbf{D}_u(v)$ is not defined.

Next, note that $\boldsymbol{\delta}_u(f)$ is a well-defined generalized random element for all u and f, while definitions of $\mathbf{D}_u(v)$ and $\mathcal{L}_u(f)$ require additional assumptions. Indeed, $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}=0$ unless $\boldsymbol{\beta}\le\boldsymbol{\alpha}$, and therefore the inner sum on the right-hand side of (3.12) always contains finitely many non-zero terms. By the same reason, the inner sums on the right-hand sides of (3.11) and (3.13) usually contain infinitely many non-zero terms and the convergence must be verified. This observation is an extension of Remark 2.4, and we illustrate it on a concrete example. The example also shows that $\mathcal{L}_u(v)$ can be defined even when $\mathbf{D}_u(v)$ is not.

Example 3.6. Consider $u = v = \dot{W}$. Then $\mathbf{D}_u(v)$ is not defined. Indeed,

$$u_{\boldsymbol{\alpha}} = v_{\boldsymbol{\alpha}} = \begin{cases} \mathfrak{u}_k, & \text{if } \boldsymbol{\alpha} = \boldsymbol{\varepsilon}(k),\ k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.17}$$

Thus, $\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}} = 0$ if $|\boldsymbol{\alpha}| > 0$, and $\big(\mathbf{D}_u(v)\big)_{(\mathbf{0})} = \sum_{k\geq 1} \mathfrak{u}_k \otimes \mathfrak{u}_k$, which is not a convergent series.

On the other hand, interpreting v as an $\mathbb{R} \otimes \mathcal{U}$-valued generalized random element, we find

$$\big(\boldsymbol{\delta}_u(v)\big)_{\boldsymbol{\alpha}} = \begin{cases} \sqrt{2}, & \boldsymbol{\alpha} = 2\boldsymbol{\varepsilon}(k),\ k \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

or, keeping in mind that $\sqrt{2}\,\xi_{2\boldsymbol{\varepsilon}(k)} = \mathrm{H}_2(\xi_k)$, $\boldsymbol{\delta}_{\dot{W}}(\dot{W}) = \sum_{k\geq 1} \mathrm{H}_2(\xi_k)$. Note that $\sum_{k\geq 1} \mathrm{H}_2(\xi_k) \in (\mathcal{S})_{0,\ell}(\mathbb{R})$ for every $\ell < -1/2$.

We conclude the example with an observation that, although $\mathbf{D}_{\dot{W}}(\dot{W})$ is not defined, $\mathcal{L}_{\dot{W}}(\dot{W})$ is. If fact, (3.13) implies that $\mathcal{L}_{\dot{W}}(\dot{W}) = \dot{W}$, which is consistent with (3.17) and the equality $\mathcal{L}_{\dot{W}}\xi_{\boldsymbol{\alpha}} = |\boldsymbol{\alpha}|\xi_{\boldsymbol{\alpha}}$.

If either u or v is a finite linear combination of $\xi_{\boldsymbol{\alpha}}$, then $\mathbf{D}_u(v)$ is defined. The following proposition gives two more sufficient conditions for $\mathbf{D}_u(v)$ to be defined.

Proposition 3.7. *(1) Assume that there exist weights $r_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{J}$ such that $\sum_{\boldsymbol{\alpha}\in\mathcal{J}} 2^{|\boldsymbol{\alpha}|} r_{\boldsymbol{\alpha}}^{-2} \|v_{\boldsymbol{\alpha}}\|_X^2 < \infty$ and $\sum_{\boldsymbol{\alpha}\in\mathcal{J}} r_{\boldsymbol{\alpha}}^2 \|u_{\boldsymbol{\alpha}}\|_{\mathcal{U}}^2 < \infty$. If*

$$\sup_{\boldsymbol{\beta}\in\mathcal{J}} \frac{r_{\boldsymbol{\alpha}+\boldsymbol{\beta}}}{r_{\boldsymbol{\beta}}} := b_{\boldsymbol{\alpha}} < \infty \tag{3.18}$$

for every $\boldsymbol{\alpha} \in \mathcal{J}$, then $\mathbf{D}_u(v)$ is well-defined and

$$\|\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}}\|_{X\otimes\mathcal{U}}^2 \leq 2^{|\boldsymbol{\alpha}|}\, b_{\boldsymbol{\alpha}}^2 \sum_{\boldsymbol{\beta}\in\mathcal{J}} 2^{|\boldsymbol{\beta}|} r_{\boldsymbol{\beta}}^{-2} \|v_{\boldsymbol{\beta}}\|_X^2 \sum_{\boldsymbol{\beta}\in\mathcal{J}} r_{\boldsymbol{\beta}}^2 \|u_{\boldsymbol{\beta}}\|_{\mathcal{U}}^2 \,. \tag{3.19}$$

(2) Assume that there exist weights $r_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{J}$ such that $\sum_{\boldsymbol{\alpha}\in\mathcal{J}} r_{\boldsymbol{\alpha}}^2 \|v_{\boldsymbol{\alpha}}\|_X^2 < \infty$ and $\sum_{\boldsymbol{\alpha}\in\mathcal{J}} 2^{|\boldsymbol{\alpha}|} r_{\boldsymbol{\alpha}}^{-2} \|u_{\boldsymbol{\alpha}}\|_{\mathcal{U}}^2 < \infty$. If

$$\sup_{\boldsymbol{\beta}\in\mathcal{J}} \frac{r_{\boldsymbol{\beta}}}{r_{\boldsymbol{\alpha}+\boldsymbol{\beta}}} := c_{\boldsymbol{\alpha}} < \infty \tag{3.20}$$

for every $\boldsymbol{\alpha} \in \mathcal{J}$, then $\mathbf{D}_u(v)$ is well-defined and

$$\|\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}}\|_{X\otimes\mathcal{U}}^2 \leq 2^{|\boldsymbol{\alpha}|}\, c_{\boldsymbol{\alpha}}^2 \sum_{\boldsymbol{\beta}\in\mathcal{J}} r_{\boldsymbol{\beta}}^2 \|v_{\boldsymbol{\beta}}\|_X^2 \sum_{\boldsymbol{\beta}\in\mathcal{J}} 2^{|\boldsymbol{\beta}|} r_{\boldsymbol{\beta}}^{-2} \|u_{\boldsymbol{\beta}}\|_{\mathcal{U}}^2 \,.$$

Proof. Using $\sum_{k\geq 0}\binom{n}{k} = 2^n$ we conclude that $\binom{n}{k} \leq 2^n$ for all $k \geq 0$ and therefore

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \prod_k \binom{\alpha_k}{\beta_k} \leq 2^{|\boldsymbol{\alpha}|} \tag{3.21}$$

for all $\boldsymbol{\beta} \in \mathcal{J}$. Therefore,

$$\begin{aligned} \big\|\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}}\big\|_{X\otimes\mathcal{U}} &= \left\|\sum_{\boldsymbol{\beta}\in\mathcal{J}} \sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\, v_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \otimes u_{\boldsymbol{\beta}}\right\|_{X\otimes\mathcal{U}} \\ &\leq \sum_{\boldsymbol{\beta}\in\mathcal{J}} 2^{|\boldsymbol{\alpha}+\boldsymbol{\beta}|/2}\, \|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|_X\, \|u_{\boldsymbol{\beta}}\|_{\mathcal{U}}. \end{aligned} \tag{3.22}$$

and the result follows by the Cauchy-Schwartz inequality. □

Remark 3.8. (a) If $r_{\boldsymbol{\alpha}} = \mathfrak{q}^{\boldsymbol{\alpha}}$ for some sequence $\mathfrak{q}$, then both (3.18) and (3.20) hold. (b) More information about the structure of u and/or v can lead to weaker sufficient conditions. For example, if $(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\beta}})_{\mathcal{U}} = 0$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, and $\|u_{\boldsymbol{\alpha}}\|_{\mathcal{U}} \leq 1$, then $\big\|\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}}\big\|^2_{X\otimes\mathcal{U}} < \infty$ if and only if $\sum_{\boldsymbol{\beta}\in\mathcal{J}}\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|^2_X < \infty$, which is a generalization of (2.6). Similarly, if $(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\beta}})_{\mathcal{U}} = 0$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\big(\mathcal{L}_u(v)\big)_{\boldsymbol{\alpha}}$ exists for all $\boldsymbol{\alpha} \in \mathcal{J}$ and $\big(\mathcal{L}_u(v)\big)_{\boldsymbol{\alpha}} = \Big(\sum_{\beta\in\mathcal{J}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\, \|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|^2_{\mathcal{U}}\Big)\, v_{\boldsymbol{\alpha}}$.

Direct computations show that

(1) If $u = \dot{W}$, with $u_{\boldsymbol{\varepsilon}(k)} = \mathfrak{u}_k$ and $u_{\boldsymbol{\alpha}} = 0$ otherwise, then (3.11), (3.12), and (3.13) become, respectively, (2.7), (2.9), and (2.11).

(2) The operators $\boldsymbol{\delta}_{\xi_k}$ and $\mathbf{D}_{\xi_k}$ are the *creation* and *annihilation* operators from quantum physics [2]:

$$\mathbf{D}_{\xi_k}(\xi_{\boldsymbol{\alpha}}) = \sqrt{\alpha_k}\, \xi_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}(k)}, \quad \boldsymbol{\delta}_{\xi_k}(\xi_{\boldsymbol{\alpha}}) = \sqrt{\alpha_k+1}\, \xi_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}(k)}.$$

More generally,

$$\mathbf{D}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}}) = \sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}\, \xi_{\boldsymbol{\alpha}-\boldsymbol{\beta}},\ \boldsymbol{\delta}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}}) = \sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\, \xi_{\boldsymbol{\alpha}+\boldsymbol{\beta}},\ \mathcal{L}_{\xi_{\boldsymbol{\beta}}}(\xi_{\boldsymbol{\alpha}}) = \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\, \xi_{\boldsymbol{\alpha}}.$$

(3) If

$$v \in L_2(\mathbb{F}; X),\ f \in L_2(\mathbb{F}; X\otimes\mathcal{U}),\ \mathbf{D}_u(v) \in L_2(\mathbb{F}; X\otimes\mathcal{U}),\ \boldsymbol{\delta}_u(f) \in L_2(\mathbb{F}; X), \tag{3.23}$$

then a simple rearrangement of terms shows that the following analogue of (2.5) holds:

$$\mathbb{E}\big(\mathbf{D}_u(v), f\big)_{X\otimes\mathcal{U}} = \mathbb{E}\big(v, \boldsymbol{\delta}_u(f)\big)_X. \tag{3.24}$$

For example, $\mathbf{D}_u(\xi_{\boldsymbol{\gamma}}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\sqrt{\binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}}\, u_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}\, \xi_{\boldsymbol{\alpha}}$, and, if we assume that u and f are such that $\boldsymbol{\delta}_u(f) \in L_2(\mathbb{F}; X)$, then $\mathbb{E}\big(\xi_{\boldsymbol{\gamma}}\boldsymbol{\delta}_u(f)\big) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\sqrt{\binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}}\, (u_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}, f_{\boldsymbol{\alpha}})_{\mathcal{U}}$ $= \mathbb{E}\big(f, \mathbf{D}_u(\xi_{\boldsymbol{\gamma}})\big)_{\mathcal{U}}$.

(4) With the notation $\mathrm{H}_{\boldsymbol{\alpha}} = \sqrt{\boldsymbol{\alpha}!}\, \xi_{\boldsymbol{\alpha}}$, $\mathbf{D}_{\xi_k}(\mathrm{H}_{\boldsymbol{\alpha}}) = \alpha_k\, \mathrm{H}_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}(k)}$, $\boldsymbol{\delta}_{\xi_k}(\mathrm{H}_{\boldsymbol{\alpha}}) = \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}(k)}$, and

$$\mathbf{D}_{\mathrm{H}_{\boldsymbol{\beta}}}\mathrm{H}_{\boldsymbol{\alpha}} = \boldsymbol{\beta}!\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\mathrm{H}_{\boldsymbol{\alpha}-\boldsymbol{\beta}},\ \boldsymbol{\delta}_{\mathrm{H}_{\boldsymbol{\alpha}}}(\mathrm{H}_{\boldsymbol{\beta}}) = \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \tag{3.25}$$

To conclude the section, we use (3.25) to establish a connection between the Skorokhod integral $\boldsymbol{\delta}$ and the *Wick product.*

Definition 3.9. *Let f be a generalized X-valued random element and η, a generalized real-valued random element. The* `Wick product` *$f \diamond \eta$ is a generalized X-valued random element defined by*

$$f \diamond \eta = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \left(\sum_{\boldsymbol{\beta}\in\mathcal{J}} \sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}} f_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\, \eta_{\boldsymbol{\beta}} \right) \xi_{\boldsymbol{\alpha}}. \tag{3.26}$$

The definition implies that $f \diamond \eta = \eta \diamond f$, $\xi_{\boldsymbol{\alpha}} \diamond \xi_{\boldsymbol{\beta}} = \sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\alpha}}}\, \xi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$, $\mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathrm{H}_{\boldsymbol{\beta}} = \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. In other words, (3.26) extends relation (3.25) by linearity to generalized random elements. Comparing (3.26) and (3.12), we get the connection between the Wick product and the Skorokhod integral.

Theorem 3.10. *If f is a generalized X-valued random element and η, a generalized real-valued random element, then $\boldsymbol{\delta}_\eta(f) = f \diamond \eta$. In particular, if η and θ are generalized real-valued random elements, then $\boldsymbol{\delta}_\eta(\theta) = \boldsymbol{\delta}_\theta(\eta) = \eta \diamond \theta$.*

The original definition of Wick product [26] is not related to the Skorokhod integral, and it is remarkable that the two coincide in some situations. The important feature of (3.26) is the presence of point-wise multiplication, which does not admit a straightforward extension to general spaces.

Definition 3.9 and Theorem 3.10 raise the following questions:

(1) Is it possible to extend the operation $\diamond$ by replacing the point-wise product on the right-hand side of (3.26) with something else and still preserve the connection with the operator $\boldsymbol{\delta}$? Clearly, simply setting $f \diamond u = \boldsymbol{\delta}_u(f)$ is not acceptable, as we expect the $\diamond$ operation to be fully symmetric.
(2) Under what conditions will the operator $v \mapsto u \diamond v$ be (a Hilbert space) adjoint or (a topological space) dual of $\mathbf{D}_u$?
(3) What is the most general construction of the multiple Wiener-Itô integral?

We will not address these questions in this paper and leave them for future investigation (see references [21, 22] for some particular cases).

4. Elements of Malliavin Calculus on special spaces

The objectives of this section are

- to establish results of the type $\|\mathcal{A}_u(v)\|_a \le C\big(\|u\|_b\big)\, \|v\|_c$, where $\|\cdot\|_i$, $i = a, b, c$ are norms in the suitable sequence or Kondratiev spaces, the function C is independent of v, and $\mathcal{A}$ is one of the operators $\mathbf{D}$, $\boldsymbol{\delta}$, $\mathcal{L}$.
- to look closer at $\mathbf{D}$ and $\boldsymbol{\delta}$ as adjoints of each other when (3.23) does not hold.

To simplify the notations, we will write $L^p_{2,\mathsf{q}}(X)$ for $L^p_{2,\mathsf{q}}(\mathbb{F}; X)$.

To begin, let us see what one can obtain with a straightforward application of the Cauchy-Schwartz inequality. The first collection of results is for the sequence spaces.

Theorem 4.1. *Let* $\mathfrak{q} = \{q_k,\ k \geq 1\}$ *be a sequence such that* $q_k > 1$ *for all* k *and* $\sum_{k\geq 1} 1/q_k^2 < \infty$. *Denote by* $\sqrt{2}\mathfrak{q}$ *the sequence* $\{\sqrt{2}\, q_k,\ k \geq 1\}$.

(a) If $u \in L^{-1}_{2,\mathfrak{q}}(\mathcal{U})$ *and* $v \in L_{2,\sqrt{2}\mathfrak{q}}(X)$, *then* $\mathbf{D}_u(v) \in L_2(\mathbb{F}; X \otimes \mathcal{U})$ *and* $\big(\mathbb{E}\|\mathbf{D}_u(v)\|^2_{X\otimes\mathcal{U}}\big)^{1/2} \leq \Big(\prod_{k\geq 1} 1 - q_k^{-2^{-1}}\Big)^{1/2} \|u\|_{L^{-1}_{2,\mathfrak{q}}(\mathcal{U})} \|v\|_{L_{2,\sqrt{2}\mathfrak{q}}(X)}$.

(b) If $u \in L^{-1}_{2,\mathfrak{q}}(\mathcal{U})$, $f \in L^{-1}_{2,\mathfrak{q}}(X \otimes \mathcal{U})$, *and* $\sum_{k\geq 1} 2^k/q_k^2 < \infty$, *then* $\boldsymbol{\delta}_u(f) \in L^{-1}_{2,\sqrt{2}\mathfrak{q}}(X)$ *and*

$$\|\boldsymbol{\delta}_u(f)\|_{L^{-1}_{2,\sqrt{2}\mathfrak{q}}(X)} \leq \left(\sum_{k\geq 1} \frac{2^k}{q_k^2}\right)^{1/2} \|u\|_{L^{-1}_{2,\mathfrak{q}}(\mathcal{U})} \|f\|_{L^{-1}_{2,\mathfrak{q}}(X\otimes\mathcal{U})}.$$

In particular, if $u \in L^{-}_{2,\mathfrak{q}}(\mathcal{U})$ *and* $f \in L^{-}_{2,\mathfrak{q}}(X \otimes U)$, *then* $\boldsymbol{\delta}_u(f) \in L^{-}_{2,\mathfrak{q}}(X)$.

(c) If $u \in L^{-1}_{2,\mathfrak{q}}(\mathcal{U})$, $v \in L_{2,\sqrt{2}\mathfrak{q}}(X)$, *and* $\sum_{k\geq 1} 2^k/q_k^2 < \infty$, *then* $\mathcal{L}_u(v) \in L^{-1}_{2,\sqrt{2}\mathfrak{q}}(X)$ *and*

$$\|\mathcal{L}_u(v)\|_{L^{-1}_{2,\sqrt{2}\mathfrak{q}}(X)} \leq \left(\prod_{k\geq 1} \frac{q_k^2}{q_k^2 - 1}\right)^{1/2} \left(\sum_{k\geq 1} \frac{2^k}{q_k^2}\right)^{1/2} \|u\|^2_{L^{-1}_{2,\mathfrak{q}}(\mathcal{U})} \|v\|_{L_{2,\sqrt{2}\mathfrak{q}}(X)}.$$

Proof. (a) By (3.19) with $r_{\boldsymbol{\alpha}} = b_{\boldsymbol{\alpha}} = \mathfrak{q}^{-\boldsymbol{\alpha}}$,

$$\|\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}}\|^2_{X\otimes\mathcal{U}} \leq \mathfrak{q}^{-2\boldsymbol{\alpha}} \|u\|^2_{L^{-1}_{2,\mathfrak{q}}(\mathcal{U})} \|v\|^2_{L_{2,\sqrt{2}\mathfrak{q}}(X)}.$$

The result then follows from (3.6).

(b) By (3.12), (3.21), and the Cauchy-Schwartz inequality,

$$\|\big(\boldsymbol{\delta}_u(f)\big)_{\boldsymbol{\alpha}}\|^2_X \leq 2^{|\boldsymbol{\alpha}|}\mathfrak{q}^{2\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} \mathfrak{q}^{-2\boldsymbol{\beta}} \|f_{\boldsymbol{\beta}}\|^2_{X\otimes\mathcal{U}} \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \mathfrak{q}^{-2(\boldsymbol{\alpha}-\boldsymbol{\beta})} \|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|^2_{\mathcal{U}},$$

and the result follows.

(c) This follows by combining the results of (a) and (b). □

Analysis of the proof shows that alternative results are possible by avoiding inequality (3.21); see Theorem 4.3 below. The next collection of results is for the Kondratiev spaces.

Theorem 4.2. *(a) If* $u \in (\mathcal{S})_{-1,-\ell}(\mathcal{U})$ *and* $v \in (\mathcal{S})_{1,\ell}(X)$ *for some* $\ell \in \mathbb{R}$, *then* $\mathbf{D}_u(v) \in (\mathcal{S})_{1,\ell-p}(X \otimes \mathcal{U})$ *for all* $p > 1/2$, *and*

$$\|\mathbf{D}_u(v)\|^{1/2}_{(\mathcal{S})_{1,\ell-p}(X\otimes\mathcal{U})} \leq \left(\prod_{k\geq 1} \frac{1}{1 - (2k)^{-2p}}\right)^{1/2} \|u\|_{(\mathcal{S})_{-1,-\ell}(\mathcal{U})} \|v\|_{(\mathcal{S})_{1,\ell}(X)}.$$

(b) If $u \in (\mathcal{S})_{-1,\ell}(\mathcal{U})$ *and* $f \in (\mathcal{S})_{-1,\ell}(X \otimes \mathcal{U})$ *for some* $\ell \in \mathbb{R}$, *then* $\boldsymbol{\delta}_u(f) \in (\mathcal{S})_{-1,\ell-p}(X)$ *for every* $p > 1/2$, *and*

$$\|\boldsymbol{\delta}_u(f)\|_{(\mathcal{S})_{-1,\ell-p}(X)} \leq \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} (2\mathbb{N})^{-2p\boldsymbol{\alpha}}\right)^{1/2} \|u\|_{(\mathcal{S})_{-1,\ell}(\mathcal{U})} \|f\|_{(\mathcal{S})_{-1,\ell}(X\otimes\mathcal{U})}.$$

In particular, if $u \in \mathcal{S}_{-1}(\mathcal{U})$ *and* $f \in \mathcal{S}_{-1}(X \otimes \mathcal{U})$*, then* $\boldsymbol{\delta}_u(f) \in \mathcal{S}_{-1}(X)$*.*

(c) If $u \in (\mathcal{S})_{-1,-\ell}(\mathcal{U})$ *and* $v \in (\mathcal{S})_{1,\ell+p}(X)$ *for some* $\ell \in \mathbb{R}$ *and* $p > 1/2$*, then* $\mathcal{L}_u(f) \in (\mathcal{S})_{-1,\ell-p}(X)$ *and*

$$\|\mathcal{L}_u(v)\|_{(\mathcal{S})_{-1,\ell-p}(X)} \leq \left(\prod_{k\geq 1} \frac{1}{1-(2k)^{-2p}}\right)^{1/2} \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} (2\mathbb{N})^{-2p\boldsymbol{\alpha}}\right)^{1/2} \|u\|^2_{(\mathcal{S})_{-1,-\ell}(\mathcal{U})} \|v\|_{(\mathcal{S})_{1,\ell}(X\otimes\mathcal{U})}.$$

Proof. To simplify the notations, we write $r_{\boldsymbol{\alpha}} = (2\mathbb{N})^{\ell\boldsymbol{\alpha}}$.

(a) By (3.11), $\big(\mathbf{D}_u(v)\big)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \left(\frac{r^2_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{r^2_{\boldsymbol{\alpha}}\, r^2_{\boldsymbol{\beta}}\, \boldsymbol{\alpha}!\boldsymbol{\beta}!}\right)^{1/2} v_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \otimes u_{\boldsymbol{\beta}}$. To get the result, use triangle inequality, followed by the Cauchy-Schwartz inequality and (3.5).

(b) By (3.12) and the Cauchy-Schwartz inequality,

$$\|\big(\boldsymbol{\delta}_u(f)\big)_{\boldsymbol{\alpha}}\|^2_X \leq r^{-2}_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! \sum_{\boldsymbol{\beta}} \frac{r^2_{\boldsymbol{\beta}}}{\boldsymbol{\beta}!} \|f_{\boldsymbol{\beta}}\|^2_{X\otimes\mathcal{U}} \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \frac{r^2_{\boldsymbol{\alpha}-\boldsymbol{\beta}}}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!} \|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|^2_{\mathcal{U}},$$

and the result follows.

(c) This follows by combining the results of (a) and (b), because $(\mathcal{S})_{1,\ell}(X) \subset (\mathcal{S})_{-1,\ell}(X)$. □

Let us now discuss the duality relation between $\boldsymbol{\delta}_u$ and $\mathcal{D}_u$. Recall that (3.24) is just a consequence of the definitions, once the terms in the corresponding sums are rearranged, *as long as the sums converge*. Condition (3.23) is one way to ensure the convergence, but is not the only possibility: one can also use duality relations between various weighted chaos spaces.

In particular, duality relation (3.2) and Theorem 4.2 lead to the following version of (3.24): if, for some $\ell \in \mathbb{R}$ and $p > 1/2$, we have $u \in (\mathcal{S})_{-1,-\ell-p}(\mathcal{U})$, $v \in (\mathcal{S})_{1,\ell+p}(X)$, and $f \in (\mathcal{S})_{-1,\ell}(X \otimes \mathcal{U})$, then $\langle \boldsymbol{\delta}_u(f), v\rangle_{1,\ell+p} = \langle f, \mathbf{D}_u(v)\rangle_{1,\ell}$.

To derive a similar result in the sequence spaces, we need a different version of Theorem 4.1.

Theorem 4.3. *Let* $\mathfrak{p}$, $\mathfrak{q}$, *and* $\mathfrak{r}$ *be sequences of positive numbers such that*

$$\frac{1}{p_k^2} + \frac{1}{q_k^2} = \frac{1}{r_k^2},\ k \geq 1. \tag{4.1}$$

(a) If $u \in L_{2,\mathfrak{p}}(\mathcal{U})$ *and* $f \in L_{2,\mathfrak{q}}(X \otimes \mathcal{U})$*, then* $\boldsymbol{\delta}_u(f) \in L_{2,\mathfrak{r}}(X)$ *and*

$$\|\boldsymbol{\delta}_u(f)\|_{L_{2,\mathfrak{r}}(X)} \leq \|u\|_{L_{2,\mathfrak{p}}(\mathcal{U})} \|f\|_{L_{2,\mathfrak{q}}(X\otimes\mathcal{U})}.$$

(b) In addition to (4.1) *assume that*

$$\sum_{k\geq 1} \frac{r_k^2}{p_k^2} < \infty. \tag{4.2}$$

Define $\bar{C} = \left(\prod_{k\geq 1} \frac{p_k^2}{p_k^2 - r_k^2}\right)^{1/2}$. *If* $u \in L_{2,\mathfrak{p}}(\mathcal{U})$ *and* $v \in L^{-1}_{2,\mathfrak{r}}(X)$, *then* $\mathbf{D}_u(v) \in L^{-1}_{2,\mathfrak{q}}(X \otimes \mathcal{U})$ *and*

$$\|\mathbf{D}_u(v)\|_{L^{-1}_{2,\mathfrak{q}}(X\otimes\mathcal{U})} \leq \bar{C}\, \|u\|_{L_{2,\mathfrak{p}}(\mathcal{U})}\, \|v\|_{L^{-1}_{2,\mathfrak{r}}(X)}.$$

Proof. (a) By (3.12),

$$\begin{aligned}\|\boldsymbol{\delta}_u(f)\|^2_{L_{2,\mathfrak{r}}(X)} &= \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \left\| \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}} (f_{\boldsymbol{\alpha}}, u_{\boldsymbol{\beta}})_{\mathcal{U}} \right\|^2_X \mathfrak{r}^{2\boldsymbol{\gamma}} \\ &\leq \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \left\| \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}} |(f_{\boldsymbol{\alpha}}, u_{\boldsymbol{\beta}})_{\mathcal{U}}|\ \mathfrak{r}^{\boldsymbol{\alpha}}\mathfrak{r}^{\boldsymbol{\beta}} \right\|^2_X.\end{aligned}$$

Define the sequence $\mathfrak{c} = \{c_k,\ k \geq 1\}$ by $c_k = p_k^2/q_k^2$, such that $(1+\mathfrak{c}^{-1})^{\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\alpha}} = \mathfrak{q}^{2\boldsymbol{\alpha}}$, $(1+\mathfrak{c})^{\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\alpha}} = \mathfrak{p}^{2\boldsymbol{\alpha}}$. Then

$$\|\boldsymbol{\delta}_u(f)\|^2_{L_{2,\mathfrak{r}}(X)} \leq \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \left(\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}} \mathfrak{c}^{\boldsymbol{\alpha}/2}\mathfrak{c}^{-\boldsymbol{\alpha}/2} \|f_{\boldsymbol{\alpha}}\|_{\mathcal{U}\otimes X} \|u_{\boldsymbol{\beta}}\|_{\mathcal{U}}\ \mathfrak{r}^{\boldsymbol{\alpha}}\mathfrak{r}^{\boldsymbol{\beta}} \right)^2.$$

By the Cauchy-Schwartz inequality and (3.8),

$$\begin{aligned}\|\boldsymbol{\delta}_u(f)\|^2_{L_{2,\mathfrak{r}}(X)} &\leq \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \left(\left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \binom{\boldsymbol{\gamma}}{\boldsymbol{\alpha}} \mathfrak{c}^{\boldsymbol{\alpha}} \right) \left(\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} \mathfrak{c}^{-\boldsymbol{\alpha}} \|f_{\boldsymbol{\alpha}}\|^2_{\mathcal{U}\otimes X} \|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\ \mathfrak{r}^{2\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\beta}} \right) \right) \\ &= \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \left((1+\mathfrak{c})^{\boldsymbol{\gamma}} \left(\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} \mathfrak{c}^{-\boldsymbol{\alpha}} \|f_{\boldsymbol{\alpha}}\|^2_{\mathcal{U}\otimes X} \|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\ \mathfrak{r}^{2\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\beta}} \right) \right) \\ &= \sum_{\boldsymbol{\gamma}\in\mathcal{J}} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} (1+\mathfrak{c}^{-1})^{\boldsymbol{\alpha}} (1+\mathfrak{c})^{\boldsymbol{\beta}} \|f_{\boldsymbol{\alpha}}\|^2_{\mathcal{U}\otimes X} \|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\ \mathfrak{r}^{2\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\beta}} \\ &= \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \|f_{\boldsymbol{\alpha}}\|^2_{\mathcal{U}\otimes X}\ (1+\mathfrak{c}^{-1})^{\boldsymbol{\alpha}}\mathfrak{r}^{2\boldsymbol{\alpha}} \right) \left(\sum_{\boldsymbol{\beta}\in\mathcal{J}} \|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\ (1+\mathfrak{c})^{\boldsymbol{\beta}}\mathfrak{r}^{2\boldsymbol{\beta}} \right) \\ &= \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \|f_{\boldsymbol{\alpha}}\|^2_{\mathcal{U}\otimes X}\ \mathfrak{p}^{2\boldsymbol{\alpha}} \right) \left(\sum_{\boldsymbol{\beta}\in\mathcal{J}} \|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\ \mathfrak{q}^{2\boldsymbol{\beta}} \right) = \|f\|^2_{L_{2,\mathfrak{p}}(X\otimes\mathcal{U})}\ \|u\|^2_{L_{2,\mathfrak{p}}(\mathcal{U})}.\end{aligned}$$

(b) By (3.11),

$$\begin{aligned}\|\mathbf{D}_u(v)\|^2_{L^{-1}_{2,\mathfrak{q}}(X\otimes\mathcal{U})} &= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \left\| \sum_{\boldsymbol{\beta}\in\mathcal{J}} \sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\ v_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \otimes u_{\boldsymbol{\beta}} \right\|^2_{X\otimes\mathcal{U}} \mathfrak{q}^{-2\boldsymbol{\alpha}} \\ &\leq \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \left(\sum_{\boldsymbol{\beta}\in\mathcal{J}} \sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\ \|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|_X\, \|u_{\boldsymbol{\beta}}\|_{\mathcal{U}} \right)^2 \mathfrak{q}^{-2\boldsymbol{\alpha}}\end{aligned}$$

Define the sequence $\mathfrak{c} = \{c_k,\ k \geq 1\}$ by $c_k = r_k^2/p_k^2 < 1$, such that

$$(\mathfrak{c}^{-1} - 1)^{\boldsymbol{\alpha}}\mathfrak{q}^{2\boldsymbol{\alpha}} = \mathfrak{p}^{2\boldsymbol{\alpha}},\ (1-\mathfrak{c})^{\boldsymbol{\alpha}}\mathfrak{q}^{2\boldsymbol{\alpha}} = \mathfrak{r}^{2\boldsymbol{\alpha}}. \tag{4.3}$$

Then

$$\|\mathbf{D}_u(v)\|^2_{L^{-1}_{2,\mathfrak{q}}(X\otimes\mathcal{U})} \le \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\sqrt{\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}}\,\mathfrak{c}^{\boldsymbol{\beta}/2}\mathfrak{q}^{-(\boldsymbol{\alpha}+\boldsymbol{\beta})}\mathfrak{c}^{-\boldsymbol{\beta}/2}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|_X\,\mathfrak{q}^{\boldsymbol{\beta}}\|u_{\boldsymbol{\beta}}\|_{\mathcal{U}}\right)^2.$$

By the Cauchy-Schwartz inequality and (3.6),

$$\begin{aligned}
\|\mathbf{D}_u(v)\|^2_{L^{-1}_{2,\mathfrak{q}}(X\otimes\mathcal{U})} &\le \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\mathfrak{c}^{\boldsymbol{\beta}}\right)\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\mathfrak{q}^{-2(\boldsymbol{\alpha}+\boldsymbol{\beta})}\mathfrak{c}^{-\boldsymbol{\beta}}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|^2_X\,\mathfrak{q}^{2\boldsymbol{\beta}}\|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\right)\\
&= \bar{C}^2\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\left((1-\mathfrak{c})^{-\boldsymbol{\alpha}}\right)\sum_{\boldsymbol{\beta}\in\mathcal{J}}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|^2_X\mathfrak{c}^{-\boldsymbol{\beta}}\,\mathfrak{q}^{-2(\boldsymbol{\alpha}+\boldsymbol{\beta})}\,\mathfrak{q}^{2\boldsymbol{\beta}}\,\|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\right)\\
= \bar{C}^2\sum_{\boldsymbol{\beta}\in\mathcal{J}}&\left(\|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}\,\mathfrak{c}^{-\boldsymbol{\beta}}(1-\mathfrak{c})^{\boldsymbol{\beta}}\,\mathfrak{q}^{2\boldsymbol{\beta}}\left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|^2_X\mathfrak{q}^{-2(\boldsymbol{\alpha}+\boldsymbol{\beta})}(1-\mathfrak{c})^{-(\boldsymbol{\alpha}+\boldsymbol{\beta})}\right)\right)\\
&\le \bar{C}^2\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\|u_{\boldsymbol{\beta}}\|^2_{\mathcal{U}}(\mathfrak{c}^{-1}-1)^{\boldsymbol{\beta}}\mathfrak{q}^{2\boldsymbol{\beta}}\right)\left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\|v_{\boldsymbol{\alpha}}\|^2_X(1-\mathfrak{c})^{-\boldsymbol{\alpha}}\mathfrak{q}^{-2\boldsymbol{\alpha}}\right)\\
&= \bar{C}^2\|u\|^2_{L_{2,\mathfrak{p}}(\mathcal{U})}\,\|v\|^2_{L^{-1}_{2,\mathfrak{r}}(X)},
\end{aligned}$$

where the last equality follows from (4.3). Note also that

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\|v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\|^2_X\mathfrak{r}^{-2(\boldsymbol{\alpha}+\boldsymbol{\beta})} \le \|v\|^2_{L^{-1}_{2,\mathfrak{r}}(X)}$$

and the equality holds if and only if $\boldsymbol{\beta} = \mathbf{(0)}$. □

Together with duality relation (3.2), Theorem 4.3 leads to the following version of (3.23): if $u \in L_{2,\mathfrak{p}}(\mathcal{U})$, $f \in L_{2,\mathfrak{q}}(X\otimes\mathcal{U})$, and $v \in L^{-1}_{2,\mathfrak{r}}(X)$, if the sequences $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ are related by (4.1), and if (4.2) holds, then $\langle\boldsymbol{\delta}_u(f), v\rangle_{\mathfrak{r}} = \langle f, \mathbf{D}_u(v)\rangle_{\mathfrak{q}}$.

Here is a general procedure for constructing sequences $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ satisfying (4.1) and (4.2). Start with an arbitrary sequence of positive numbers $\mathfrak{p}$ and a sequence $\mathfrak{c}$ such that $0 < c_k < 1$ and $\sum_{k\ge1} c_k < \infty$. Then set $r_k^2 = c_k p_k^2$ and $q_k^2 = p_k^2/(c_k^{-1}-1)$. If the space $\mathcal{U}$ is n-dimensional, then condition (4.2) is not necessary because in this case the sequences $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ are finite.

The next result is for the Ornstein-Uhlenbeck operator.

Theorem 4.4. *Let $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ be sequences of positive numbers such that*

$$\left(\frac{1}{r_k^2}-\frac{1}{p_k^2}\right)\left(q_k^2-\frac{1}{p_k^2}\right) = 1,\ \ k\ge 1, \tag{4.4}$$

and also, $p_k^2 q_k^2 > 1$, $k \ge 1$, $\sum_{k\ge1}\frac{1}{p_k^2q_k^2} < \infty$. If $u \in L_{2,\mathfrak{p}}(\mathcal{U})$ and $v \in L_{2,\mathfrak{q}}(X)$, then $\mathcal{L}_u(v) \in L_{2,\mathfrak{r}}(X)$ and

$$\|\mathcal{L}_u(v)\|_{L_{2,\mathfrak{r}}(X)} \le \left(\prod_{k\ge1}\frac{p_k^2q_k^2}{p_k^2q_k^2-1}\right)^{1/2}\|u\|^2_{L_{2,\mathfrak{p}}(\mathcal{U})}\,\|v\|_{L_{2,\mathfrak{q}}(X)}. \tag{4.5}$$

Proof. It follows from (3.13) that

$$\|\big(\mathcal{L}_u(v)\big)_{\boldsymbol{\alpha}}\|_X^2 \leq \left(\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\gamma}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X\ \|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{\mathcal{U}}\ \|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}\right)^2.$$

Let $\mathfrak{h} = \{h_k,\ k \geq 1\}$ be a sequence of positive numbers such that that $h_k < 1,\ k \geq 1,\ \sum_k h_k < \infty$. Define $C_h = \prod_k (1-h_k)^{-1/2}$. Then

$$\begin{aligned}\sum_{\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\gamma}}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}} &\leq \left(\sum_{\boldsymbol{\gamma}}\binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\gamma}}\mathfrak{h}^{\boldsymbol{\gamma}}\right)^{1/2}\left(\sum_{\boldsymbol{\gamma}}\mathfrak{h}^{-\boldsymbol{\gamma}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X^2\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}^2\right)^{1/2}\\ &= C_h\left(\frac{1}{(1-\mathfrak{h})^{\boldsymbol{\beta}}}\right)^{1/2}\left(\sum_{\boldsymbol{\gamma}}\mathfrak{h}^{-\boldsymbol{\gamma}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X^2\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}^2\right)^{1/2}\end{aligned}$$

Next, take another sequence $\mathfrak{w} = \{w_k,\ k \geq 1\}$ of positive numbers and define the sequence $\mathfrak{c} = \{c_k,\ k \geq 1\}$ by

$$c_k = \frac{w_k}{1-h_k}. \tag{4.6}$$

Then

$$\begin{aligned}&\left(\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \sqrt{\binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\gamma}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X\|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{\mathcal{U}}\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}\right)^2\\ &\quad\leq C_h^2\left(\sum_{\boldsymbol{\beta}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\mathfrak{c}^{\boldsymbol{\beta}}\right)\left(\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{\mathcal{U}}^2\mathfrak{w}^{-\boldsymbol{\beta}}\left(\sum_{\boldsymbol{\gamma}}\mathfrak{h}^{-\boldsymbol{\gamma}}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X^2\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}^2\right)\right).\end{aligned}$$

As a result,

$$\begin{aligned}\sum_{\boldsymbol{\alpha}}\|\big(\mathcal{L}_u(v)\big)_{\boldsymbol{\alpha}}\|_X^2\mathfrak{r}^{2\boldsymbol{\alpha}} \leq & C_h^2\sum_{\boldsymbol{\gamma}}\mathfrak{r}^{-2\boldsymbol{\gamma}}(1+\mathfrak{c})^{-\boldsymbol{\gamma}}\mathfrak{w}^{\boldsymbol{\gamma}}\mathfrak{h}^{-\boldsymbol{\gamma}}\|u_{\boldsymbol{\gamma}}\|_{\mathcal{U}}^2\\ &\sum_{\boldsymbol{\beta}}\mathfrak{r}^{2(\boldsymbol{\beta}+\boldsymbol{\gamma})}(1+\mathfrak{c})^{\boldsymbol{\beta}+\boldsymbol{\gamma}}\mathfrak{w}^{-(\boldsymbol{\beta}+\boldsymbol{\gamma})}\|v_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\|_X^2\\ &\sum_{\boldsymbol{\alpha}}\mathfrak{r}^{2(\boldsymbol{\alpha}-\boldsymbol{\beta})}(1+\mathfrak{c})^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{\mathcal{U}}^2\end{aligned}$$

Then (4.5) holds if

$$r_k^2(1+c_k) = \frac{w_k}{r_k(1+c_k)h_k} = p_k^2, \quad \frac{r_k^2(1+c)}{w_k} = q_k^2. \tag{4.7}$$

The three equalities in (4.7) imply $(1+c_k) = p_k^2/r_k^2$, $w_k = p_k^2/q_k^2$, $h_k = 1/(p_k^2q_k^2)$, and then (4.4) follows from (4.6). Note that a particular case of (4.4) is $q_k = 1/r_k$, $p_k^{-2}+1 = r_k^{-2}$, which is consistent with Theorem 4.3 if we require the range of $\mathbf{D}_u$ to be in the domain of $\boldsymbol{\delta}_u$. □

Example 4.5. Let $\mathcal{U} = X = \mathbb{R}$. Then $\boldsymbol{\alpha} = n \in \{0,1,2,\ldots\}$, $\xi_{\boldsymbol{\alpha}} := \xi_{(n)} = \frac{\mathrm{H}_n(\xi)}{\sqrt{n}}$, $\xi := \xi_{(1)}$, $u = \sum_{n\geq 0} u_n\xi_{(n)}$, $v = \sum_{n\geq 0} u_n\xi_{(n)}$, $f = \sum_{n\geq 0} f_n\xi_{(n)}$, $u_n, v_n, f_n \in \mathbb{R}$. To begin, take $u=\xi$. Then

$$\mathbf{D}_u(v)=\sum_{n\geq 1}\sqrt{n}v_n\xi_{(n-1)},\ \boldsymbol{\delta}_u(f)=\sum_{n\geq 0}\sqrt{n+1}f_{n+1}\xi_{(n)},$$
$$\mathcal{L}_u(v)=\sum_{n\geq 1}nv_n\xi_{(n)}.$$

Next, let us illustrate the results of Theorems 4.3 and 4.4. Let p,q,r be positive real numbers such that $p^{-1}+q^{-1}=r^{-1}$, for example, $p=q=1,\ r=1/2$. By Theorem 4.3, if $\sum_{n\geq 0}p^nu_n^2<\infty$ and $\sum_{n\geq 0}\frac{v_n^2}{r^n}<\infty$, then $\sum_{n\geq 1}\big(\mathbf{D}_u(v)\big)_n^2q^{-n}<\infty$. If $\sum_{n\geq 0}p^nu_n^2<\infty$ and $\sum_{n\geq 0}q^nf_n^2<\infty$, then $\sum_{n\geq 1}r^n\big(\boldsymbol{\delta}_u(f)\big)_n^2<\infty$. If p,q,r are positive real numbers such that $(r^{-1}-p^{-1})\,(q-p^{-1})=1$ (for example, $p=1,\ q=2,\ r=1/2$) and $\sum_{n\geq 0}p^nu_n^2<\infty$, $\sum_{n\geq 0}q^nv_n^2<\infty$, then, by Theorem 4.4, $\sum_{n\geq 0}r^n\big(\mathcal{L}_u(v)\big)_n^2<\infty$.

5. Higher Order Noise and the Stochastic Exponential

Recall that the Gaussian white noise $\dot{W}$ on a Hilbert space $\mathcal{U}$ is a zero-mean Gaussian family $\{\dot{W}(h),\ h\in\mathcal{U}\}$ such that $\mathbb{E}\big(\dot{W}(g)\dot{W}(h)\big)=(g,h)_{\mathcal{U}}$. Given an orthonormal basis $\{\mathfrak{u}_k,\ k\geq 1\}$ in $\mathcal{U}$, we get a chaos expansion of $\dot{W}$, $\dot{W}=\sum_{k\geq 1}\mathfrak{u}_k\,\xi_k,\ \xi_k=\dot{W}(\mathfrak{u}_k)$. This expansion can be generalized to higher-order chaos spaces:

$$\dot{Z}_N=\sum_{0<|\boldsymbol{\alpha}|\leq N}\mathfrak{u}^{\otimes\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}},\quad \dot{Z}_\infty=\sum_{|\boldsymbol{\alpha}|>0}\mathfrak{u}^{\otimes\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}. \tag{5.1}$$

We call $\dot{Z}_N$ the N-`th order noise` and we call $\dot{Z}_\infty$ the `infinite order noise`; clearly, $\dot{W}=\dot{Z}_1$. For technical reasons, it is more convenient to work with the un-normalized expansion using the basis function $\mathrm{H}_{\boldsymbol{\alpha}}$.

In this and the following sections, we will work with sequence Kondratiev spaces $(\mathcal{S})_{\rho,\mathfrak{q}}(X)$, defined in Section 3, and denote by $\|\cdot\|_{\rho,\mathfrak{q},X}$ the corresponding norm. Recall that if $f\in(\mathcal{S})_{\rho,\mathfrak{q}}(X)$ and $f=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}f_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}$, then

$$\|f\|_{\rho,\mathfrak{q},X}^2=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\|f_{\boldsymbol{\alpha}}\|_X^2\mathfrak{q}^{2\boldsymbol{\alpha}}(\boldsymbol{\alpha}!)^{\rho+1}. \tag{5.2}$$

We also use the following notation: $Y:=\bigoplus_{k\geq 1}\mathcal{U}^{\otimes k}$. The collection $\{\mathfrak{u}^{\otimes\boldsymbol{\beta}},\ |\boldsymbol{\beta}|>0\}$ is an orthonormal basis in the space Y. It follows from (5.1) that $\dot{Z}_N$ and $\dot{Z}_\infty$ are generalized Y-valued random elements.

Next, we derive the expressions for the derivative, divergence, and the Ornstein-Uhlenbeck operators.

The derivatives $\mathbf{D}_{\dot{Z}_N}$ and $\mathbf{D}_{\dot{Z}_\infty}$. If $v=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} v_{\boldsymbol{\alpha}}\,\mathrm{H}_{\boldsymbol{\alpha}}$ is an X-valued generalized random element, then (3.14) implies

$$\mathbf{D}_{\dot{Z}_N}(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{0<|\boldsymbol{\beta}|\leq N}\frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!}v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\otimes\mathfrak{u}^{\otimes\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}}, \tag{5.3}$$

$$\mathbf{D}_{\dot{Z}_\infty}(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{|\boldsymbol{\beta}|>0}\frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!}v_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\otimes\mathfrak{u}^{\otimes\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}}. \tag{5.4}$$

The divergence operators $\boldsymbol{\delta}_{\dot{Z}_N}$ and $\boldsymbol{\delta}_{\dot{Z}_\infty}$. If $f=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}>0} f_{\boldsymbol{\alpha},\boldsymbol{\beta}}\otimes\mathfrak{u}^{\otimes\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}}$ is an $X\otimes Y$-valued generalized random element, then the second equality in (3.15) implies

$$\boldsymbol{\delta}_{\dot{Z}_N}(f)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\substack{|\boldsymbol{\beta}|\leq N\\0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}}} f_{\boldsymbol{\alpha}-\boldsymbol{\beta},\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}},\quad \boldsymbol{\delta}_{\dot{Z}_\infty}(f)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}-\boldsymbol{\beta},\boldsymbol{\beta}}\right)\mathrm{H}_{\boldsymbol{\alpha}}. \tag{5.5}$$

Proposition 5.1. *Assume that the sequence $\mathfrak{q}$ is such that $0<q_k<1$, $\sum_k q_k^2<\infty$. Then $\boldsymbol{\delta}_{\dot{Z}_N}$ and $\boldsymbol{\delta}_{\dot{Z}_\infty}$ are bounded linear operator from $(\mathcal{S})_{-1,\mathfrak{q}}(X\otimes Y)$ to $(\mathcal{S})_{-1,\mathfrak{q}}(X)$.*

Proof. It is enough to consider $\boldsymbol{\delta}_{\dot{Z}}$. Let $f\in(\mathcal{S})_{-1,\mathfrak{q}}(X\otimes Y)$. By (5.2),

$$\|f\|^2_{-1,\mathfrak{q},X\otimes Y}=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}}\|f_{\boldsymbol{\alpha},\boldsymbol{\beta}}\|_X^2\mathfrak{q}^{2\boldsymbol{\alpha}}<\infty.$$

By (5.5) and the Cauchy-Schwartz inequality,

$$\begin{aligned}\|\boldsymbol{\delta}_{\dot{Z}_\infty}f\|^2_{-1,\mathfrak{q},X}&=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left\|\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}f_{\boldsymbol{\alpha}-\boldsymbol{\beta},\boldsymbol{\beta}}\mathfrak{q}^{\boldsymbol{\beta}}\right\|_X^2\\&\leq\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\mathfrak{q}^{2\boldsymbol{\alpha}-2\boldsymbol{\beta}}\|f_{\boldsymbol{\alpha}-\boldsymbol{\beta},\boldsymbol{\beta}}\|_X^2\right)\left(\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\mathfrak{q}^{2\boldsymbol{\beta}}\right)\\&\leq\left(\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}}\|f_{\boldsymbol{\alpha},\boldsymbol{\beta}}\|_X^2\mathfrak{q}^{2\boldsymbol{\alpha}}\right)\left(\sum_{\boldsymbol{\beta}\in\mathcal{J}}\mathfrak{q}^{2\boldsymbol{\beta}}\right),\end{aligned}$$

and the result follows from (3.6). □

The OU operators $\mathcal{L}_{\dot{Z}_N}$ and $\mathcal{L}_{\dot{Z}_\infty}$. If v is an X-valued generalized random element with expansion $v=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} v_{\boldsymbol{\alpha}}\,\mathrm{H}_{\boldsymbol{\alpha}}$, then (3.16) implies

$$\mathcal{L}_{\dot{Z}_N}(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{0<|\boldsymbol{\alpha}-\boldsymbol{\beta}|\leq N}\frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!}\right)v_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}, \tag{5.6}$$

$$\mathcal{L}_{\dot{Z}_\infty}(v)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!}\right)v_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}. \tag{5.7}$$

Indeed, in (3.16) we have $\bar{u}_{\boldsymbol{\gamma}} = \mathfrak{u}^{\otimes\boldsymbol{\gamma}}$, $|\boldsymbol{\gamma}| > 0$, and $(\mathfrak{u}^{\otimes\boldsymbol{\gamma}}, \mathfrak{u}^{\otimes(\boldsymbol{\alpha}-\boldsymbol{\beta})})_Y = \mathbf{1}_{(\boldsymbol{\gamma}=\boldsymbol{\alpha}-\boldsymbol{\beta})}$. In particular, each $\mathrm{H}_{\boldsymbol{\alpha}}$ is an eigenfunction of $\mathcal{L}_{\dot{Z}_\infty}$, with the corresponding eigenvalue

$$\lambda(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!}.$$

Similarly, each $\mathrm{H}_{\boldsymbol{\alpha}}$ is an eigenfunction of $\mathcal{L}_{\dot{Z}_N}$, with the corresponding eigenvalue

$$\lambda_N(\boldsymbol{\alpha}) = \sum_{0<|\boldsymbol{\alpha}-\boldsymbol{\beta}|\leq N} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!}.$$

If $N = 1$, then we recover the familiar result $\lambda_1(\boldsymbol{\alpha}) = |\boldsymbol{\alpha}|$, and if $N = 2$, then

$$\lambda_2(\boldsymbol{\alpha}) = |\boldsymbol{\alpha}| + \sum_{k\neq l} \alpha_k\alpha_l + \sum_k \alpha_k(\alpha_k - 1) = |\boldsymbol{\alpha}|^2.$$

In general, though, $\lambda_N(\boldsymbol{\alpha}) \neq |\boldsymbol{\alpha}|^N$ if $N > 2$.

Next, we introduce the stochastic exponential $\mathcal{E}_h$ and show that the random variable $\mathcal{E}_h$ is, in some sense, an eigenfunction of every $\mathbf{D}_{\dot{Z}_N}$ and, under additional conditions, also of $\mathbf{D}_{\dot{Z}_\infty}$.

Consider the complexification $\mathcal{U}_{\mathbb{C}}$ of $\mathcal{U}$. Given $h = f + ig,\ i = \sqrt{-1},\ f, g \in \mathcal{U}$, denote by h^* the complex conjugate: $h^* = f - ig$. If $h_1 = f_1 + ig_1$, $h_2 = f_2 + ig_2$, then

$$(h_1, h_2)_{\mathcal{U}_{\mathbb{C}}} = (f_1, f_2)_{\mathcal{U}} + (g_1, g_2)_{\mathcal{U}} + i\big((f_2, g_1)_{\mathcal{U}} - (f_1, g_2)_{\mathcal{U}}\big).$$

The Gaussian white noise $\dot{W}$ extends to $\mathcal{U}_{\mathbb{C}}$: if $h = f+ig,$, then $\dot{W}(h) = \dot{W}(f)+i\dot{W}(g)$. Given $h \in \mathcal{U}_{\mathbb{C}}$, consider the complex-valued random variable

$$\mathcal{E}_h = \exp\left(\dot{W}(h) - \frac{1}{2}(h, h^*)_{\mathcal{U}_{\mathbb{C}}}\right), \tag{5.8}$$

also known as the stochastic exponential. It is a standard fact that the collection of the (real) random variables $\mathcal{E}_h,\ h \in \mathcal{U}$, is dense in $L_2(\mathbb{F})$.

Writing

$$h = \sum_{k\geq 1} z_k\, \mathfrak{u}_k,\ \ z_k \in \mathbb{C}, \tag{5.9}$$

we get

$$\mathcal{E}_h = \exp\left(\sum_{k\geq 1} \big(z_k\, \xi_k - (z_k^2/2)\big)\right). \tag{5.10}$$

Using the generating function formula for the Hermite polynomials and the notation $\mathfrak{z} = (z_1, z_2, \ldots)$, (5.10) becomes $\mathcal{E}_h = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\mathfrak{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\,\mathrm{H}_{\boldsymbol{\alpha}} := \mathcal{E}_{\mathfrak{z}}$. In other words, if the emphasis is on the function h, then the stochastic exponential will be denoted by $\mathcal{E}_h$. If the emphasis is on the sequence $\mathfrak{z}$ from representation (5.9), then the stochastic exponential will be denoted by $\mathcal{E}_{\mathfrak{z}}$.

Proposition 5.2. *(a) For every $N \geq 1$, the random variable $\mathcal{E}_h$ is in the domain of $\mathbf{D}_{\dot{Z}_N}$ and*

$$\mathbf{D}_{\dot{Z}_N}(\mathcal{E}_h) = h_N\, \mathcal{E}_h,\ \text{ where } h_N = \left(\sum_{n=1}^N h^{\otimes n}\right). \tag{5.11}$$

(b) Let $\mathfrak{q} = (q_1, q_2, \ldots)$ be a sequence of positive numbers. If the function h with expansion (5.9) *is such that*

$$|z_k|\, q_k < 1 \text{ for all } k \text{ and } \sum_{k\geq 1} |z_k|^2\, q_k^2 < \infty, \tag{5.12}$$

then $\mathcal{E}_h \in (\mathcal{S})_{1,\mathfrak{q}}(\mathbb{C})$.

Proof. (a) To prove (5.11), recall that $\mathbf{D}_{\mathrm{H}_{\boldsymbol{\beta}}} \mathrm{H}_{\boldsymbol{\alpha}} = \boldsymbol{\beta}! \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathrm{H}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}$. Therefore,

$$\mathbf{D}_{\mathrm{H}_{\boldsymbol{\beta}}}(\mathcal{E}_h) = \sum_{\boldsymbol{\alpha}\geq\boldsymbol{\beta}} \frac{\mathfrak{z}^{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\, \mathrm{H}_{\boldsymbol{\alpha}-\boldsymbol{\beta}} = \mathfrak{z}^{\boldsymbol{\beta}} \mathcal{E}_h. \tag{5.13}$$

By (5.3), $\mathbf{D}_{\dot{Z}_N}(\mathcal{E}_h) = \left(\sum_{|\boldsymbol{\beta}|\leq N} \mathfrak{z}^{\boldsymbol{\beta}}\, \mathfrak{u}^{\otimes \boldsymbol{\beta}} \right) \mathcal{E}_h$, which is the same as (5.11).

(b) The result follows from (3.6) and the equality $\|\mathcal{E}_h\|^2_{1,\mathfrak{q},\mathbb{C}} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} |\mathfrak{z}|^{2\boldsymbol{\alpha}} \mathfrak{q}^{2\boldsymbol{\alpha}}$.

□

Corollary 5.3. *If $\|h\|_{\mathcal{U}} < 1$, then the random variable $\mathcal{E}_h$ is in the domain of $\mathbf{D}_{\dot{Z}_\infty}$ and*

$$\mathbf{D}_{\dot{Z}_\infty}(\mathcal{E}_h) = h_\infty\, \mathcal{E}_h, \text{ where } h_\infty = \left(\sum_{n=1}^{\infty} h^{\otimes n} \right). \tag{5.14}$$

Proof. By definition of the norm in the space Y, $\|h_N\|_Y^2 = \sum_{k=1}^N \|h\|^2_{\mathcal{U}}$. As a result, if $\|h\|_{\mathcal{U}} < 1$, then the series $\sum_{n=1}^\infty h^{\otimes n}$ converges in Y and $\sum_{n=1}^\infty h^{\otimes n} = \sum_{|\boldsymbol{\beta}|>0} \mathfrak{z}^{\boldsymbol{\beta}} \mathfrak{u}^{\otimes \boldsymbol{\beta}}$. Equality (5.14) now follows from (5.4) and (5.13). □

Corollary 5.4. *Assume that $\|h\|_{\mathcal{U}} < 1$, $f \in (\mathcal{S})_{-1,\mathfrak{q}}(X \otimes Y)$, and $0 < q_k < 1$, $\sum_{k\geq 1} q_k < \infty$. Then, using notation* (3.3),

$$\langle\!\langle \boldsymbol{\delta}_{\dot{Z}_\infty}(f), \mathcal{E}_h \rangle\!\rangle = \langle\!\langle f, \mathbf{D}_{\dot{Z}_\infty}(\mathcal{E}_h) \rangle\!\rangle = \big(\langle\!\langle f, \mathcal{E}_h \rangle\!\rangle, h_\infty \big)_Y. \tag{5.15}$$

Remark 5.5. (a) Since non-random objects play the role of constants for the Malliavin derivative, equalities (5.11) and (5.14) can indeed be interpreted as eigenvalue/eigenfuntion relations for $\mathbf{D}_N$ and $\mathbf{D}_{\dot{Z}}$.

(b) Given a sequence $\mathfrak{z}$, one can always find a sequence $\mathfrak{q}$ to satisfy (5.12), for example, by taking $q_k = 2^{-k-1}/(|z_k|+1)$.

If $\eta \in (\mathcal{S})_{-1,\mathfrak{q}}(X)$, then we can define the X-valued function $\widetilde{\eta}(\mathfrak{z}) = \langle\!\langle \eta, \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle$. The next two theorems establish some useful properties of the function $\widetilde{\eta}$. Before we state the results, let us introduce another notation: Given a sequence $\mathfrak{c} = (c_1, c_2 \ldots)$ of positive numbers, $\mathbb{K}(\mathfrak{c})$ denotes the collection of complex sequences $\mathfrak{z} = (z_1, z_2, \ldots)$ such that $z_k \in \mathbb{C}$ and $|z_k| < c_k$ for all $k \geq 1$:

$$\mathbb{K}(\mathfrak{c}) = \big\{ \mathfrak{z} = (z_1, z_2, \ldots) \mid z_k \in \mathbb{C},\ |z_k| < c_k,\ k \geq 1 \big\} \tag{5.16}$$

Theorem 5.6. *If $\eta = \sum_{\boldsymbol{\alpha}} \eta_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}} \in (\mathcal{S})_{-1,\mathfrak{q}}(X)$ then there exists a region $\mathbb{K}(\mathfrak{c})$ such that $\mathcal{E}_{\mathfrak{z}} \in (\mathcal{S})_{1,\mathfrak{q}^{-1}}(\mathbb{C})$ for $\mathfrak{z} \in \mathbb{K}(\mathfrak{c})$ and $\widetilde{\eta}(\mathfrak{z}) = \langle\!\langle \eta, \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle$ is an analytic function in $\mathbb{K}(\mathfrak{c})$.*

Proof. In the setting of the theorem, (3.3) implies $\widetilde{\eta}(\mathfrak{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \eta_{\boldsymbol{\alpha}}\, \mathfrak{z}^{\boldsymbol{\alpha}}$, which is a power series and hence analytic inside its region of convergence. By the Cauchy-Schwartz inequality,

$$|\widetilde{u}(\mathfrak{z})|^2 \leq \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \|u_{\boldsymbol{\alpha}}\|_X^2 \mathfrak{q}^{2\boldsymbol{\alpha}}\right)\left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{|\mathfrak{z}|^{2\boldsymbol{\alpha}}}{\mathfrak{q}^{2\boldsymbol{\alpha}}}\right). \tag{5.17}$$

According to (3.6), the right-hand side of (5.17) is finite if $|z_k| < q_k$ for all k and $\sum_k |z_k|^2/q_k^2 < 1$, for example, if $|z_k| \leq 2^{-k-1}q_k := c_k$. With this choice of the sequence $\mathfrak{c}$, the function $\widetilde{u}$ is analytic in $\mathbb{K}(\mathfrak{c})$. □

Theorem 5.7. *Let $\mathfrak{q} = (q_1, q_2, \ldots)$ be a sequence such that $0 < q_k < 1$ and $\sum_{k\geq 1} q_k^2 < 1$, and let X be a Hilbert space.*

(a) If $\eta = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \eta_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}} \in (\mathcal{S})_{-1,\mathfrak{q}}(X)$, then $\widetilde{\eta}(\mathfrak{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \eta_{\boldsymbol{\alpha}}\mathfrak{z}^{\boldsymbol{\alpha}}$ is an X-valued analytic function in the region $\mathbb{K}(\mathfrak{q}^2)$ and

$$\sup_{\mathfrak{z}\in\mathbb{K}(\mathfrak{q}^2)} \|\widetilde{\eta}(\mathfrak{z})\|_X \leq \left(\prod_{k\geq 1}\frac{1}{1-q_k^2}\right)^{1/2} \|\eta\|_{-1,\mathfrak{q},X}. \tag{5.18}$$

(b) If $\widetilde{\eta} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \eta_{\boldsymbol{\alpha}}\mathfrak{z}^{\boldsymbol{\alpha}}$ is an X-valued function, analytic in $\mathbb{K}(\mathfrak{q})$ and such that

$$\sup_{\mathfrak{z}\in\mathbb{K}(\mathfrak{q})} \|\widetilde{\eta}(\mathfrak{z})\|_X \leq B, \tag{5.19}$$

then $\eta := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \eta_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}} \in (\mathcal{S})_{-1,\mathfrak{q}^2}(X)$ and

$$\|\eta\|_{-1,\mathfrak{q}^2,X} \leq \left(\prod_{k\geq 1}\frac{1}{1-q_k^2}\right)^{1/2} B. \tag{5.20}$$

Proof. (a) If $|z_k| \leq q_k^2$, then, by the triangle inequality, followed by the Cauchy-Schwartz inequality,

$$\|\widetilde{\eta}(\mathfrak{z})\|_X \leq \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \|\eta_{\boldsymbol{\alpha}}\|_X\, |\mathfrak{z}|^{\boldsymbol{\alpha}} \leq \|\eta\|_{-1,\mathfrak{q},X} \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{q}^{2\boldsymbol{\alpha}}\right)^{1/2},$$

and then (5.18) follows from (3.6).

(b) Inequality (5.19) and general properties of analytic functions imply $\|\eta_{\boldsymbol{\alpha}}\|_X\, \mathfrak{q}^{\boldsymbol{\alpha}} \leq B$ for all $\boldsymbol{\alpha} \in \mathcal{J}$, which is derived starting with a single complex variable and applying the Cauchy integral formula, and then doing induction on the number of variables; see [4, Lemma 2.6.10]. After that,

$$\|\eta\|_{-1,\mathfrak{q}^2,X}^2 = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \|\eta_{\boldsymbol{\alpha}}\|_X^2 = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \left(\|\eta_{\boldsymbol{\alpha}}\|_X^2 \mathfrak{q}^{2\boldsymbol{\alpha}}\right) \mathfrak{q}^{2\boldsymbol{\alpha}} \leq B^2 \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{q}^{2\boldsymbol{\alpha}},$$

and (5.20) follows. □

6. Stochastic evolution equations driven by infinite-order noise

The results of the previous sections allow us to develop basic solvability theory in the sequence Kondratiev spaces $(\mathcal{S})_{-1,\mathfrak{q}}(X)$ for stochastic equations driven by $\dot{Z}_\infty$. In what follows, we assume that $0 < q_k < 1$ and $\sum_k q_k^p < 1$ for some $p > 0$. Since a smaller sequence $\mathfrak{q}$ corresponds to a larger space $(\mathcal{S})_{-1,\mathfrak{q}}(X)$, there is no loss of generality.

6.1. **Evolution equations.** To begin, we introduce the set-up to study evolution equations. Let (V, H, V') be a normal triple of Hilbert spaces and

$$\mathcal{V}(T) = L_2((0,T);V),\ \mathcal{H}(T) = L_2((0,T);H),\ \mathcal{V}'(T) = L_2((0,T);V'). \tag{6.1}$$

Let $\mathbf{A}(t) : V \to V'$ and $\mathbf{M}(t) : V \to V' \otimes Y$ be bounded linear operators for every $t \in [0,T]$. Given $v \in V$, $\mathbf{M}(t)v = \sum_{|\boldsymbol{\beta}|>0} v_{\boldsymbol{\beta}}(t) \otimes \mathfrak{u}^{\otimes\boldsymbol{\beta}}$, and we define $\mathbf{M}_{\boldsymbol{\beta}}(t)v = v_{\boldsymbol{\beta}}(t) = \big(\mathbf{M}(t)v, \mathfrak{u}^{\otimes\boldsymbol{\beta}}\big)_Y,\ |\boldsymbol{\beta}| > 0$. Recall that $Y = \bigoplus_{k\ge1} \mathcal{U}^{\otimes k}$.

The objective of this section is to study the stochastic evolution equation

$$u(t) = u^\circ + \int_0^t \Big(\mathbf{A}(s)u(s) + f(s) + \boldsymbol{\delta}_{\dot{Z}_\infty}\big(\mathbf{M}(s)u(s)\big)\Big)ds,\ 0 \le t \le T, \tag{6.2}$$

where $u^\circ \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(H)$, $f \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(\mathcal{V}'(T))$.

To ensure that $\mathbf{M}\eta \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}'(T) \otimes Y\big)$ for every $\eta \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}'(T)\big)$, we impose the following condition.

Condition (ME): *There exists a sequence $\mathfrak{b} = (b_1, b_2, \dots)$ of positive numbers such that, for every $v \in \mathcal{V}(T)$ and every multi-index $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| > 0$,*

$$\|\mathbf{M}_{\boldsymbol{\alpha}} v\|_{\mathcal{V}'(T)} \le \mathfrak{b}^{\boldsymbol{\alpha}} \|v\|_{\mathcal{V}(T)}. \tag{6.3}$$

Remark 6.1. To consider a more general noise $\dot{Z}^a = \sum_{|\boldsymbol{\alpha}|>0} a_{\boldsymbol{\beta}} \mathrm{H}_{\boldsymbol{\beta}} \mathfrak{u}^{\otimes\boldsymbol{\beta}},\ a_{\boldsymbol{\beta}} \in \mathbb{R}$, in equation (6.2), it is enough to replace $\mathbf{M}_{\boldsymbol{\beta}}$ with $a_{\boldsymbol{\beta}} \mathbf{M}_{\boldsymbol{\beta}}$.

Recall the following standard definition.

Definition 6.2. *A solution of equation*

$$U(t) = U_0 + \int_0^t \mathbf{A}(s)U(s)ds + \int_0^t F(s)ds, \tag{6.4}$$

with $U_0 \in H$ and $F \in \mathcal{V}'(T)$, is an element of $\mathcal{V}(T)$ for which equality (6.4) *holds in $\mathcal{V}'(T)$. In other words, for every v from a dense subset of V,*

$$[U(t), v] = [U_0, v] + \int_0^t [\mathbf{A}(s)U(s), v]ds + \int_0^t [F(s), v]ds \tag{6.5}$$

for almost all $t \in [0,T]$. In (6.5), *$[\cdot,\cdot]$ denotes the duality between V and V' relative to the inner product in H.*

Definition 6.3. *The solution of equation* (6.2) *is an element of $\bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$ with the following properties:*

(1) *There exists a region* $\mathbb{K}(\mathfrak{c})$ *inside which the function* $\widetilde{u}(\mathfrak{z}) = \langle\!\langle u, \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle$ *is defined and analytic with values in* $\mathcal{V}(T)$;
(2) *For every* $\mathfrak{z} \in \mathbb{K}(\mathfrak{c})$, *the following equality holds in* $\mathcal{V}'(T)$:

$$\begin{aligned} \langle\!\langle u(t), \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle = \langle\!\langle u^{\circ}, \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle + \int_0^t \langle\!\langle \mathbf{A}(s)u(s)), \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle ds + \int_0^t \langle\!\langle f(s), \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle ds \\ + \int_0^t \langle\!\langle \boldsymbol{\delta}_{\dot{Z}_\infty}\big(\mathbf{M}(s)u(s)\big), \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle ds. \end{aligned} \tag{6.6}$$

Remark 6.4. (a) The solutions from both Definition 6.3 and Definition 6.2 are examples of variational solutions. The key feature of the variational solution is the use of duality to make sense out of the corresponding equation. In particular, in (6.5), the duality is between V and V', while in (6.6) the duality is between $(\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$ and $(\mathcal{S})_{1,\mathfrak{q}^{-n}}(\mathbb{C})$. (b) Theorem 5.7(a) and assumption $u \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$ ensure existence of a region $\mathbb{K}(\mathfrak{c})$ inside which the function $\widetilde{u}$ is analytic.

The following theorem gives a characterization of the solution in terms of the chaos expansion.

Theorem 6.5. *A* $\mathcal{V}'(T)$*-valued generalized process* $u(t) = \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$ *is a solution of* (6.2) *if and only if the collection* $\{u_{\boldsymbol{\alpha}}(t),\ \boldsymbol{\alpha} \in \mathcal{J}\}$ *is a solution of the system*

$$\begin{aligned} u_{(\mathbf{0})}(t) &= u^{\circ}_{(\mathbf{0})} + \int_0^t \mathbf{A}u_{(\mathbf{0})}(s)ds + \int_0^t f_{(\mathbf{0})}(s)ds,\ |\boldsymbol{\alpha}| = 0, \\ u_{\boldsymbol{\alpha}}(t) &= u^{\circ}_{\boldsymbol{\alpha}} + \int_0^t \mathbf{A}u_{\boldsymbol{\alpha}}(s)ds + \int_0^t f_{\boldsymbol{\alpha}}(s)ds + \sum_{0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \int_0^t \mathbf{M}_{\boldsymbol{\beta}}(s)u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(s)ds,\ |\boldsymbol{\alpha}| > 0. \end{aligned} \tag{6.7}$$

Remark 6.6. The system of equations (6.7) is called the `propagator` corresponding to (6.2), and is solvable by induction on $|\boldsymbol{\alpha}|$: once every $u_{\boldsymbol{\alpha}}(t)$ is known for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = n$, then all $u_{\boldsymbol{\alpha}}(t)$ corresponding to $|\boldsymbol{\alpha}| = n+1$ can be recovered. Each equation of the system is of the type (6.4), and its solution is understood in the sense of Definition 6.2.

Proof of Theorem 6.5. With no loss of generality, we can assume that $\sum_{k\geq 1}|z_k|^2 < 1$. By Corollary 5.3, $\mathbf{D}_{\dot{Z}}(\mathcal{E}_{\mathfrak{z}}) = h_\infty(\mathfrak{z})\mathcal{E}_{\mathfrak{z}}$, where $h_\infty(\mathfrak{z}) = \sum_{n=1}^{\infty} h^{\otimes n}(\mathfrak{z})$ and $h(\mathfrak{z}) = \sum_k z_k\, \mathfrak{u}_k$. Then (5.15) implies

$$\begin{aligned} \widetilde{u}(t;\mathfrak{z}) = \langle\!\langle u^{\circ}, \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle + \int_0^t \mathbf{A}(s)\widetilde{u}(s;\mathfrak{z})ds + \int_0^t \langle\!\langle f(s), \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle ds \\ + \int_0^t \big(\mathbf{M}(s)\widetilde{u}(s;\mathfrak{z}), h_\infty(\mathfrak{z})\big)_Y ds. \end{aligned} \tag{6.8}$$

By the definition of the operators $\mathbf{M}_{\boldsymbol{\beta}}$,

$$\big(\mathbf{M}(s)\widetilde{u}(s;\mathfrak{z}), h_\infty(\mathfrak{z})\big)_Y = \sum_{|\boldsymbol{\beta}|>0} \mathfrak{z}^{\boldsymbol{\beta}}\mathbf{M}_{\boldsymbol{\beta}}(s)\widetilde{u}(s;\mathfrak{z}), \tag{6.9}$$

and condition (ME) ensures convergence of the series on the right-hand side in a suitable region $\mathbb{K}(\mathfrak{c})$.

Define the operator $D_0^{\boldsymbol{\alpha}}$ by $D_0^{\boldsymbol{\alpha}}F(\mathfrak{z}) = \frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{|\boldsymbol{\alpha}|}F(\mathfrak{z})}{\partial\mathfrak{z}^{\boldsymbol{\alpha}}}\Big|_{\mathfrak{z}=0}$. Representation $\widetilde{u}(t;\mathfrak{z}) = \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}(t)\mathfrak{z}^{\boldsymbol{\alpha}}$ and analyticity of $\widetilde{u}(t;\mathfrak{z})$ imply $u_{\boldsymbol{\alpha}}(t) = D_0^{\boldsymbol{\alpha}}\widetilde{u}(t;\mathfrak{z})$ Application of $D_0^{\boldsymbol{\alpha}}$ to both sides of (6.8) results in (6.7).

This completes the proof of Theorem 6.5. □

When $N = 1$, equation (6.2) was investigated in [12]. In particular, it was shown that (6.2) with $N = 1$ includes as particular cases a variety of evolution equations, ordinary or with partial derivatives, driven by standard Brownian motion, fractional Brownian motion, Brownian sheet, and many other Gaussian processes.

Let us illustrate possible behavior of the solution of (6.2) for $N > 1$ when $\mathcal{U} = V = H = V' = \mathbb{R}$, so that $\dot{Z}_N = \sum_{k=1}^N \mathrm{H}_k(\xi)$ for a standard normal random variable ξ.

First, let us take $N = 2$ and consider the equation $u(t) = 1 + \int_0^t \boldsymbol{\delta}_{\dot{Z}_2}\big(\mathbf{M}u(s)\big)ds$, where $\mathbf{M}_1 = 0$, and $\mathbf{M}_2 = 1$. Direct computations show that $u(t) = 1 + \sum_{k=1}^\infty \frac{t^k}{k!}\mathrm{H}_{2k}(\xi)$. Using Stirling's formula, we conclude that the second moment of the solution blows up in finite time:

$$\mathbb{E}u^2(t) = 1 + \sum_{k\geq 1}\frac{t^{2k}(2k)!}{(k!)^2} \sim \sum_{k\geq 1}(2t)^{2k} = +\infty \text{ if } t \geq 1/2.$$

Next, let us take $N = 4$ and consider the equation $u(t) = 1 + \int_0^t \boldsymbol{\delta}_{\dot{Z}_4}\big(\mathbf{M}u(s)\big)ds$, where $\mathbf{M}_k = 0$, $k = 1, 2, 3$, and $\mathbf{M}_4 = 1$. Direct computations show that $u(t) = 1 + \sum_{k=1}^\infty \frac{t^k}{k!}\mathrm{H}_{4k}(\xi)$. Using Stirling's formula, we conclude that the second moment of the solution is infinite for all $t > 0$:

$$\mathbb{E}u^2(t) = 1 + \sum_{k\geq 1}\frac{t^{2k}(4k)!}{(k!)^2} \sim \sum_{k\geq 1}\left(\frac{16t}{e}\right)^{2k} k^{2k} = +\infty \text{ if } t > 0.$$

More complicated example involving stochastic partial differential equations are given later in the section.

To prove existence and uniqueness of solution of (6.2), we impose the following conditions on the operator $\mathbf{A}$.

Condition (AE): *For every $U_0 \in H$ and $F \in \mathcal{V}'(T)$, there exists a unique function $U \in \mathcal{V}$ that solves the deterministic equation*

$$\partial_t U(t) = \mathbf{A}(t)U(t) + F(t),\ U(0) = U_0 \tag{6.10}$$

and there exists a constant $C_A = C_A(\mathbf{A}, T)$ such that $\|U\|_{\mathcal{V}(T)} \leq C_A\big(\|U_0\|_H + \|F\|_{\mathcal{V}'(T)}\big)$. In particular, the operator $\mathbf{A}$ generates a semi-group $\Phi = \Phi_{t,s}$, $t \geq s \geq 0$, and

$$U(t) = \Phi_{t,0}U_0 + \int_0^t \Phi_{t,s}F(s)ds.$$

Remark 6.7. There are various types of assumptions on the operator $\mathbf{A}$ that imply condition (AE). In particular, (AE) holds if the operator $\mathbf{A}$ is coercive in (V, H, V'): $[\mathbf{A}(t)v, v] + \gamma\|v\|_V^2 \leq C\|v\|_H^2$ for every $v \in V$, $t \in [0, T]$, where $\gamma > 0$ and $C \in \mathbb{R}$ are both independent of v, t.

The following is the main result about existence, uniqueness, and regularity of the solution of (6.2).

Theorem 6.8. *Assume that conditions (AE) and (ME) hold,* $u^\circ \in (\mathcal{S})_{-1,\mathfrak{q}}(H)$, $f \in (\mathcal{S})_{-1,\mathfrak{q}}\big(\mathcal{V}'(T)\big)$, *and assume that the sequence* $\mathfrak{q} = (q_1, q_2, \ldots)$ *has the following properties:*

$$0 < q_k < 1,\ \sum_{k\geq 1} q_k < 1,\ C_0 := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{b}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}} < \frac{1}{C_A}, \tag{6.11}$$

where the number C_A *and the sequence* $\mathfrak{b}$ *are from conditions (AE) and (ME), respectively. Define the number* $C_1 = \prod_{k\geq 1}(1-q_k^2)^{-1/2}$.

Then equation (6.2) *has a unique solution in* $\bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$. *The solution is an element of* $(\mathcal{S})_{-1,\mathfrak{q}^4}\big(\mathcal{V}(T)\big)$ *and*

$$\|u\|_{-1,\mathfrak{q}^4,\mathcal{V}(T)} \leq \frac{C_A C_1^2}{1-C_0C_A}\Big(\|u^\circ\|_{-1,\mathfrak{q},H} + \|f\|_{-1,\mathfrak{q},\mathcal{V}'(T)}\Big). \tag{6.12}$$

Remark 6.9. Analysis of the equation $du = u_{xx}(dt + dw(t))$ on the real line with initial condition $u^\circ(x) = e^{-x^2/2}$ shows that the conclusion of the theorem is sharp in the following sense: if $\rho > -1$, then, in general, one cannot find a sequence $\mathfrak{c}$ to ensure that the solution of (6.2) belongs to $(\mathcal{S})_{\rho,\mathfrak{c}}(\mathcal{V}(T))$; see [10] for details.

Proof of Theorem 6.8. Define the functions $\widetilde{u}^\circ(\mathfrak{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u^\circ_{\boldsymbol{\alpha}}\mathfrak{z}^{\boldsymbol{\alpha}}$, $\widetilde{f}(t;\mathfrak{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} f_{\boldsymbol{\alpha}}(t)\mathfrak{z}^{\boldsymbol{\alpha}}$. By Theorem 5.7(a), both $\widetilde{u}^\circ$ and $\widetilde{f}$ are analytic in $\mathbb{K}(\mathfrak{q}^2)$. For fixed $\mathfrak{z} \in \mathbb{K}(\mathfrak{q}^2)$ consider the following deterministic equation with the unknown function $\widetilde{u}(t;\mathfrak{z})$:

$$\begin{aligned}\widetilde{u}(t;\mathfrak{z}) = \widetilde{u}^\circ(\mathfrak{z}) + \int_0^t \mathbf{A}(s)\widetilde{u}(t;\mathfrak{z})ds + \int_0^t \widetilde{f}(t;\mathfrak{z})ds \\ + \sum_{|\boldsymbol{\beta}|>0} \mathfrak{z}^{\boldsymbol{\beta}} \int_0^t \mathbf{M}_{\boldsymbol{\beta}}(s)\widetilde{u}(s;\mathfrak{z})\, ds.\end{aligned} \tag{6.13}$$

The sum in $\boldsymbol{\beta}$ is well defined because of inequality (6.3) and the assumptions on $\mathfrak{q}$.

If $u \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$, then, by Theorem 5.7(a), the corresponding function $\widetilde{u} = \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}\mathfrak{z}^{\boldsymbol{\alpha}}$ is analytic in some region $\mathbb{K}(\mathfrak{c})$. We already know (see (6.8) and (6.9)) that if u is a solution of (6.2), then $\widetilde{u}$ is an element of $\mathcal{V}(T)$ and satisfies (6.13). Therefore, to prove the theorem, we need to show that (6.13) has a unique solution and to derive a suitable bound on $\|\widetilde{u}\|_{\mathcal{V}(T)}$.

Condition (AE) implies the following *a priori* bound on $\|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}'(T)}$:

$$\|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}(T)} \leq C_A\Big(\|\widetilde{u}^\circ(\mathfrak{z})\|_H + \|\widetilde{f}(\cdot;\mathfrak{z}) + \sum_{|\boldsymbol{\beta}|\leq N} \mathfrak{z}^{\boldsymbol{\beta}}\mathbf{M}_{\boldsymbol{\beta}}(s)\widetilde{u}(s;\mathfrak{z})\|_{\mathcal{V}'(T)}\Big).$$

By the triangle inequality and (6.3),

$$\|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}(T)} \leq C_A\Big(\|\widetilde{u}^\circ(\mathfrak{z})\|_H + \|\widetilde{f}(\cdot;\mathfrak{z})\|_{\mathcal{V}'(T)} + C_0\|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}(T)}\Big)$$

or

$$\|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}(T)} \leq \frac{C_A}{1-C_AC_0}\Big(\|\widetilde{u}^{\circ}(\mathfrak{z})\|_H + \|\widetilde{f}(\cdot;\mathfrak{z})\|_{\mathcal{V}'(T)}\Big). \tag{6.14}$$

Applying a fixed-point iteration, we conclude that equation (6.13) has a unique solution in $\mathcal{V}(T)$ and (6.14) holds.

By Theorem 5.7(a),

$$\sup_{\mathfrak{z}\in\mathbb{K}(\mathfrak{q}^2)} \|\widetilde{u}^{\circ}(\mathfrak{z})\|_H \leq C_1\|u^{\circ}\|_{-1,\mathfrak{q},H} \text{ and } \sup_{\mathfrak{z}\in\mathbb{K}(\mathfrak{q}^2)} \|\widetilde{f}(\cdot;\mathfrak{z})\|_{\mathcal{V}'(T)} \leq C_1\|f\|_{-1,\mathfrak{q},\mathcal{V}'(T)}.$$

Therefore, (6.14) implies

$$\sup_{\mathfrak{z}\in\mathbb{K}(\mathfrak{q}^2)} \|\widetilde{u}(\cdot;\mathfrak{z})\|_{\mathcal{V}(T)} \leq \frac{C_1C_A}{1-C_0C_A}\Big(\|u^{\circ}\|_{-1,\mathfrak{q},H} + \|f\|_{-1,\mathfrak{q},\mathcal{V}'(T)}\Big).$$

By Theorem 5.7(b), we conclude that $\widetilde{u}(\cdot;\mathfrak{z})$ corresponds to an element $u \in (\mathcal{S})_{-1,\mathfrak{q}^4}\big(\mathcal{V}(T)\big)$ and (6.12) holds.

This completes the proof of Theorem 6.8. □

Corollary 6.10. *Standard results about deterministic evolution equations imply that, in the setting of Theorem 6.8, we have $u(t) \in (\mathcal{S})_{-1,\mathfrak{q}^4}(H)$ for every $t \in [0,T]$.*

If both $u^{\circ} \in H$ and $f \in \mathcal{V}'(T)$ are non-random, then, of course, $u \in \bigcap_{n,\rho}(\mathcal{S})_{\rho,\mathfrak{q}^n}(H)$ and $f \in \bigcap_{n,\rho}(\mathcal{S})_{\rho,\mathfrak{q}^n}\big(\mathcal{V}'(T)\big)$ for every sequence $\mathfrak{q}$. Nonetheless, it was shown in [10] that, in general, the solution of (6.2) with $N=1$ will not be an element of any space smaller than $(\mathcal{S})_{-1,\mathfrak{q}^n}\big(\mathcal{V}(T)\big)$ for some sequence $\mathfrak{q}$ with $0<q_k<1$. Moreover, conditions of the type (6.11) are in general necessary as well ([12, Proposition 4.8]).

The attractive special feature of equation (6.2) with deterministic input is that the solution admits a representation in multiple Skorohod integrals of deterministic kernels:

Theorem 6.11. *Consider equation* (6.2) *and assume that $u^{\circ} \in H$ and $f \in \mathcal{V}'(T)$ are non-random. Assume that the operators $\mathbf{A}$ and $\mathbf{M}$ satisfy conditions (AE) and (ME), respectively, and let $\mathfrak{q}$ be a sequence satisfying* (6.11). *Define the sequence $U_n = U_n(t)$, $n \geq 0$, $t \in [0,T]$, by*

$$U_0(t) = u_{(0)}(t) = \Phi_{t,0}u^{\circ} + \int_0^t \Phi_{t,s}f(s)ds,$$

$$U_{n+1}(t) = \int_0^t \Phi_{t,s}\boldsymbol{\delta}_N\big({}_{\mathfrak{q}}\mathbf{M}(s)U_n(s)\big)ds,\ n>0,$$

where $\big({}_{\mathfrak{q}}\mathbf{M}\big)_{\boldsymbol{\beta}} = \mathfrak{q}^{\boldsymbol{\beta}}\mathbf{M}_{\boldsymbol{\beta}}$. Then

$$\sum_{|\boldsymbol{\alpha}|=n} \mathfrak{q}^{\boldsymbol{\alpha}}u_{\boldsymbol{\alpha}}(t)\,\mathrm{H}_{\boldsymbol{\alpha}} = U_n(t).$$

Proof. By (6.7) with $u^{\circ}_{\boldsymbol{\alpha}}=0$ and $f_{\boldsymbol{\alpha}}=0$ for $|\boldsymbol{\alpha}|>0$,

$$\mathfrak{q}^{\boldsymbol{\alpha}}u_{\boldsymbol{\alpha}}(t) = \int_0^t \mathbf{A}\big(\mathfrak{q}^{\boldsymbol{\alpha}}u_{\boldsymbol{\alpha}}(s)\big)ds + \sum_{0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\int_0^t \mathfrak{q}^{\boldsymbol{\beta}}\mathbf{M}_{\boldsymbol{\beta}}\big(\mathfrak{q}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(s)\big)ds.$$

Therefore, for $n > 0$,

$$\big(U_n(t)\big)_{\boldsymbol{\alpha}} = \sum_{0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \int_0^t \Phi_{t-s}\big({}_{\mathfrak{q}}\mathbf{M}\big)_{\boldsymbol{\beta}}\big(U_{n-1}(s)\big)_{\boldsymbol{\alpha}-\boldsymbol{\beta}}ds.$$

By (5.5),

$$U_n(t) = \int_0^t \Phi_{t,s}\boldsymbol{\delta}_{\dot{Z}}\big({}_{\mathfrak{q}}\mathbf{M}(s)U_{n-1}(s)\big)ds,$$

and the result follows. □

Theorem 6.11 is a further extension of a formula of Krylov and Veretennikov (see [12, Corolloary 4.10] for the case $N = 1$ and [8] for the original result).

As a first application of Theorem 6.8, consider the equation

$$u_t(t,x) = \big(a(x) \diamond u_x(t,x)\big)_x + f(t,x),\ t > 0,\ x \in \mathbb{R}, \tag{6.15}$$

where $a(x) = e^{\diamond \dot{W}(x)}$ The function $a = a(x)$ is known as the positive noise process and has a representation $a(x) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{e}^{\boldsymbol{\alpha}}(x)\mathrm{H}_{\boldsymbol{\alpha}}/\boldsymbol{\alpha}!$, where $\{\mathfrak{e}_k(x),\ k \geq 1\}$ are the Hermite functions; see [4, Section 2.6] for details. Equation (6.15) can model diffusion in a random media with very irregular diffusion coefficient.

By Theorem 3.10, equation (6.15) is a particular case of (6.2), with $\mathbf{A}v = v_{xx}$, $\mathbf{M}_{\boldsymbol{\alpha}}v = \big(\mathfrak{e}^{\boldsymbol{\alpha}}v_x\big)_x/\boldsymbol{\alpha}!$. Properties of the Hermite functions imply that $\sup_x|\mathfrak{e}_k(x)| \leq C_e$, $\sup_x|\mathfrak{e}'_k(x)| \leq C_e\,k$ for some constant C_e independent of k, and therefore condition (ME) can be satisfied by taking $b_k = Ck$, $C = \max(2, C_e)$. Then Theorem 6.8 gives a Hilbert-space version of the result of Gjerde (see [4, Theorem 4.7.4]). Theorem 6.8 also makes it possible to consider more general diffusion coefficients $a = a(x)$ of the form $a(x) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} a_{\boldsymbol{\alpha}}(x)\mathrm{H}_{\boldsymbol{\alpha}}$ where $a_{(\mathbf{0})}(x)$ is a strictly positive bounded measurable function, and each $a_{\boldsymbol{\alpha}}(x)$, $|\boldsymbol{\alpha}| > 0$, is a continuously differentiable function such that $\sup_x|a_{\boldsymbol{\alpha}}(x)| + \sup_x|a'_{\boldsymbol{\alpha}}(x)| \leq \mathfrak{b}^{\boldsymbol{\alpha}}$ for some positive sequence $\mathfrak{b}$. Extensions to $x \in \mathbb{R}^d$ and a matrix-valued function $a = a(x)$ are straightforward.

As a motivation of another application of Theorem 6.8, let us consider the following three equations:

$$du = u_{xx}dt + \sigma\, u\, dw(t), \tag{6.16}$$

$$du = u_{xx}dt + \sigma\, u_x\, dw(t), \tag{6.17}$$

$$du = u_{xx}dt + \sigma\, u_{xx}\, dw(t). \tag{6.18}$$

In all three equations, w is a standard Brownian motion, σ is a positive real number, $x \in \mathbb{R}$, and, for simplicity, $u(0,x) = e^{-x^2/2}$. Note that all three equations are solvable in closed form using the Fourier transform. In particular, it is known that

(1) for equation (6.16), $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})} < \infty$ for all $t \geq 0$ and all $\sigma > 0$;
(2) for equation (6.17), if $\sigma^2 \leq 2$, then $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})} < \infty$ for all $t \geq 0$; if $\sigma^2 > 2$, then $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})} < \infty$ for $t \leq 1/(\sigma^2 - 2)$ and, for $t > 1/(\sigma^2 - 2)$, $u(t) \in (\mathcal{S})_{0,\mathfrak{q}}(L_2(\mathbb{R}))$ with $q_k^2 = 2/\sigma^2$ for all k, (see [9, Theorem 4.1]);

(3) for equation (6.18), $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})} = \infty$ for all $t > 0$ and $\sigma > 0$, and $u(t) \in (\mathcal{S})_{-1,\mathfrak{q}}(L_2(\mathbb{R}))$, for a suitable sequence $\mathfrak{q}$ (see [10, Section 2]).

All these results are consistent with the conclusions of Theorem 6.8.

To proceed, recall that the standard Brownian motion on $[0, T]$ has a representation

$$w(t) = \sum_{k\geq 1} M_k(t)\, \xi_k, \tag{6.19}$$

where $\xi_k,\ k \geq 1$ are iid standard Gaussian random variables, $M_k(t) = \int_0^t m_k(s)ds$, and $m_k,\ k \geq 1$, are the elements of an orthonormal basis in $L_2((0, T))$.

We now modify (6.19) as follows. For an integer $n \geq 1$, define the process $w^{[n]} = w^{[n]}(t),\ t \in [0, T]$ by

$$w^{[n]} = \sum_{k\geq 1} M_k(t)\, \mathrm{H}_n(\xi_k).$$

Clearly, $w^{[1]} = w$. From Parseval's identity

$$\sum_{k\geq 1} M_k^2(t) = \sum_{k\geq 1} \left(\int_0^t m_k(s)ds \right)^2 = \int_0^t ds = t \tag{6.20}$$

we conclude that $w^{[n]}$ is well-defined for all $n \geq 1$ and $t > 0$, with $\mathbb{E}w^{[n]}(t) = 0$ and $\mathbb{E}\Big(w^{[n]}(t)\Big)^2 = (n!)\, t$. Figure 1 presents sample trajectories of $w^{[n]}$ for $n = 1, 2, 5$.

Detailed analysis of the process $w^{[n]}$, while potentially an interesting problem, is beyond the scope of this paper.

Let us replace w with $w^{[n]}$ in equations (6.16)–(6.18):

$$du = u_{xx}dt + \sigma\, u dw^{[n]}(t), \tag{6.21}$$

$$du = u_{xx}dt + \sigma\, u_x dw^{[n]}(t), \tag{6.22}$$

$$du = u_{xx}dt + \sigma\, u_{xx} dw^{[n]}(t), \tag{6.23}$$

and investigate how the properties of the solution change with n. All three equations can be written as

$$du = u_{xx}dt + \sigma\, \partial^p u\, dw^{[n]}(t),\ 0 < t \leq T, \quad u(0, x) = e^{-x^2/2}, \tag{6.24}$$

where $\partial^0 = 1$, $\partial^p = \partial^p/\partial x^p$, $p = 1, 2$.

To define the stochastic integral with respect to $w^{[n]}$ we use the divergence operator $\boldsymbol{\delta}_{\dot{Z}_n}$ and the setting of Theorem 6.8 with $\mathcal{U} = L_2((0, T))$, $V = H^1(\mathbb{R})$, $H = L_2(\mathbb{R})$, $V' = H^{-1}(\mathbb{R})$, $\mathbf{A} = \partial^2/\partial x^2$,

$$\mathbf{M}_{\boldsymbol{\beta}}(t) = \begin{cases} \sigma\, m_k(t)\partial^p & \text{if } \boldsymbol{\beta} = n\boldsymbol{\varepsilon}(k), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 6.8, there is a unique solution of (6.24) for every $p = 0, 1, 2$, $\sigma > 0$ and $T > 0$, and, with a suitable sequence $\mathfrak{q}$, $\|u(t,\cdot)\|^2_{-1,\mathfrak{q},L_2(\mathbb{R})} < \infty$ for all $t > 0$. In what

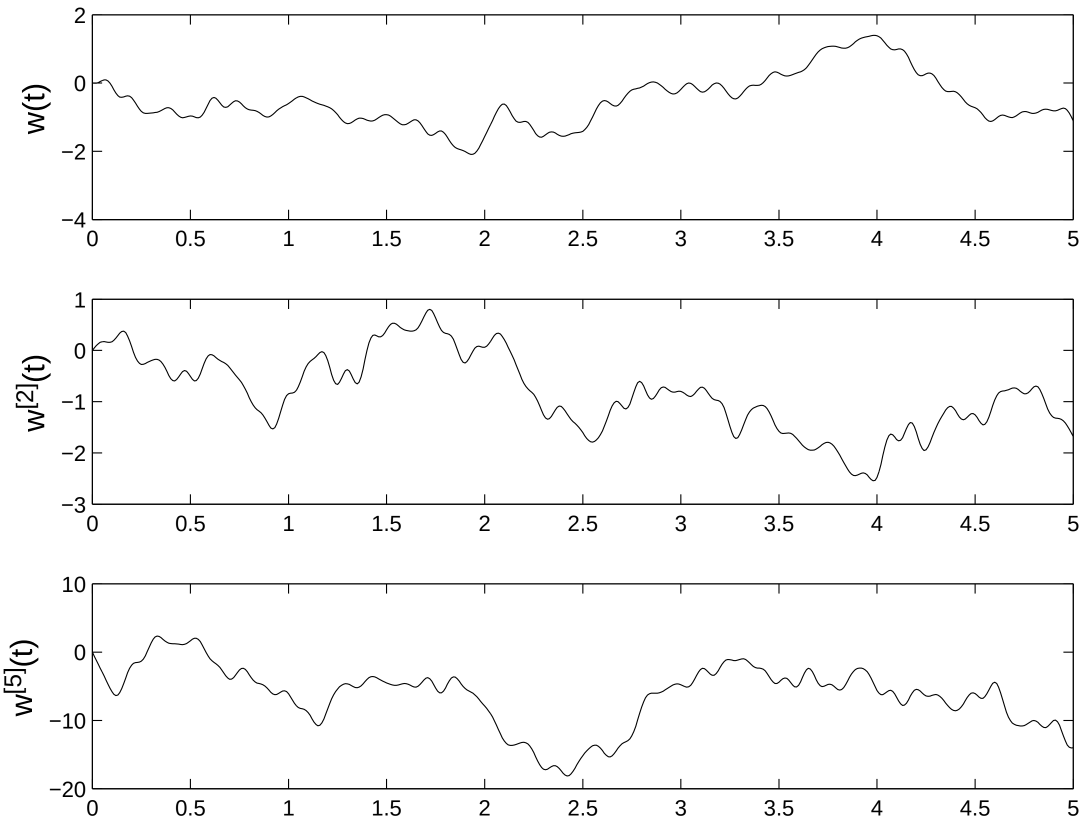

FIGURE 1. Sample trajectories of the process $w^{[n]}$

follows, we show that this result can be improved and the solution of (6.24) has the following properties:

- $u(t,\cdot) \in (\mathcal{S})_{-1+(2-p)/n,\mathfrak{q}}\big(L_2(\mathbb{R})\big)$ for a suitable sequence $\mathfrak{q}$;
- if $p=0$, then $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})} < \infty$ for $t < 1/(4\sigma^2)$.

If $U = U(t,y)$ is the Fourier transform of u, then

$$dU = -y^2 U dt + \sigma\,(iy)^p\, U\, dw^{[n]}(t),\ U(0,y) = e^{-y^2/2}. \tag{6.25}$$

As a result, to solve (6.25), we first need to solve

$$X(t) = 1 + \int_0^t b\, X(s) dw^{[n]}(s),\ b \in \mathbb{C}.$$

By Theorem 6.5, $X(t) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} X_{\boldsymbol{\alpha}}(t)\mathrm{H}_{\boldsymbol{\alpha}}$, where

$$X_{(\mathbf{0})}(t) = 1,\ X_{\boldsymbol{\alpha}}(t) = \sum_k \mathbf{1}_{\alpha_k \ge n} \int_0^s b\, M_k(s) X_{\boldsymbol{\alpha}-n\boldsymbol{\varepsilon}(k)}(s) ds. \tag{6.26}$$

With the notation $M^{\boldsymbol{\alpha}}(t) = \prod_k M_k^{\alpha_k}(t)$, the solution of the system (6.26) is $X_{n\boldsymbol{\alpha}} = b^{|\boldsymbol{\alpha}|} M^{\boldsymbol{\alpha}}(t)/\boldsymbol{\alpha}!$, $X_{\boldsymbol{\beta}} = 0$ otherwise. This can either be verified by direct computation or derived by noticing that the system of equation satisfied by the functions $x_{\boldsymbol{\alpha}} = X_{n\boldsymbol{\alpha}}$

is the same as the propagator for the geometric Brownian motion. As a result,

$$X(t)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{b^{|\boldsymbol{\alpha}|}M^{\boldsymbol{\alpha}}(t)}{\boldsymbol{\alpha}!}\,\mathrm{H}_{n\boldsymbol{\alpha}}.$$

Then $U(t,y)=e^{-y^2/2}e^{-y^2t}\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{\sigma^{|\boldsymbol{\alpha}|}(iy)^{p|\boldsymbol{\alpha}|}M^{\boldsymbol{\alpha}}(t)}{\boldsymbol{\alpha}!}\,\mathrm{H}_{n\boldsymbol{\alpha}}$. By the Fourier isometry,

$$\begin{aligned}\|u(t,\cdot)\|^2_{-\rho,\mathfrak{q}^r,L_2(\mathbb{R})}&=\|U(t,\cdot)\|^2_{-\rho,\mathfrak{q}^r,L_2(\mathbb{R})}\\&=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\left(\int_{\mathbb{R}}e^{-y^2(1+2t)/2}y^{2p|\boldsymbol{\alpha}|}dy\right)\sigma^{2|\boldsymbol{\alpha}|}M^{2\boldsymbol{\alpha}}(t)\frac{\mathfrak{q}^{2rn\boldsymbol{\alpha}}(n\boldsymbol{\alpha})!}{(\boldsymbol{\alpha}!)^2((n\boldsymbol{\alpha})!)^{\rho}}.\end{aligned}$$

Since

$$\int_{\mathbb{R}}e^{-y^2(1+2t)/2}|y|^Ndy=\frac{2^{(N+1)/2}\,\Gamma\left(\frac{N+1}{2}\right)}{(1+2t)^{(N+1)/2}},$$

we find

$$\|u(t,\cdot)\|^2_{-\rho,\mathfrak{q}^r,L_2(\mathbb{R})}=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{\sigma^{2|\boldsymbol{\alpha}|}\,2^{(2p|\boldsymbol{\alpha}|+1)/2}\,\Gamma\left(\frac{2p|\boldsymbol{\alpha}|+1}{2}\right)}{(1+2t)^{(2p|\boldsymbol{\alpha}|+1)/2}}M^{2\boldsymbol{\alpha}}(t)\frac{\mathfrak{q}^{2rn\boldsymbol{\alpha}}(n\boldsymbol{\alpha})!}{(\boldsymbol{\alpha}!)^2((n\boldsymbol{\alpha})!)^{\rho}}.$$

Equation (6.21) ($p=0$)

If $n=2$, then

$$\|u(t,\cdot)\|^2_{-\rho,\mathfrak{q},L_2(\mathbb{R})}=\sqrt{\frac{2\pi}{1+2t}}\sum_{\boldsymbol{\alpha}}\binom{2\boldsymbol{\alpha}}{\boldsymbol{\alpha}}\sigma^{2|\boldsymbol{\alpha}|}M^{2\boldsymbol{\alpha}}(t)\frac{\mathfrak{q}^{4\boldsymbol{\alpha}}}{((2\boldsymbol{\alpha})!)^{\rho}}. \tag{6.27}$$

It follows that $u(t,\cdot)\in(\mathcal{S})_{0,\mathfrak{q}}(H)$, $0<t<T$, for every sequence $\mathfrak{q}$ with

$$q_k<1/(2\sigma\sqrt{T}). \tag{6.28}$$

In particular, $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})}<\infty$ if $t<1/(4\sigma^2)$. Indeed, using (3.21) and $\Gamma(1/2)=\sqrt{\pi}$,

$$\|u(t,\cdot)\|^2_{-\rho,\mathfrak{q},L_2(\mathbb{R})}\le\sqrt{2\pi}\sqrt{\frac{2\pi}{1+2t}}\sum_{\boldsymbol{\alpha}}(2\sigma)^{2|\boldsymbol{\alpha}|}M^{2\boldsymbol{\alpha}}(t)\mathfrak{q}^{4\boldsymbol{\alpha}}$$

and then (6.20) and (3.6) imply that the right-hand side of the last equality is finite if (6.28) holds. This argument also leads to an upper bound on the blow-up time t^* of $\mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbb{R})}$: since $(2\boldsymbol{\alpha})!\ge(\boldsymbol{\alpha}!)^2$, we have $t^*\le 1/\sigma^2$.

If $n>2$, then factorial terms dominate the right-hand side of (6.27). As a result, we cannot take $\rho=0$, and it becomes more complicated (and less important) to find the best possible sequence $\mathfrak{q}$. Accordingly, we assume that $\mathfrak{q}$ is such that $q_k>0$ and $\sum_k q_k$ is sufficiently small, and will concentrate on finding the best ρ.

With Stirling's formula not easily applicable to $\boldsymbol{\alpha}!$, we will use the inequalities

$$\boldsymbol{\alpha}!\le|\boldsymbol{\alpha}|!\le\frac{\boldsymbol{\alpha}!}{\mathfrak{q}^{\boldsymbol{\alpha}}},$$

where the first one is obvious and the second follows from the multinomial formula

$$\left(\sum_k q_k\right)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!}\,\mathfrak{q}^{\boldsymbol{\alpha}}.$$

Then

$$\frac{(n\boldsymbol{\alpha})!}{(\boldsymbol{\alpha}!)^2((n\boldsymbol{\alpha})!)^{\rho}} \leq \frac{(n|\boldsymbol{\alpha}|)!}{(|\boldsymbol{\alpha}|!)^2((n|\boldsymbol{\alpha}|)!)^{\rho}}\,\mathfrak{q}^{-(n\rho+2)}. \tag{6.29}$$

A very rough version of the Stirling formula $N! \sim N^N$ shows that the term $|\boldsymbol{\alpha}|^{c|\boldsymbol{\alpha}|}$ will have $c=0$ if $n-2-n\rho=0$ or if $\rho = 1-(2/n)$. By (3.6) convergence of the remaining geometric series can be achieve by choosing q_k sufficiently small. As a result, $u(t,\cdot) \in (\mathcal{S})_{-1+(2/n),\mathfrak{q}^r}(L_2(\mathbb{R}))$ if $r \geq n+4$.

Equation (6.22) ($p=1$)

Since the function $\Gamma(x)$ is increasing for $x \geq 2$,

$$(|\boldsymbol{\alpha}|-1)! \leq \Gamma\left(\frac{2|\boldsymbol{\alpha}|+1}{2}\right) \leq |\boldsymbol{\alpha}|! \text{ when } |\boldsymbol{\alpha}| \geq 3.$$

Using (6.29), we conclude that $u(t,\cdot) \in (\mathcal{S})_{-1+(1/n),\mathfrak{q}^r}(L_2(\mathbb{R}))$ if $r \geq n+4$ and q_k are sufficiently small.

Equation (6.23) ($p=2$)

Since the function $\Gamma(x)$ is increasing for $x \geq 2$,

$$(2|\boldsymbol{\alpha}|-1)! \leq \Gamma\left(\frac{4|\boldsymbol{\alpha}|+1}{2}\right) \leq |2\boldsymbol{\alpha}|! \text{ when } |\boldsymbol{\alpha}| \geq 2.$$

Using (6.29), we conclude that $u(t,\cdot) \in (\mathcal{S})_{-1,\mathfrak{q}^r}(L_2(\mathbb{R}))$ if $r \geq n+4$ and q_k are sufficiently small.

6.2. **Stationary equations.** Let V and V' be two Hilbert spaces such that V is densely and continuously embedded into V'. Let $\mathbf{A} : V \to V'$ and $\mathbf{M}(t) : V \to V' \otimes Y$ be bounded linear operators. Similar to the time-dependent case, we define the operators $\mathbf{M}_{\boldsymbol{\beta}} : V \to V'$, $|\boldsymbol{\beta}|>0$, by $\mathbf{M}_{\boldsymbol{\beta}} v = \big(\mathbf{M}(t)v, \mathfrak{u}^{\otimes\boldsymbol{\beta}}\big)_Y,\ |\boldsymbol{\beta}|>0$.

The objective of this section is to study the equation

$$\mathbf{A}u = \boldsymbol{\delta}_{\dot{Z}_\infty}(\mathbf{M}u) + f,\ \ f \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(V'). \tag{6.30}$$

Note that, similar to the evolution case, Remark 6.1 applies to equation (6.30).

Even though equation (6.30) was investigated in [13] in the particular case of deterministic f, the proofs in [13] are not easily extendable to more general f. On the other hand, the tools developed in this paper work equally well for evolutionary and stationary equations, with either deterministic or random input. In particular, all theorems and proofs for stationary equations are identical to the corresponding theorems and proofs for evolution equations. Below, we outline the main ideas and leave the details to an interested reader.

To ensure that $\mathbf{M}\eta \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}\big(V' \otimes Y\big)$ for every $\eta \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(V')$, we impose the following condition.

Condition (MS): *There exists a sequence $\mathfrak{b} = (b_1, b_2, \ldots)$ of positive numbers such that, for every $v \in V$ and every multi-index $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| > 0$,*

$$\|\mathbf{M}_{\boldsymbol{\alpha}} v\|_{V'} \leq \mathfrak{b}^{\boldsymbol{\alpha}} \|v\|_V. \tag{6.31}$$

The following is the stationary version of Definition 6.2.

Definition 6.12. *The solution of equation* (6.30) *is an element of* $\bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(V)$ *with the following properties:*

(1) *There exists a region $\mathbb{K}(\mathfrak{c})$ inside which the function $\widetilde{u}(\mathfrak{z}) = \langle\!\langle u, \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle$ is defined and analytic with values in V;*
(2) *For every $\mathfrak{z} \in \mathbb{K}(\mathfrak{c})$, the following equality holds in $\mathcal{V}'(T)$:*

$$\langle\!\langle \mathbf{A}u, \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle = \langle\!\langle \boldsymbol{\delta}_{\dot{Z}_\infty}\big(\mathbf{M}(s)u\big), \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle + \langle\!\langle f, \mathcal{E}_{\mathfrak{z}} \rangle\!\rangle. \tag{6.32}$$

Note that Theorem 5.7(a) and assumption $u \in \bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(V)$ ensure existence of a region $\mathbb{K}(\mathfrak{c})$ inside which the function $\widetilde{u}$ is analytic.

The following theorem gives a characterization of the solution in terms of the chaos expansion.

Theorem 6.13. *A V'-valued generalized process $u = \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$ is a solution of* (6.2) *if and only if the collection $\{u_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{J}\}$ is a solution of the system*

$$\begin{aligned} \mathbf{A}u_{(\mathbf{0})} &= f_{(\mathbf{0})},\ |\boldsymbol{\alpha}| = 0, \\ \mathbf{A}u_{\boldsymbol{\alpha}} &= \sum_{0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \mathbf{M}_{\boldsymbol{\beta}} u_{\boldsymbol{\alpha}-\boldsymbol{\beta}} + f_{\boldsymbol{\alpha}},\ |\boldsymbol{\alpha}| > 0. \end{aligned} \tag{6.33}$$

The proof is identical to the proof of Theorem 6.5.

To prove existence and uniqueness of solution of (6.2), we impose the following condition on the operator $\mathbf{A}$.

Condition (AS) *The operator $\mathbf{A}$ has a bounded inverse $\mathbf{A}^{-1} : V' \to V$.*

We denote by C_A the operator norm of $\mathbf{A}^{-1}$.

The following is the main result about existence, uniqueness, and regularity of the solution of (6.2).

Theorem 6.14. *Assume that conditions (AS) and (MS) hold, $f \in (\mathcal{S})_{-1,\mathfrak{q}}(V')$, and assume that the sequence $\mathfrak{q} = (q_1, q_2, \ldots)$ has the following properties:*

$$0 < q_k < 1,\ \sum_{k\geq1} q_k < 1,\ C_0 := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{b}^{\boldsymbol{\alpha}} \mathfrak{q}^{\boldsymbol{\alpha}} < \frac{1}{C_A}, \tag{6.34}$$

Define the number $C_1 = \prod_{k\geq1}(1 - q_k^2)^{-1/2}$.

Then equation (6.30) *has a unique solution in* $\bigcup_n (\mathcal{S})_{-1,\mathfrak{q}^n}(V)$. *The solution is an element of* $(\mathcal{S})_{-1,\mathfrak{q}^4}(V)$ *and*

$$\|u\|_{-1,\mathfrak{q}^4,V} \leq \frac{C_A C_1^2}{1-C_0 C_A}\|f\|_{-1,\mathfrak{q},V'} \tag{6.35}$$

Proof. The steps are the same as in the proof of Theorem 6.14. First, we use (6.34) to show that, for $\mathfrak{z} \in \mathbb{K}(\mathfrak{q}^2)$, the function $\widetilde{u}\big((z)\big) = \langle\!\langle u, \mathcal{E}_{\mathfrak{z}}\rangle\!\rangle$ is the unique solution of $\mathbf{A}\widetilde{u} = \sum_{\boldsymbol{\beta}>0} \mathfrak{z}^{\boldsymbol{\beta}} \mathbf{M}_{\boldsymbol{\beta}}\widetilde{u} + \widetilde{f}$ and satisfies $\|\widetilde{u}\|_V \leq C_A\|\widetilde{f}\|_{V'}/(1-C_0C_A)$. Then (6.35) follows from Theorem 5.7(b). □

Remark 6.15. The solution of equation $u = 1+\boldsymbol{\delta}_{\xi}(u)$, where ξ is a standard Gaussian random variable, is $u = \sum_{k\geq 0} \mathrm{H}_k(\xi)$. This example shows that the conclusion of Theorem 6.14 is sharp in the following sense: if $\rho > -1$, then, in general, one cannot find a sequence $\mathfrak{c}$ to ensure that the solution of (6.2) belongs to $(\mathcal{S})_{\rho,\mathfrak{c}}(\mathcal{V}(T))$. For more examples of this kind, see [13].

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