WICK PRODUCT IN THE STOCHASTIC BURGERS EQUATION: A CURSE OR A CURE?*

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Abstract. It has been known for a while that a nonlinear equation driven by singular noise must be interpreted in the renormalized, or Wick, form. For the stochastic Burgers equation, Wick nonlinearity forces the solution to be a generalized process no matter how regular the random perturbation is, whence the curse. On the other hand, certain multiplicative random perturbations of the deterministic Burgers equation can only be interpreted in the Wick form, whence the cure. The analysis is based on the study of the coefficients of the chaos expansion of the solution at different stochastic scales.

1. Introduction

Burgers equation,

$$u_t + uu_x = u_{xx},\ t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

first suggested as a simplified model of turbulence (Bateman [1], Burgers [2, 3]) is now used to study problems such as traffic flows (Chowdhury et al. [4]) and mass distribution of the large scale structure of the universe (Molchanov et al. [5]). The equation also appears in the study of interacting particle systems (Sznitman [6]).

Considering random initial condition and/or driving force in equation (1.1) is one way to model and investigate turbulence. The idea goes back to Burgers himself and the result is known as the Burgers turbulence. A popular option for the driving force is additive space-time white noise $\dot{W}(t,x)$. Mathematical theory of the resulting equation,

$$u_t + uu_x = u_{xx} + \dot{W}(t,x) \tag{1.2}$$

has been developed (Bertini et al. [7]). A more general version of (1.2) with multiplicative noise,

$$u_t + uu_x = u_{xx} + f(t,x) + \big(g(u(t,x))\big)_x + h(u(t,x))\dot{W}(t,x), \tag{1.3}$$

has also been studied (Gyöngy and Nualart [8]). It turns out that (1.2) cannot be generalized much further while staying within the same mathematical framework, when the solution of the equation is a square-integrable random variable with sufficiently regular sample paths. In particular, equation

$$u_t + uu_x = u_{xx} + u_x\dot{W}(t,x) \tag{1.4}$$

makes no sense, because the solution, if it existed, cannot be regular enough to define the pointwise multiplications uu_x and $u_x\dot{W}$.

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Using the Wick product $\diamond$ instead of the usual pointwise multiplication makes it possible to study (1.4) and similar equations in the framework of white noise theory. The operation $\diamond$ was first introduced in quantum field theory (Wick [9]). In stochastic analysis, the operation is essentially a convolution (Hida and Ikeda [10]), and is closely connected with the Itô and Skorokhod integrals (Holden et al. [11, Chapter 2]). The Wick version of (1.4),

$$u_t + u \diamond u_x = u_{xx} + u_x \diamond \dot{W}(t,x),\ x \in \mathbb{R}, \tag{1.5}$$

has been studied (Benth et al. [12, Section 5.2, Example 2]), and is known to have a solution in the space of generalized random processes. The tools of white noise analysis and availability of generalized random processes make it possible to consider even more singular noise than $\dot{W}$, both additive (Holden et al. [13, 14]) and multiplicative (Benth et al. [12]).

Our objective in this paper is to study the equation

$$\begin{aligned} u_t(t,x) + u(t,x) \diamond u_x(t,x) &= u_{xx}(t,x) + f(t,x) \\ &+ \sum_{k\geq 1} \big(a_k(t,x)u_{xx} + b_k(t,x)u_x + c_k(t,x)u + g_k(t,x)\big) \diamond \xi_k, \end{aligned} \tag{1.6}$$

$0 < t \leq T,\ u(0,x) = \varphi(x)$, where $\{\xi_k,\ k \geq 1\}$ are independent identically distributed (iid) standard normal random variables, $T < \infty$ is non-random, the coefficients a_k, b_k, c_k and the free terms f, g_k are non-random, and $x \in G$, with

- $G = \mathbb{R}$ (whole line) or
- $G = S^1$ (the circle, which corresponds to periodic boundary conditions).

Equation (1.6) includes (1.5) as a particular case because the space-time white noise $\dot{W}(t,x)$ can be written as

$$\dot{W}(t,x) = \sum_{k\geq 1} h_k(t,x)\xi_k, \tag{1.7}$$

where $\{h_k,\ k \geq 1\}$ is an orthonormal basis in $L_2((0,T)\times\mathbb{R})$ (see, for example, Holden et al. [11, Definition 2.3.9]). From the physical point of view, (1.6) is a natural random perturbation of the original equation (1.1). Indeed, one possible interpretation of (1.1) is the motion of a one-dimensional fluid, and then the basic laws of fluid dynamics suggest a more general version of (1.1):

$$u_t + uu_x = \frac{1}{\rho}(\mu u_x)_x + \frac{1}{\rho}F(t,x), \tag{1.8}$$

where ρ is the density of the fluid, μ is the (dynamic) viscosity, and F is the external force (see, for example, Gallavotti [15, Section 1.2.2]). Thus, (1.1) is (1.8) with constant μ and ρ. If μ is not known, then (assuming ρ is still constant) a possible approach is to consider

$$\mu(t,x) = \mu_0 + \varepsilon\dot{W}(t,x),\ \mu_0 = const; \tag{1.9}$$

given the time and space scales of the model, one can choose $\varepsilon > 0$ small enough to have the right-hand side of (1.9) positive with probability arbitrarily close to one. Substituting (1.9) into (1.8) and using (1.7) (and interpreting all multiplications in the Wick sense) leads to a particular case of (1.6) with $a_k = \mu_0 + h_k$, $b_k = \partial h_k/\partial x$,

$f = c_k = g_k = 0$. Note that we must interpret multiplications in the Wick sense: no matter how small the ε is in (1.9), the equation $u_t = (1+\varepsilon\dot{W}(t,x))u_{xx}$ is ill-posed path-wise. An extra benefit of this interpretation is preservation of the mean dynamic, something we would never get with the usual Burgers equation. Indeed, a remarkable property of the Wick product is that the (generalized) expectation of the product is the product of expectations: $\mathbb{E}(u \diamond v) = (\mathbb{E}u)(\mathbb{E}v)$. As a result, the (generalized) expected value $U = \mathbb{E}u$ of the solution of (1.6) satisfies the usual Burgers equation

$$U_t + UU_x = U_{xx} + f,\ U|_{t=0} = \mathbb{E}\varphi.$$

The main results of the paper are as follows:

(1) If the functions a_k, b_k, c_k, f, g_k are non-random, bounded and measurable and the initial condition φ is a generalized random field on G, then (under some addition technical conditions on f and φ) there exists a unique generalized process solution of (1.6);
(2) The Wick product $u \diamond u_x$ forces the solution of (1.6) to be a generalized process even when the random perturbation is very well-behaved (such as $a_k = b_k = c_k = g_k = 0$ for all $k \geq 1$, and $\varphi(x) = \xi_1\, h(x)$ for a smooth compactly supported h).

Our approach to proving existence and uniqueness of solution is to derive the chaos expansion of the solution of (1.6) and to establish explicit bounds on the coefficient of the expansion. The proof has a combinatorial component due to the appearance of the Catalan numbers. Mikulevicius and Rozovskii [16] use the same approach to study the Wick-stochastic version of the Navier-Stokes equations.

The overall conclusion, which also explains the title of the paper, is that, while solutions of any stochastic Burgers equation in the Wick form are generalized random processes (even when the use of the point-wise multiplication leads to a classical solution — whence the curse), certain random perturbations can only be considered for the Wick form of the equation (because point-wise multiplication cannot be defined — whence the cure). In our opinion, the cures (the ability to consider more general stochastic perturbations, preservation of the mean dynamics, and an easy access to the chaos coefficients of the solution) outweigh the curses (necessity to work with generalized random elements and questions about physical interpretation of the model).

Although there are some similarities between our work and that of Benth et al. [12], there are also several important differences:

(1) The random perturbations in (1.6) and in [12] are different and there is no obvious way to reduce one to the other;
(2) The solution of (1.6) is global in time (can be constructed for all $T > 0$);
(3) The solution of (1.6) is constructed on an arbitrary probability space, not just on the white noise space;
(4) Our analysis of the chaos expansion of the solution leads to more detailed information about the solution space and can be used to compute the solution numerically.

In Section 2 we outline the main constructions in the theory of generalized processes. In Section 3 we define the chaos solution for equation (1.6) and state the main result about existence and uniqueness of solution (Theorem 3.3). We also show (Theorem 3.2) that the Wick product indeed forces the solution to be a generalized process. In Section 4 we derive the propagator for equation (1.6) (the system of deterministic partial differential equations describing the propagation of random perturbation in the equation through different stochastic scales) and outline the proof of Theorem 3.3. We also discuss briefly the general technical issues related to the study of (1.6) and similar equations. The details of the proof of Theorem 3.3 are in Section 5.

Throughout the paper, we fix a probability space $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ and a countable collection of independent and identically distributed (iid) standard Gaussian random variables $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots)$ on $\mathbb{F}$. We also assume that $\mathcal{F}$ is generated by $\boldsymbol{\xi}$. By $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of real and complex numbers, respectively; $\mathcal{C}$ and $\mathcal{C}^1$ denote the spaces of bounded continuous and bounded continuously differentiable functions (with the derivative also bounded).

2. Gaussian polynomial chaos and the Wick product

We start with a review of some constructions from the white noise theory.

Definition 2.1. *A generalized Gaussian chaos space is collection of the following four objects* $(\mathbb{F}, \boldsymbol{\xi}, H, \mathrm{Q})$*:*

- *A probability space* $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$*;*
- *a collection of iid standard Gaussian random variables* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots)$ *such that* $\mathcal{F}$ *is generated by* $\boldsymbol{\xi}$*;*
- *A separable Hilbert space* H*;*
- *An unbounded, self-adjoint positive-definite operator* Q *on* H *such that* Q *has a pure point spectrum:*

$$\mathrm{Q}\mathfrak{h}_k = q_k \mathfrak{h}_k,\ k \geq 1,$$

the eigenfunctions $\mathfrak{h}_k$ *of* Q *form an orthonormal basis in* H*, and the eigenvalues* q_k *of* Q *satisfy*

$$\sum_{k\geq 1} \frac{1}{q_k^{\gamma}} < \infty \tag{2.1}$$

for some $\gamma > 0$*.*

Condition (2.1) ensures that the projective limit of the domains of Q^n, $n \geq 1$, is a nuclear space.[1] To simplify some of the future computations, we will assume that

$$1 < q_1 \leq q_2 \leq \ldots \quad \text{and} \quad \sum_k \frac{1}{q_k} < 1. \tag{2.2}$$

There is no loss of generality involved, as we can always replace the original operator Q with $c\mathrm{Q}^{\gamma}$, with c sufficiently large. By $\mathfrak{q}$ we denote the sequence $(q_1, q_2, \ldots)$ of the eigenvalues of Q.

[1]In the white noise approach, the dual of that nuclear space with the Borel sigma-algebra and a Gaussian measure is the probability space. We are able to use an arbitrary probability space without any topological structure. For us, the numbers q_k are just weights to ensure convergence of certain infinite sums.

Example 2.2. (1) The traditional white noise constructions correspond to $H = L_2(\mathbb{R}^d)$ and $\mathrm{Q} = -\boldsymbol{\Delta} + |x|^2 + 1$. If $d = 1$, then $q_k = 2k$. (2) Stochastic partial differential equations in $\mathbb{R}^d$ correspond to $H = L_2((0,T)\times\mathbb{R}^d)$, $\mathrm{Q} = -\partial^2/\partial t^2 - \boldsymbol{\Delta} + |x|^2 + 1$, with periodic boundary conditions in time. (3) Stochastic partial differential equations in domains or on a manifold correspond to $H = L_2((0,T)\times G)$, where G is a smooth bounded domain or a smooth compact manifold in $\mathbb{R}^d$, and $\mathrm{Q} = -\partial^2/\partial t^2 - \boldsymbol{\Delta}$, with periodic boundary conditions in time and appropriate boundary conditions for the Laplace operator $\boldsymbol{\Delta}$ on G.

Next, we review some of the notations related to multi-indices.

A *multi-index* $\boldsymbol{\alpha}$ is a sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers, such that only finitely many of α_k are different from 0. The collection of all multi-indices is denoted by $\mathcal{J}$. By definition,

- $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ if $\beta_k \le \alpha_k$ for all k;
- $\boldsymbol{\beta} < \boldsymbol{\alpha}$ if $\beta_k \le \alpha_k$ for all k and $\beta_k < \alpha_k$ for at least one k.

If $\boldsymbol{\beta} \le \boldsymbol{\alpha}$, then $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is the multi-index $(\alpha_k - \beta_k,\ k \ge 1)$. For $\boldsymbol{\alpha} \in \mathcal{J}$ define

$$|\boldsymbol{\alpha}| = \sum_k \alpha_k,\ \ \boldsymbol{\alpha}! = \prod_k \alpha_k!.$$

Special multi-indices and the corresponding notations are (i) $\mathbf{(0)}$, the multi-index with all zeros: $|\mathbf{(0)}| = 0$; (ii) $\boldsymbol{\epsilon(k)}$, the multi-index with 1 at position k and zeroes elsewhere: $|\boldsymbol{\epsilon(k)}| = 1$.

For a sequence $\boldsymbol{z} = (z_1, z_2, \ldots)$ of complex numbers and $\boldsymbol{\alpha} \in \mathcal{J}$ we write

$$\boldsymbol{z}^{\boldsymbol{\alpha}} = \prod_k z_k^{\alpha_k},\ \ \boldsymbol{z}^{r\boldsymbol{\alpha}} = \big(\boldsymbol{z}^{\boldsymbol{\alpha}}\big)^r,\ r \in \mathbb{R}.$$

Here are some useful technical results.

Proposition 2.3. *Let* $\mathfrak{q} = (q_1, q_2, \ldots)$ *be the sequence of eigenvalues of the operator* Q*. Under the assumptions* (2.2),

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\mathfrak{q}^{-\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = \exp\left(\sum_{k\ge1}\frac{1}{q_k}\right), \tag{2.3}$$

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{q}^{-\boldsymbol{\alpha}} = \prod_{k\ge1}\frac{q_k}{q_k-1}, \tag{2.4}$$

$$|\boldsymbol{\alpha}|! \le \mathfrak{q}^{\boldsymbol{\alpha}}\boldsymbol{\alpha}!. \tag{2.5}$$

Proof. Define $p_k = 1/q_k$. To establish (2.3) and (2.4), note that, since $\lim_{k\to\infty} p_k = 0$, we have

$$\exp\big(\sum_k p_k\big) = \prod_k e^{p_k} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\prod_k \frac{p_k^{\alpha_k}}{\alpha_k!},\quad \prod_{k\ge1}\frac{1}{1-p_k} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\prod_k p_k^{\alpha_k}.$$

To establish (2.5), let $n = |\boldsymbol{\alpha}|$ and use the multinomial formula and (2.2) to find

$$1 \ge \left(\sum_k \frac{1}{q_k}\right)^n = \sum_{\boldsymbol{\alpha}\in\mathcal{J};|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!\,\mathfrak{q}^{-\boldsymbol{\alpha}}}.$$

This concludes the proof of Proposition 2.3. □

By a theorem of Cameron and Martin [17], every $F(\boldsymbol{\xi}) \in L^2(\mathbb{F})$ with values in a (complex) Hilbert space V can be written as

$$F(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} F_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}, \tag{2.6}$$

where

$$\xi_{\boldsymbol{\alpha}} = \frac{1}{\sqrt{\boldsymbol{\alpha}!}}\prod_k \mathrm{H}_{\alpha_k}(\xi_k),$$

$$\mathrm{H}_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$$

and

$$F_{\boldsymbol{\alpha}} = \mathbb{E}\big(F(\boldsymbol{\xi})\xi_{\boldsymbol{\alpha}}\big).$$

Then

$$\mathbb{E}\|F(\boldsymbol{\xi})\|_V^2 = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\|F_{\boldsymbol{\alpha}}\|_V^2.$$

We denote by $\mathbb{L}(\boldsymbol{\xi};V)$ the collection of all square-integrable V-valued functions $F(\boldsymbol{\xi})$.

Next, we construct spaces of stochastic test functions and generalized random elements.

Definition 2.4. *For $\rho \in [0,1]$ and $\ell \geq 0$,*

- *the space $(\mathcal{S})_{\rho,\ell}(V)$ is the collection of $\Phi \in \mathbb{L}_2(\boldsymbol{\xi};V)$ such that*
$$\|\Phi\|^2_{\rho,\ell;V} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}(\boldsymbol{\alpha}!)^{\rho}\mathfrak{q}^{\ell\boldsymbol{\alpha}}\|\Phi_{\boldsymbol{\alpha}}\|_V^2 < \infty;$$
- *the space $(\mathcal{S})_{-\rho,-\ell}(V)$ is the closure of $\mathbb{L}_2(\boldsymbol{\xi};V)$ with respect to the norm*
$$\|\Phi\|^2_{-\rho,-\ell;V} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}(\boldsymbol{\alpha}!)^{-\rho}\mathfrak{q}^{-\ell\boldsymbol{\alpha}}\|\Phi_{\boldsymbol{\alpha}}\|_V^2;$$
- *the space $(\mathcal{S})_{\rho}(V)$ is the projective limit (intersection endowed with a special topology) of the spaces $(\mathcal{S})_{\rho,\ell}(V)$, as ℓ varies over all non-negative integers;*
- *the space $(\mathcal{S})_{-\rho}(V)$ is the inductive limit (union endowed with a special topology) of the spaces $(\mathcal{S})_{-\rho,-\ell}(V)$, as ℓ varies over all non-negative integers.*

A stochastic test function is an element of $(\mathcal{S})_{\rho}(\mathbb{C})$ for some $\rho \geq 0$.

A (V-valued) generalized random element is an element of $(\mathcal{S})_{-1}(V)$.

Thus, every $\Phi \in (\mathcal{S})_{-1}(V)$ has a *chaos expansion*

$$\Phi = \sum_{\boldsymbol{\alpha}}\Phi_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}, \tag{2.7}$$

and the coefficients $\Phi_{\boldsymbol{\alpha}}$ provide information about Φ at different stochastic scales (that is, in the linear spans of $\xi_{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| = n$).

In the white noise theory, the spaces $(\mathcal{S})_{-0}(\mathbb{C})$ and $(\mathcal{S})_{-1}(\mathbb{C})$ are known, respectively, as the spaces of Hida and Kondratiev distributions (note that indeed $(\mathcal{S})_{-0}$ is not the same as $(\mathcal{S})_0$).

It follows from (2.5) that $\alpha!$ in the definitions of the spaces can be replaced with $|\alpha|!$: it will not change $(\mathcal{S})_{\rho}$ and $(\mathcal{S})_{-\rho}$ and will shift the index ℓ in the individual $(\mathcal{S})_{\pm\rho,\pm\ell}$. We will see below why the values of ρ are restricted to $[0,1]$.

For $\Psi \in (\mathcal{S})_{-\rho}(V)$ and $\eta \in (\mathcal{S})_{\rho}(\mathbb{C})$, define $\langle \Psi, \eta \rangle \in V$ by

$$\langle \Psi, \eta \rangle = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \Psi_{\boldsymbol{\alpha}} \eta_{\boldsymbol{\alpha}}. \tag{2.8}$$

By the Cauchy-Schwarz and the triangle inequalities,

$$\| \langle \Psi, \eta \rangle \|_V \leq \left(\sum_{\boldsymbol{\alpha} \in \mathcal{J}} \| \Psi_{\boldsymbol{\alpha}} \|_V^2 (\boldsymbol{\alpha}!)^{-\rho} \mathfrak{q}^{-\ell \boldsymbol{\alpha}} \right)^{1/2} \left(\sum_{\boldsymbol{\alpha} \in \mathcal{J}} |\eta_{\boldsymbol{\alpha}}|^2 (\boldsymbol{\alpha}!)^{\rho} \mathfrak{q}^{\ell \boldsymbol{\alpha}} \right)^{1/2}. \tag{2.9}$$

Let $\boldsymbol{z} = (z_1, z_2, \ldots)$ be a sequence of complex numbers such that $\sum_{k \geq 1} |z_k|^2 < \infty$. The *stochastic exponential* of $\boldsymbol{z}$ is the random variable

$$\mathcal{E}(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}}{\sqrt{\boldsymbol{\alpha}!}} \xi_{\boldsymbol{\alpha}}.$$

Direct computations using (2.3), (2.4) and the generating function formula for the Hermite polynomials show that

(1) If $\sum_k q_k^{\ell} |z_k|^2 < \infty$, then $\mathcal{E}(\boldsymbol{z}) \in (\mathcal{S})_{\rho,\ell}(\mathbb{C})$ for every $0 \leq \rho < 1$;
(2) $\sum_k q_k^{\ell} |z_k|^2 < 1$, then $\mathcal{E}(\boldsymbol{z}) \in (\mathcal{S})_{1,\ell}(\mathbb{C})$.

Definition 2.5. *The S-transform* $\widetilde{\Phi}$ *of* $\Phi \in (\mathcal{S})_{-\rho,-\ell}(V)$ *with chaos expansion* (2.7) *is*

$$\widetilde{\Phi}(\boldsymbol{z}) = \langle \Phi, \mathcal{E}(\boldsymbol{z}) \rangle = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \frac{\Phi_{\boldsymbol{\alpha}}}{\sqrt{\boldsymbol{\alpha}!}} \boldsymbol{z}^{\boldsymbol{\alpha}},$$

defined for $\boldsymbol{z}$ *such that* $\mathcal{E}(\boldsymbol{z}) \in (\mathcal{S})_{\rho,\ell}(\mathbb{C})$.

Note that $\Phi_{(\mathbf{0})} = \widetilde{\Phi}(0)$; $\Phi_{(\mathbf{0})}$ is called a *generalized expectation* of Φ and even denoted by $\mathbb{E}\Phi$, although it is clear from the corresponding definitions that a typical element of $(\mathcal{S})_{-\rho,-\ell}$ for $\rho, \ell > 0$ cannot have any moments in the usual sense.

The following is a (partial) characterization of the spaces $\mathcal{S}_{-\rho}(V)$ in terms of the S-transform.

Theorem 2.6 (Characterization Theorem)**.** *(a) If* $\Phi \in (\mathcal{S})_{-\rho,-\ell}$ *and* $0 \leq \rho < 1$ *then, for every* real *sequences* $\boldsymbol{p}$ *and* $\boldsymbol{r}$*, such that* $\mathcal{E}(\boldsymbol{p}) \in (\mathcal{S})_{\rho,\ell}(\mathbb{C})$*,* $\mathcal{E}(\boldsymbol{r}) \in (\mathcal{S})_{\rho,\ell}(\mathbb{C})$*, the function* $f(z) = \widetilde{\Phi}(z\boldsymbol{p} + \boldsymbol{q})$ *is an entire function of* $z \in \mathbb{C}$.

(b) For $0 < R, \ell < \infty$ *let*

$$\mathbb{K}_{\ell}(R) = \left\{ \boldsymbol{z} \; : \; \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \mathfrak{q}^{\ell \boldsymbol{\alpha}} |\boldsymbol{z}^{\boldsymbol{\alpha}}|^2 < R^2 \right\}. \tag{2.10}$$

If $\Phi \in (\mathcal{S})_{-1}(V)$*, then there exist* R, ℓ *such that* $\widetilde{\Phi}(\boldsymbol{z})$ *is analytic in* $\mathbb{K}_{\ell}(R)$*. Conversely, if* $f = f(\boldsymbol{z})$ *is a function analytic in* $\mathbb{K}_{\ell}(R)$ *for some* $0 < R, \ell < \infty$*, then there exists a unique* $\Phi \in (\mathcal{S})_{-1}(V)$ *such that* $\widetilde{\Phi} = f$.

Proof. (a) See Kuo [18, Theorem 8.10]. There is also a converse statement characterizing entire functions with certain growth at infinity as S-transforms of elements from $(\mathcal{S})_{-\rho}(V)$.

(b) See Holden et al. [11, Theorem 2.6.11]. Note that, given $f = \widetilde{\Phi}$, we recover $\Phi = \sum_{\boldsymbol{\alpha}} \Phi_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}$ by

$$\Phi_{\boldsymbol{\alpha}} = \frac{1}{\sqrt{\boldsymbol{\alpha}!}} \frac{\partial^{|\boldsymbol{\alpha}|}\widetilde{\Phi}(\boldsymbol{z})}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots}\Big|_{\boldsymbol{z}=0}. \tag{2.11}$$

□

In what follows, we refer to $\mathbb{K}_\ell(R)$ from (2.10) as a *neighborhood* of $\boldsymbol{z} = 0$.

The significance of Theorem 2.6 is that it provides an *intrinsic* characterization of the spaces $(\mathcal{S})_{-\rho}(V)$ in terms of the S-transform and the spaces of analytic functions, as opposed to the *extrinsic* characterization of Definition 2.4 using a particular orthonormal basis in $\mathbb{L}_2(\boldsymbol{\xi}; V)$. We also see why the values of ρ are restricted to $[0, 1]$: for $\rho > 1$, stochastic exponentials are never test functions, making it impossible to define the S-transform.

Finally, we define the Wick product $\diamond$ for two elements of $(\mathcal{S})_{-1}(\mathbb{C})$.

Definition 2.7. *Given $\Phi, \Psi \in (\mathcal{S})_{-1}(\mathbb{C})$, $\Phi \diamond \Psi$ is the unique element of $(\mathcal{S})_{-1}(\mathbb{C})$ such that*

$$\widetilde{\Phi \diamond \Psi}(\boldsymbol{z}) = \widetilde{\Phi}(\boldsymbol{z}) \widetilde{\Psi}(\boldsymbol{z}) \tag{2.12}$$

An immediate consequence of (2.12) is a useful property of generalized expectations:

$$\mathbb{E}(\Phi \diamond \Psi) = \big(\mathbb{E}\Phi\big)\big(\mathbb{E}\Psi\big).$$

Formula (2.12) is motivated by the following property of the Wick product (which can be traced back to the original paper by Wick [9]):

$$\mathrm{H}_m(\xi_k) \diamond \mathrm{H}_n(\xi_l) = \begin{cases} \mathrm{H}_m(\xi_k)\mathrm{H}_n(\xi_l), & \text{if } k \neq l; \\ \mathrm{H}_{m+n}(\xi_k), & \text{if } k = l. \end{cases} \tag{2.13}$$

Writing $\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sqrt{\boldsymbol{\alpha}!}\xi_{\boldsymbol{\alpha}}$, we conclude from (2.13) that

$$\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \diamond \mathrm{H}_{\boldsymbol{\beta}}(\boldsymbol{\xi}) = \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\boldsymbol{\xi}). \tag{2.14}$$

If

$$\Phi = \sum_{\boldsymbol{\alpha}} \bar{\Phi}_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \quad \Psi = \sum_{\boldsymbol{\alpha}} \bar{\Psi}_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}),$$

then formal term-by-term multiplication using (2.14) implies

$$\Phi \diamond \Psi = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \bar{\Phi}_{\boldsymbol{\alpha}} \bar{\Psi}_{\boldsymbol{\beta}} \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\boldsymbol{\xi}), \tag{2.15}$$

which is consistent with (2.12). Note also that if

$$\Phi = \sum_{\boldsymbol{\alpha}} \Phi_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}} \quad \Psi = \sum_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}},$$

then $\Psi_{\boldsymbol{\alpha}} = \bar{\Phi}_{\boldsymbol{\alpha}} \sqrt{\boldsymbol{\alpha}!}$ and

$$\Phi \diamond \Psi = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \sqrt{\frac{(\boldsymbol{\alpha} + \beta)!}{\boldsymbol{\alpha}!\boldsymbol{\beta}!}} \Phi_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\beta}} \xi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \tag{2.16}$$

which is not as convenient as (2.15).

3. Existence and uniqueness of solution

Let $G = \mathbb{R}$ or $G = S^1$ (the circle, which corresponds to periodic boundary conditions and which we identify with $\mathbb{R}/(0,\pi)$). Denote by $H^\gamma(G)$, $\gamma \in \mathbb{R}$, the Sobolev spaces on G and by $\|\cdot\|_\gamma$ the corresponding norms; $H^0(G) = L_2(G)$. Recall that, for $G = \mathbb{R}$,

$$\|f\|_\gamma^2 = \int_{\mathbb{R}} (1+y^2)^\gamma |\hat{f}(y)|^2 dy,$$

where $\hat{f}$ is the Fourier transform of f, and for $G = S^1$,

$$\|f\|_\gamma^2 = \sum_k (1+k^2)^\gamma |f_k|^2,$$

where f_k are the Fourier coefficients of f. Also recall that, by the Sobolev embedding theorem, every element (equivalence class) from $H^1(G)$ has a representative that is a bounded and Hölder continuous function on G. To avoid unnecessary technical complications, we will simply say that if $f \in H^1(G)$, then f is bounded and Hölder continuous, and $\sup_{x\in G} |f(x)| \le C\|f\|_1$ for some C independent of f.

Consider the equation

$$\begin{aligned} u_t(t,x) + u(t,x) \diamond u_x(t,x) &= u_{xx}(t,x) + f(t,x) \\ &+ \sum_{k\ge 1} (a_k(t,x)u_{xx} + b_k(t,x)u_x + c_k(t,x)u + g_k(t,x)) \diamond \xi_k, \\ &0 < t \le T,\ x \in G, \end{aligned} \tag{3.1}$$

with initial condition $u(0,x) \in (\mathcal{S})_{-1}(L_2(G))$, where ξ_k are iid standard normal random variables, and f, a_k, b_k, c_k and g_k are *non-random* measurable functions. To define the solution we assume that $f, g_k \in L_2((0,T)\times G)$ and

$$\sup_{t,x} |a_k(t,x)| + \sup_{t,x} |b_k(t,x)| + \sup_{t,x} |c_k(t,x)| + \|g_k\|_{L_2((0,T)\times G)} \le q_k^r \tag{3.2}$$

for q_k from Definition 2.1 and some $r > 0$.

Definition 3.1. *The process $u \in (\mathcal{S})_{-1}\big((L_2((0,T); H^2(G))\big)$, is called a* strong chaos solution *of (3.1) if there exist $0 < R, \ell < \infty$ such that, for all $\boldsymbol{z} = (z_1, z_2, \ldots) \in \mathbb{K}_\ell(R)$ and all $t \in [0,T]$, the equality*

$$\begin{aligned} \widetilde{u}(t,x;\boldsymbol{z}) + \int_0^t \widetilde{u}(s,x;\boldsymbol{z})\widetilde{u}_x(s,x;\boldsymbol{z})ds &= \widetilde{u}(0,x;\boldsymbol{z}) + \int_0^t \widetilde{u}_{xx}(s,x;\boldsymbol{z})ds \\ &+ \sum_{k\ge 1} \int_0^t \Big(a_k(s,x)\widetilde{u}_{xx} + b_k(s,x)\widetilde{u}_x + c_k(t,x)\widetilde{u} + g_k(t,x)\Big) z_k ds \end{aligned} \tag{3.3}$$

holds in $L_2(G)$.

The a priori assumption $u \in (\mathcal{S})_{-1}\big((L_2((0,T); H^2(G))\big)$ implies

$$\langle u, \eta \rangle \in L_2((0,T); H^2(G))$$

for all $\eta \in (\mathcal{S})_1(\mathbb{C})$. The S-transform $\widetilde{u}$ of u is thus an element of $L_2((0,T); H^2(G))$, so that $\widetilde{u}_x \in L_2((0,T); H^1(G))$ and $\widetilde{u}_x \in L_2((0,T)\times G)$. By the Sobolev embedding theorem, $\widetilde{u}, \widetilde{u}_x \in L_2((0,T); \mathcal{C}(G))$. Condition (3.2) ensures uniform in (t,x) convergence of all infinite sums.

The chaos solution is a variational solution in the space $(\mathcal{S})_{-1}\big(L_2((0,T);H^2(G))\big)$, and the characterization theorem (Theorem 2.6) makes it possible to restrict the set of test functions to stochastic exponentials. The resulting deterministic equation for the S-transform (3.3) (which follows from (3.1) by the linearity of the S-transform and the definition of the Wick product (2.12)) can be satisfied in a variety of ways (classical, variational, viscosity, etc.) Our a priori assumptions on u allow us to satisfy the equation for $\widetilde{u}$ in the *strong* sense, because all the necessary partial derivatives of $\widetilde{u}$ in t and x exist as L_2 functions. This is why the resulting solution of (3.1) is called a *strong chaos solution*.

While our main objective is to establish existence and uniqueness of solution of (3.1), we will start by showing that the space $(\mathcal{S})_{-1}$ is indeed the natural solution space. In other words, we should not expect the solution of (3.1) to belong to any $(\mathcal{S})_{-\rho}$ if $\rho < 1$.

Theorem 3.2. *Let $\phi = \phi(x),\ x \in \mathbb{R}$ be a smooth compactly supported function and let ξ be a standard Gaussian random variable. Then equation*

$$u_t + u \diamond u_x = u_{xx},\ 0 < t \le T,\ x \in \mathbb{R}, \tag{3.4}$$

with initial condition $u(0,x) = \xi\,\phi(x)$ cannot have a solution that is an element of $(\mathcal{S})_{-\rho,-\ell}(L_2((0,T);H^2(\mathbb{R}))$ for some $0 \le \rho < 1$ and $\ell > 0$.

Proof. We will show that the S-transform $\widetilde{u}$ of u cannot be an entire function, as required by Theorem 2.6 for u to be an element of $(\mathcal{S})_{-\rho,\ell}$, $\rho < 1$.

By (3.3),

$$\widetilde{u}_t(t,x;z) + \widetilde{u}(t,x;z)\widetilde{u}_x(t,x;z) = \widetilde{u}_{xx}(t,x;z), \tag{3.5}$$

$\widetilde{u}(0,x;z) = z\phi(x)$; with only one random variable ξ, we have only one complex parameter z.

Equation (3.5) is the usual Burgers equation and has a closed-form solution via the Hopf-Cole transformation (see Evans [19, Section 4.4.1]): writing $F(x) = \int_{-\infty}^{x}\phi(y)dy$, we find

$$\widetilde{u}(t,x;z) = \frac{\int_{-\infty}^{+\infty} z\phi(y)\exp\left(-\frac{(x-y)^2}{4t} - \frac{zF(y)}{2}\right)dy}{\int_{-\infty}^{+\infty}\exp\left(-\frac{(x-y)^2}{4t} - \frac{zF(y)}{2}\right)dy}. \tag{3.6}$$

This is a classical solution of (3.5) and leads to a classical chaos solution of (3.4); the solution of (3.5) is unique in the class $L_2((0,T);H^2(\mathbb{R}))$ (see Biler et al. [20, Theorem 2.1]). If $u(t,x) \in (\mathcal{S})_{-\rho,-\ell}(\mathbb{R})$ for every t,x, then, by Theorem 2.6(a), $\widetilde{u}(t,x;z)$ must be an analytic function of z for all $t \in [0,T]$, $x \in \mathbb{R}$, $z \in \mathbb{C}$. Since it is not immediately clear from (3.6) whether the dependence of $\widetilde{u}$ on z is analytic, we will transform equation (3.6) further by going back to the derivation of the Hopf-Cole transformation.

Consider the equation

$$v_t + \frac{1}{2}|v_x|^2 = v_{xx},\ 0 < t \le T,\ x \in \mathbb{R}, \tag{3.7}$$

with initial condition $v(0,x;z) = zF(x) = z\int_{-\infty}^{x}\phi(y)dy$. Information about v leads to information about $\widetilde{u}$ because of the relation $v_x = \widetilde{u}$. Note that F is a smooth bounded function. Direct computations (see [19, Section 4.4.1]) show that the function V defined by

$$V(t,x;z) = \exp\Big(-v(t,x;z)/2\Big) \tag{3.8}$$

satisfies the heat equation

$$V_t = V_{xx},\ V(0,x;z) = \exp\Big(-zF(x)/2\Big). \tag{3.9}$$

Equation (3.9) has a classical solution which is unique in the class of bounded twice continuously differentiable functions. Interpreting the heat kernel as the normal density, the solution of (3.9) can be written as

$$V(t,x;z) = \mathbb{E}e^{-z\zeta(t,x)}, \tag{3.10}$$

where $\zeta(t,x) = (1/2)F\big(x+\sqrt{2}w(t)\big)$ and w is a standard Brownian motion.

We are now ready to show that the function $\widetilde{u}$ from (3.6) cannot be entire (that is, analytic for all $z \in \mathbb{C}$). The argument goes as follows:

(1) The function V defined in (3.10) is an extension to the complex plane of the characteristic function of a uniformly bounded non-degenerate random variable ζ and is therefore an entire function of the form

$$V(t,x;z) = V_0(t,x;z)e^{g(t,x)z}, \tag{3.11}$$

where $V_0(t,x;z)$ is an entire function with infinitely many zeroes (Lukacs [21, Theorem 7.2.3]).

(2) Representation (3.8) then implies that the function v cannot be an entire function of z; otherwise V would have no zeroes;

(3) By uniqueness of solutions of (3.5) and (3.7) we conclude that $v_x(t,x;z) = \widetilde{u}(t,x;z)$, because both functions satisfy the same equation with the same initial condition.

(4) By the fundamental theorem of calculus, $v(t,x;z) = v(t,0;z) + \int_0^x \widetilde{u}(t,y;z)dy$, which means that if $\widetilde{u}$ were an entire function of z, so would be $v(t,x;z) - v(t,0;z)$.

(5) If $v(t,x;z) - v(t,0;z)$ is entire, then, by (3.8), so is

$$\bar{V}(t,x;z) = \frac{V(t,x;z)}{V(t,0;z)}$$

and moreover, $\bar{V}$, as a function of z, has no zeros (because $1/\bar{V}$ corresponds to $v(t,0;z) - v(t,x;z)$ and must be entire as well); then (3.11) and some basic complex analysis imply

$$\bar{V}(t,x;z) = h(t,x)e^{zf(t,x)} \text{ or } V(t,x;z) = V(t,0,z)h(t,x)e^{zf(t,x)}. \tag{3.12}$$

(6) Finally, (3.12) and the equality $\widetilde{u} = -2V_x/V$ imply that (if we assume $\widetilde{u}$ is entire, forcing $v(t,x;z) - v(t,0;z)$ to be entire, in turn forcing V to have the form (3.12)) the solution $\widetilde{u}$ of (3.5) must be a linear function of z, which is impossible and provides the required contradiction.

Note that the key step in the argument, namely, showing that the function $V(t,x;z)$ cannot have the same zeros in z for all x, can be carried out in several different ways. *This concludes the proof of Theorem 3.2.* $\square$

Note that equation (3.4) without the Wick product,

$$U_t + UU_x = U_{xx},\ U(0,x) = \xi\,\phi(x), \tag{3.13}$$

has a classical solution which is a very regular stochastic process. Indeed, the solution is given by the right-hand side of (3.6) with ξ instead of z, from which it immediately follows that $U(t,x)$ is a smooth function of (t,x) and

$$|U(t,x)| \leq c_1|\xi|e^{c_2|\xi|}$$

for some positive numbers c_1, c_2. That is, as a random variable, U has moments of all orders. By contrast, the solution of (3.4) (provided it exists) is a generalized random element from $(\mathcal{S})_{-1}(\mathbb{R})$. In other words, Theorem (3.2) delivers the "curse" part of paper's title: Wick product in the Burgers equation indeed forces the solution into the largest space of generalized random elements. The following observation provides a bit of relief: with all the nice properties of U, there is no easy way to get $\mathbb{E}U(t,x)$; on the other hand, for the solution of (3.4), the generalized expectation $\widetilde{u}(t,x;0)$ solves the deterministic Burgers equation with zero initial condition and is therefore equal to zero.

The next theorem delivers the "cure" part by establishing solvability of (3.1) in $(\mathcal{S})_{-1}\big(L_2((0,T);H^2(G))\big)$ For technical reasons, the conditions are slightly different for $G=\mathbb{R}$ and $G=S^1$; in particular, we have to consider the "homogeneous" case $f=0$ when $G=S^1$. We discuss these and other questions in the following section. Recall that, for a generalized random element Φ, $\mathbb{E}\Phi$ denotes its generalized expectation, and $\mathbb{E}\Phi = \Phi_{(\mathbf{0})} = \widetilde{\Phi}(0)$.

Theorem 3.3. *Assume that*

- *condition* (3.2) *holds;*
- $u_0 \in (\mathcal{S})_{-1,-\ell}(H^1(G))$ *for some* $\ell > 0$ *and* $\mathbb{E}u_0 \in H^2(G)$*;*
- *If* $G=\mathbb{R}$*, then also* $\mathbb{E}u_0 \in L_1(\mathbb{R})$ *and* $f(t,x) = F_x(t,x)$ *for a function* F *such that both* F *and* F_x *are bounded and Hölder continuous in* (t,x)*;*
- *If* $G=S^1$*, then also* $f=0$.

Under these assumptions, equation (3.1) *has a unique solution and there exists an* $n>0$ *such that*

$$\|u\|^2_{-1,-n;L_2((0,T);H^2(\mathbb{R}))} + \sup_{0\leq t\leq T}\|u(t,\cdot)\|^2_{-1,-n;H^1(\mathbb{R})} < \infty. \tag{3.14}$$

It turns out that the generalized expectation $u_{(\mathbf{0})}$ of the solution has special significance. In particular, unlike the other chaos coefficients, $u_{(\mathbf{0})}$ and its partial derivative in x have to be bounded continuous functions, and the conditions of the theorem ensure that. This is yet another technical issue which we will discuss later.

Example 3.4. Let us apply Theorem 3.3 to the Burgers equation with random viscosity

$$u_t + u\diamond u_x = \Big(\big(\mu_0 + \dot{W}(t,x)\big)u_x\Big)_x, \tag{3.15}$$

which was mentioned in the introduction as one of the motivations for investigating (3.1). To concentrate on the effects of the noise in the viscosity, we assume that the initial condition is a non-random smooth function with compact support.

For the space-time white noise $\dot{W}(t,x)$, we have representation

$$\dot{W}(t,x)=\sum_{k\geq 1}h_k(t,x)\xi_k, \tag{3.16}$$

where $\{h_k,\ k\geq 1\}$ is an orthonormal basis in $L_2((0,T)\times G)$ (see, for example, Holden et al. [11, Definition 2.3.9]). Then (3.15) becomes a particular case of (3.1) with $a_k=\mu_0+h_k$, $b_k=\partial h_k/\partial x$, $c_k=g_k=f=0$.

If $G=\mathbb{R}$, then we take

$$h_k(t,x)=h_{k_1,k_2}(t,x)=m_{k_1}(t)w_{k_2}(x),\ k_1,k_2\geq 1,$$

where m_l are normalized sines and cosines, and w_l are Hermite functions (the standard reference for Hermite functions is Hille and Phillips [22, Section 21.3]). In particular, it is known that $\sup_x|w_l(x)|\leq cl^{-1/12}$ and therefore $\sup_x|w_l'(x)|\leq cl^{5/12}$ (recall that $-w_l''+(1+|x|^2)w_l=2lw_l$). Then (3.2) holds with $q_l=2l,\ r=1$.

If $G=S^1$, then we take

$$h_k(t,x)=h_{k_1,k_2}(t,x)=m_{k_1}(t)m_{k_2}(x),\ k_1,k_2\geq 1,$$

where m_l are normalized sines and cosines. In this case $\sup_x|m_l'(x)|\leq cl$, and again (3.2) holds with $q_l=2l,\ r=1$.

As a result, we can apply Theorem 3.3 with $q_l=2l$ in both cases, which is the standard choice in the traditional white noise setting.

We can generalize the model further and introduce, in addition to the white noise viscosity, a white noise random forcing and a white-in-space initial condition, with arbitrary correlation between the three (by suitably parsing the sequence $\boldsymbol{\xi}$ for the construction of the corresponding random term).

The eigenvalues of the type $q_l=(2l)^r$, $r\geq 1$ work in most models. The option to select other q_l can help, for example, to deal with exotic stochastic forcing terms such as $\sum_k\big(e^{kt}\sin x\big)\xi_k$.

This concludes Example 3.4.

4. Outline of the proof and further directions

Recall that our objective is to study the equation

$$\begin{aligned}u_t(t,x)+u(t,x)\diamond u_x(t,x)&=u_{xx}(t,x)+f(t,x)\\&+\sum_{k\geq 1}(a_k(t,x)u_{xx}+b_k(t,x)u_x+c_k(t,x)u+g_k(t,x))\diamond\xi_k,\\&0<t\leq T,\ x\in G.\end{aligned} \tag{4.1}$$

By definition, the solution is a generalized random element with S-transform $\widetilde{u}$ satisfying

$$\begin{aligned}\widetilde{u}(t,x;\boldsymbol{z})+\int_0^t\widetilde{u}(s,x;\boldsymbol{z})\widetilde{u}_x(s,x;\boldsymbol{z})ds&=\widetilde{u}(0,x;\boldsymbol{z})+\int_0^t\widetilde{u}_{xx}(s,x;\boldsymbol{z})ds\\&+\sum_{k\geq 1}\int_0^t\Big(a_k(s,x)\widetilde{u}_{xx}+b_k(s,x)\widetilde{u}_x+c_k(t,x)\widetilde{u}+g_k(t,x)\Big)z_kds.\end{aligned} \tag{4.2}$$

The most natural approach to proving existence and uniqueness of solution for (4.1) is to prove existence and uniqueness of solution for (4.2). This approach is used, for example, by Benth et al. [12]. Here are some of the reasons why we choose a different approach:

- The complex-valued Burgers equation (4.2) is not easily handled by the Hilbert-space methods (a major reason is that, for complex numbers, $z^2 \neq |z|^2$), whereas our goal is to get solvability in the Sobolev spaces H^γ.
- The Hölder spaces could be a natural choice for (4.2), but the available methods lead either to existence that is local in time or require additional assumptions about the initial conditions (in the form of smallness of certain norms). We would like to avoid either of these restrictions.
- The problem of uniqueness of solution for (4.2) is non-trivial.
- Even if we succeed in finding $\widetilde{u}$ and proving that it is unique, there is still no easy way to get any regularity information about the solution, such as estimate (3.14).

Our approach is to get information about the solution from the chaos expansion. In what follows, we will use the notation

$$\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sqrt{\boldsymbol{\alpha}!}\,\xi_{\boldsymbol{\alpha}} \tag{4.3}$$

and an equivalent chaos expansion

$$u = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}). \tag{4.4}$$

By (2.15), representation (4.4) is more convenient for the computation of the Wick product.

Our first step is to derive equations for the coefficients $u_{\boldsymbol{\alpha}}$. As a bonus, we also get an alternative way to prove uniqueness of solution.

By (2.11) (and keeping track of the factorial terms), we get the following relation between the chaos coefficients $u_{\boldsymbol{\alpha}}$ from (4.4) and the S-transform $\widetilde{u}$:

$$\widetilde{u}(t, x, \boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u_{\boldsymbol{\alpha}}(t, x)\boldsymbol{z}^{\boldsymbol{\alpha}}. \tag{4.5}$$

Before we proceed, let us introduce several notations. To reduce the number of subscripts, it is convenient to rename the initial condition: $u_0(x) = \varphi(x)$, and to use alternative notations for derivatives: $\dot{u} = u_t$, and $Du = u_x$. With the generalized expectation $u_{(\mathbf{0})}$ having special significance (which also means frequent appearance in complicated formulas), we re-name it as well: $u_{(\mathbf{0})} = \mathfrak{u}$. Now we are ready to derive the equations for $u_{\boldsymbol{\alpha}}$.

Theorem 4.1. *The function* $\widetilde{u}$ *is a (strong) solution of* (4.2) *if and only if the chaos coefficients* $\{u_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{J}\}$ *satisfy (also in the strong sense) the system of equations*

$$\dot{\mathfrak{u}} + \mathfrak{u}D\mathfrak{u} = D^2\mathfrak{u},\ \ \mathfrak{u}(0,x) = \varphi_{(\mathbf{0})}(x); \tag{4.6}$$

$$\begin{aligned} \dot{u}_{\boldsymbol{\epsilon}(\boldsymbol{k})} &= D\big(Du_{\boldsymbol{\epsilon}(\boldsymbol{k})} - \mathfrak{u}u_{\boldsymbol{\epsilon}(\boldsymbol{k})}\big) + a_k D^2\mathfrak{u} + b_k D\mathfrak{u} + c_k\mathfrak{u} + g_k, \\ u_{\boldsymbol{\epsilon}(\boldsymbol{k})}(0,x) &= \varphi_{\boldsymbol{\epsilon}(\boldsymbol{k})}(x); \end{aligned} \tag{4.7}$$

$$\begin{aligned} \dot{u}_{\boldsymbol{\alpha}} &= D\big(Du_{\boldsymbol{\alpha}} - \mathfrak{u}u_{\boldsymbol{\alpha}}\big) - \sum_{(\mathbf{0})<\boldsymbol{\beta}<\boldsymbol{\alpha}} u_{\boldsymbol{\beta}}Du_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\ &\quad + \sum_{k:\alpha_k>0} \big(a_k D^2 u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})} + b_k D u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})} + c_k u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})}\big), \\ u_{\boldsymbol{\alpha}}(0,x) &= \varphi_{\boldsymbol{\alpha}}(x),\ \ |\boldsymbol{\alpha}| > 1, \end{aligned} \tag{4.8}$$

and there exists a $p > 0$ *such that, for all* $\boldsymbol{\alpha} \in \mathcal{J}$,

$$\|u_{\boldsymbol{\alpha}}\|^2_{L_2((0,T);H^2(G))} \leq \mathfrak{q}^{p\boldsymbol{\alpha}}. \tag{4.9}$$

Proof. Note that (4.9) is equivalent to the condition $u \in (\mathcal{S})_{-1}(L_2((0,T);H^2(G)))$.

Assume that $\widetilde{u}$ satisfies (4.2) and is an analytic function of $\boldsymbol{z}$ so that the series (4.5) converges uniformly in $(t,x,\boldsymbol{z})$ in some neighborhood of $\boldsymbol{z} = 0$. We then substitute (4.5) into (4.2), compare the coefficients of $\boldsymbol{z}^{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha}$, and get (4.6)–(4.8). Uniform convergence ensures that all manipulations are legitimate.

For the proof in the opposite direction, we reverse the above argument. If (4.9) holds, then the series (4.5) converges uniformly in $(t,x,\boldsymbol{z})$ in some neighborhood of $\boldsymbol{z} = 0$. We then combine equalities (4.6)–(4.8) into power series and get (4.2).
This concludes the proof of Theorem 4.1. □

The system of PDEs (4.6)–(4.8) is called the `propagator` for equation (4.1) and describes how the stochastic equation propagates chaos through difference stochastic scales. Note that this propagation of chaos has no connection with the similar term used for deterministic equations in the study of particle systems (e.g. Sznitman [6]).

Before we proceed, let us note that:

(1) Since $\lim_{k\to\infty} q_k = +\infty$, an upper bound $\mathfrak{q}^{p\boldsymbol{\alpha}}$ is equivalent to $C^{|\boldsymbol{\alpha}|}\mathfrak{q}^{p\boldsymbol{\alpha}}$: in (4.9), we can always take a larger p, or, in the original construction of the chaos space, we can switch from q_k to Cq_k.
(2) The propagator is a lower triangular system and can be solved by induction on $|\boldsymbol{\alpha}|$. Only the first equation, describing $\mathfrak{u} = u_{(\mathbf{0})}$ is non-linear: it is a deterministic Burgers equation. All other equations are linear, but with variable coefficients in the lower-order derivatives; these coefficients are determined by $\mathfrak{u}$.
(3) By Theorem 4.1, uniqueness of solution for the propagator implies uniqueness of solution of (4.1) in the sense of Definition 3.1.

To prove Theorem 3.3, it is enough to show that the propagator has a unique solution and

$$\|u_{\boldsymbol{\alpha}}\|^2_{L_2((0,T);H^2(G))} + \sup_{0\leq t\leq T} \|u_{\boldsymbol{\alpha}}(t,\cdot)\|^2_{H^1(G)} \leq C^{|\boldsymbol{\alpha}|}\mathfrak{q}^{p\boldsymbol{\alpha}},\ \ p > 0; \tag{4.10}$$

this is actually stronger than (3.14).

A key step in the analysis of the propagator is the study of the Burgers equation (4.6). The solution of (4.6) must be

- a function, continuously differentiable in x, to act as a variable coefficient in the subsequent propagator equations, and also
- an element of $L_2((0,T);H^2(G))$, to produce a sufficiently regular free term in (4.7).

On the real line, the necessary solvability results can be obtained using the Hopf-Cole transformation, with sufficiently many regularity assumptions on φ and f. On the circle, we have to go with the best result we could find: Kiselev et al. [25, Theorem 1.1], who consider $f=0$.

We will now present the precise results about solvability of the deterministic Burgers equation.

Consider the equation

$$U_t(t,x)+U(t,x)U_x(t,x)=U_{xx}(t,x)+F_x(t,x) \tag{4.11}$$

with the initial condition $U(0,x)=U_0(x)$, for $x\in G$ where $G=\mathbb{R}$ (real line) or $G=S^1$ (unit circle or periodic boundary conditions). To define the solution of the equation, we assume that U_0 is a continuous function and $F\in L_2((0,T)\times G))$.

A *weak* solution of (4.11), U, is a continuous function in t,x such that for all smooth functions ψ with compact support in G, the equality

$$\begin{aligned}\int_G U(t,x)\psi(x)dx-\frac{1}{2}\int_0^t\int_G U^2(s,x)\psi'(x)dxdt&=\int_G U_0(x)\psi(x)dx\\ +\int_0^t\int_G U(s,x)\psi''(x)dxdt&-\int_0^t\int_G F(s,x)\psi'(x)dxds\end{aligned} \tag{4.12}$$

holds for all $t\in[0,T]$.

Theorem 4.2 (Solvability of the deterministic Burgers equation)**.**

(a) If $U_0\in\mathcal{C}^1(\mathbb{R})\bigcap L_1(\mathbb{R})$, U_0' is Hölder continuous (any non-zero order will work), and F, F_x, are bounded and Hölder continuous in (t,x) functions, then (4.11) has a weak solution $U=U(t,x)$ such that U is bounded and has a continuous bounded derivative in x. The solution is unique in the class of bounded continuous functions.

(b) If $U_0\in\mathcal{C}^1(\mathbb{R})\bigcap H^1(\mathbb{R})\bigcap L_1(\mathbb{R})$, U_0' is Hölder continuous (any non-zero order will work), and F and F_x are bounded and Hölder continuous in (t,x) functions, then (4.11) has a weak solution $U=U(t,x)$ such that

$$U\in L_2\big((0,T);H^2(\mathbb{R})\big)\bigcap\mathcal{C}((0,T);H^1(\mathbb{R})). \tag{4.13}$$

The solution is unique in the class of bounded continuous functions.

(c) If $U_0\in H^2(S^1)$ and $F=0$, then (4.11) has a weak solution $U=U(t,x)$ such that $U\in\mathcal{C}\big((0,T);H^2(S^1)\big)$ (by the Sobolev embedding, this implies continuous differentiability in x). The solution is unique in the class of bounded continuous function.

Proof. For the case $G=\mathbb{R}$, existence, uniqueness, and regularity of the solution are established using the Hopf-Cole transformation: we have

$$U=-2V_x/V, \tag{4.14}$$

where V is the solution of the heat equation

$$V_t = V_{xx} - \frac{1}{2}F(t,x)\,V, \quad V(0,x) = \exp\Big(-\frac{1}{2}\int_{-\infty}^{x} U_0(y)dy\Big). \tag{4.15}$$

Thus, the study of equation (4.12) is reduced to the study of equation (4.15).

(a) A result from Ladyzhensakaya et al. [23, Theorem IV.5.1] implies that, under our assumptions, equation (4.15) has a classical solution and the solution is unique in the class of bounded continuous functions. Then a result from Rozovskii [24, Theorem 5.1.1] provides the following probabilistic representation of the solution, with $w = w(t)$ denoting a standard Brownian motion:

$$V(t,x) = \mathbb{E}\Big(V_0\big(x+\sqrt{2}w(t)\big)\exp\big(-\frac{1}{2}\int_0^t F(s, x+\sqrt{2}w(t)-\sqrt{2}w(s))ds\big)\Big); \tag{4.16}$$

Since F is bounded and

$$e^{-(1/2)\|U_0\|_{L_1(\mathbb{R})}} \le V(x) \le e^{(1/2)\|U_0\|_{L_1(\mathbb{R})}},$$

it follows from (4.16) that there exist numbers $0 < p_1 < p_2$ such that, for all t, x, $p_1 \le V(t,x) \le p_2$. Therefore the functions $U = -2V_x/V$ and $U_x = -2(V_{xx}V - V_x^2)/V^2$ are bounded and continuous. Uniqueness of U follows from the uniqueness of V. The reader can also note that considering the Burgers equation in the form $U_t + UU_x = U_{xx}/2$ would eliminate a lot of various powers of 2 in subsequent computations.

(b) Under the assumed conditions, equation (4.15) still has a classical solution, but the standard parabolic regularity theorem in Sobolev spaces cannot be applied to V because V_0 is not integrable, and so V does not belong to any Sobolev space. On the other hand, (4.14) implies that (4.13) will follow if we can show that $1/V$ is bounded and has two continuous and bounded derivatives in x (which we did in part (a)), and that

$$V_x \in L_2\big((0,T); H^2(\mathbb{R})\big)\bigcap \mathcal{C}((0,T); H^1(\mathbb{R})), \tag{4.17}$$

which we show next.

Denoting V_x by $\bar{V}$, we find from (4.15) that

$$\bar{V}_t = \bar{V}_{xx} - \frac{1}{2}F\,\bar{V} - \frac{1}{2}VF_x,$$

and $\bar{V}(0,x) = -(1/2)U_0(x)V_0(x)$. Assumptions of the theorem and the properties of V imply that $\bar{V}(0,\cdot) \in H^1$ and $VF_x \in L_2((0,T)\times\mathbb{R})$. Then the linear parabolic regularity theorem (see Theorem 4.3 below) in the normal triple $(H^2(\mathbb{R}), H^1(\mathbb{R}), L_2(\mathbb{R}))$ implies (4.17).

By the Sobolev embedding, the conditions $U_0 \in \mathcal{C}^1(\mathbb{R})\bigcap H^1(\mathbb{R})$ plus Hölder continuity of U_0' can be replaced with a stronger but shorter condition $U_0 \in H^2(\mathbb{R})$, and this is what happens in the statement of Theorem 3.3. In general, we do have to add the condition $U_0 \in L_1(\mathbb{R})$ (for example, the function $(1+|x|^2)^{-1/2}$ is in every $H^\gamma(\mathbb{R})$, but is not in $L_1(\mathbb{R})$.) Finally, note that the resulting weak solution U, being an element of $L_2((0,T); H^2(\mathbb{R}))$, is in fact strong.

(c) See Kiselev et al. [25, Theorem 1.1].

This completes the proof of Theorem 4.2. □

We were unable to find satisfactory solvability results for the circle when an inhomogeneous term $f(t,x)$ is present, or for the Dirichlet or Neumann boundary value problems. As soon as these results are available, the corresponding analogue of Theorem 3.3 will follow immediately from Theorem 4.1. In fact, one can go even further and investigate Wick-stochastic versions of other one-dimensional equations with quadratic nonlinearity, such as Camassa-Holm, KdV, KPP, and Kuramoto-Sivashinskii equations, *provided there is a satisfactory solvability result for the corresponding deterministic nonlinear equation.*

Extension to higher dimensions presents an additional challenge, as the Sobolev space $H^1(\mathbb{R}^d)$, $d \geq 2$, is no longer embedded into the space of continuous functions. Nevertheless, Mikulevicius and Rozovskii [16] recently studied the Wick-stochastic versions of the Navier-Stokes equations using L_p solvability results with $p > d$. Extension of our results to a d-dimensional system of Burgers equations seems feasible, especially on $\mathbb{R}^d$, where the Hopf-Cole transformation reduces the system to a scalar heat equation.

Similar to [16], one can study other properties of the chaos solution of (particular cases of) (4.1), such as adaptedness and Markov property. For a number of reasons, these questions fell outside the scope of the current paper (in fact, to simplify presentation, we do not even consider a filtration and introduce random perturbation simply as a countable set of iid standard Gaussian random variables).

We now continue with an outline of the proof of Theorem 3.3. The difference between $G = \mathbb{R}$ and $G = S^1$ stops with Theorem 4.2. After that, analysis of the propagator is all about linear parabolic equations of the form

$$u_t = \mathcal{A}u + F,$$

where $\mathcal{A}$ is a linear partial differential operator. The analysis relies on the following result.

Theorem 4.3 (Linear Parabolic Regularity). *Let (V, H, V') be a normal triple of separable Hilbert spaces and denote by $[w, v]$, $w \in V'$, $v \in V$, the duality between V and V' relative to the inner product in H. Consider a collection of bounded linear operators $\mathcal{A}(t)$, $0 \leq t \leq T$, from V to V' with the following properties:*

- *For all $x, y \in V$, the function $t \mapsto [\mathcal{A}(t)x, y]$ is measurable;*
- *There exist $\varepsilon > 0$ and $C_0 \in \mathbb{R}$ such that, for all $t \in [0,T]$ and $v \in V$,*

$$[\mathcal{A}(t)v, v] + \varepsilon \|v\|_V^2 \leq C_0 \|v\|_H^2.$$

Then, for every $u_0 \in H$ and $F \in L_2((0,T);V')$, there exists a unique $u \in L_2((0,T);V)$ such that the equality

$$u(t) = u_0 + \int_0^t \mathcal{A}(s)u(s)ds + \int_0^t F(s)ds$$

holds in $L_2((0,T);V')$. *Moreover,* $u \in \mathcal{C}((0,T);H)$ *and*

$$\int_0^T \|u(t)\|_V^2 dt + \sup_{0<t<T} \|u(t)\|_H^2 \le C(\varepsilon, C_0, T)\big(\|u_0\|_H^2 + \int_0^T \|F(s)\|_{V'}^2 ds\big).$$

Proof. See, for example, Rozovskii [24, Theorem 3.1.2]. □

To study the propagator equations (4.8), we apply Theorem 4.3 with

- $V = H^2(G)$, $H = H^1(G)$, $V' = L_2(G)$;
- $\mathcal{A} = \partial^2/\partial x^2 - \mathfrak{u}(t,x)\partial/\partial x - \mathfrak{u}_x(t,x)$, where $\mathfrak{u}$ is the solution of (4.6);
- A complicated but manageable free term

$$F = F_{\boldsymbol{\alpha}} = -\sum_{(\mathbf{0})<\boldsymbol{\beta}<\boldsymbol{\alpha}} u_{\boldsymbol{\beta}} D u_{\boldsymbol{\alpha}-\boldsymbol{\beta}} + \sum_{k:\alpha_k>0} \big(a_k D^2 u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})} + b_k D u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})} + c_k u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})}\big). \tag{4.18}$$

By Theorem 4.2, the functions $\mathfrak{u}$ and $\mathfrak{u}_x$ are continuous and bounded, so that the operator $\mathcal{A}$ is indeed bounded from $H^2(G)$ to $L_2(G)$. The terms $u_{\boldsymbol{\beta}}$ have $\boldsymbol{\beta} < \boldsymbol{\alpha}$ and come from the earlier propagator equations. In particular, by Theorem 4.3 and the Sobolev embedding, $u_{\boldsymbol{\beta}} \in L^2((0,T);H^2(G)) \bigcap \mathcal{C}((0,T)\times G)$, which means that expressions $u_{\boldsymbol{\beta}} D u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}$ and $a_k D^2 u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(\boldsymbol{k})}$ are square-integrable, and therefore $F \in L_2((0,T)\times G)$.

The main technical complication is that the number of terms on the right-hand side of (4.18) grows with $|\boldsymbol{\alpha}|$, and, to satisfy (4.10), we have to control this growth. In the following section, we derive a recursive relation between the L_2 norms of $F_{\boldsymbol{\alpha}}$ for different $\boldsymbol{\alpha}$ and show that the rate of growth in $|\boldsymbol{\alpha}|$ is controlled by the *Catalan numbers* $C_{|\boldsymbol{\alpha}|-1}$ (along with the admissible factor $\mathfrak{q}^{p\boldsymbol{\alpha}}$). The exponential growth rate of the Catalan numbers ($C_k \le 4^k$) allows us to complete the proof.

5. Analysis of the propagator

In this section we carry out the analysis of the propagator (4.6)–(4.8) and complete the proof of Theorem 3.3. Our main goal is to establish (4.10).

Equation (4.6) is covered by Theorem 4.2: under the assumptions on $\varphi_{(\mathbf{0})} = \mathbb{E}u_0$ and f, we know that $\mathfrak{u}$ and $\mathfrak{u}_x$ are bounded and continuous, and

$$\mathfrak{u} \in L_2((0,T);H^2(G)).$$

Next, consider (4.7). If $\boldsymbol{\alpha} = \boldsymbol{\epsilon}_k$, we have

$$\dot{u}_{\boldsymbol{\epsilon}_k} = D\big(Du_{\boldsymbol{\epsilon}_k} - u_{(\mathbf{0})}u_{\boldsymbol{\epsilon}_k}\big) + F_{\boldsymbol{\epsilon}_k}(t,x), u_{\boldsymbol{\epsilon}_k}(0,x) = \varphi_{\boldsymbol{\epsilon}_k},$$

where

$$F_{\boldsymbol{\epsilon}_k}(t,x) = a_k D^2 u_{(\mathbf{0})} + b_k D u_{(\mathbf{0})} + c_k u_{(\mathbf{0})} + g_k$$

By Theorem 4.2 $F_{\boldsymbol{\epsilon}_k} \in L_2\left((0,T)\times G\right)$. We then solve for $u_{\boldsymbol{\epsilon}_k}$ in the normal triple $(H^2(G), H^1(G), L_2(G))$ using Theorem 4.3 and deduce (4.10) for $|\boldsymbol{\alpha}| = 1$.

As was mentioned earlier, induction on $|\boldsymbol{\alpha}|$, Sobolev embedding, and Theorem 4.3 imply that $F_{\boldsymbol{\alpha}}$, the free term in (4.8) (see also (4.18)), is an element of $L_2((0,T)\times G)$. All we need is a bound on the norm of $F_{\boldsymbol{\alpha}}$ that is consistent with (4.10).

By the Sobolev embedding, the L_2 norm of $F_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}|>1$ is controlled by the norms of $u_{\boldsymbol{\beta}}$, $\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}$, in the space $L_2((0,T);H^2(G))\bigcap \mathcal{C}((0,T);H^1(G))$. Accordingly, for $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}|\geq 1$, we define

$$L_{\boldsymbol{\alpha}}^2=\int_0^T \|u_{\boldsymbol{\alpha}}(t,\cdot)\|_2^2dt+\sup_{0<t<T}\|u_{\boldsymbol{\alpha}}(t,\cdot)\|_1^2=\|u_{\boldsymbol{\alpha}}\|^2_{L_2((0,T);H^2(G))}+\|u_{\boldsymbol{\alpha}}\|^2_{\mathcal{C}((0,T);H^1(G))}.$$

The next step is to derive a recursion for $L_{\boldsymbol{\alpha}}$.

By Theorem 4.3 applied to (4.8),

$$L_{\boldsymbol{\alpha}}^2\leq \lambda\left(\|\varphi_{\boldsymbol{\alpha}}\|_1^2+\|F_{\boldsymbol{\alpha}}\|^2_{L_2((0,T)\times G)}\right), \tag{5.1}$$

and λ, throughout, is a number depending only on T and some norms of $\mathfrak{u}$. The value of λ can change from line to line. With no loss of generality, we assume that $\lambda>1$, and, with $m=\max(\ell,r)$, put

$$L_{\boldsymbol{\epsilon(k)}}=\lambda q_k^m.$$

For $|\boldsymbol{\alpha}|>1$, (5.1), (4.18), and assumptions of Theorem 3.3 imply

$$\begin{aligned} L_{\boldsymbol{\alpha}} &\leq \lambda\Big(\mathfrak{q}^{\ell\boldsymbol{\alpha}}+\sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}\|u_{\boldsymbol{\beta}}\|_{\mathcal{C}([0,T]\times G)}\ \|u_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{L_2((0,T);H^1(G))}\\ &\qquad +\sum_{k:\alpha_k>0} q_k^r\|u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon(k)}}\|_{L_2((0,T);H^2(G))}\Big)\\ &\leq \lambda\left(\mathfrak{q}^{\ell\boldsymbol{\alpha}}+\sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}L_{\boldsymbol{\beta}}L_{\boldsymbol{\alpha}-\boldsymbol{\beta}}+\sum_{k:\alpha_k>0}q_k^rL_{\boldsymbol{\alpha}-\boldsymbol{\epsilon(k)}}\right).\end{aligned}$$

Define

$$\tilde{L}_{\boldsymbol{\alpha}}=2\lambda\left(\frac{L_{\boldsymbol{\alpha}}}{\mathfrak{q}^{m\boldsymbol{\alpha}}}+1\right).$$

Then

$$\tilde{L}_{\boldsymbol{\alpha}}\leq\sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}\tilde{L}_{\boldsymbol{\beta}}\tilde{L}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}.$$

Let $\{A_{\boldsymbol{\alpha}}:\boldsymbol{\alpha}\in\mathcal{J}\}$ be a sequence such that $A_{\boldsymbol{\epsilon_k}}=\tilde{L}_{\boldsymbol{\epsilon(k)}}$ and

$$A_{\boldsymbol{\alpha}}=\sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}A_{\boldsymbol{\beta}}A_{\boldsymbol{\alpha}-\boldsymbol{\beta}},\ |\boldsymbol{\alpha}|>1. \tag{5.2}$$

It follows by induction that

$$\tilde{L}_{\boldsymbol{\alpha}}\leq A_{\boldsymbol{\alpha}} \tag{5.3}$$

for all $\boldsymbol{\alpha}\in\mathcal{J}$. Indeed, (5.3) is true by definition if $|\boldsymbol{\alpha}|=1$. Assume that (5.3) holds for all $|\boldsymbol{\alpha}|<N$. For $|\boldsymbol{\alpha}|=N$,

$$\begin{aligned}\tilde{L}_{\boldsymbol{\alpha}} &\leq \sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}\tilde{L}_{\boldsymbol{\beta}}\tilde{L}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\\ &\leq\sum_{\mathbf{(0)}<\boldsymbol{\beta}<\boldsymbol{\alpha}}A_{\boldsymbol{\beta}}A_{\boldsymbol{\alpha}-\boldsymbol{\beta}}=A_{\boldsymbol{\alpha}},\end{aligned}$$

proving (5.3) for all $\boldsymbol{\alpha} \in \mathcal{J}$.

Our next step is to find a bound on $A_{\boldsymbol{\alpha}}$, which we do using generating functions. Since multi-indices $\boldsymbol{\alpha}$ can have non-zero entries in arbitrary positions, the argument requires an additional construction.

Let $d \geq 1$ be an integer and let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d)$ be a $d-$dimensional vector of nonnegative integers with at least one $\beta_i > 0$. Let $M_{\epsilon_i} \in \mathbb{R}$, $i = 1, 2, \ldots, d$, be prescribed positive numbers and let $M_{\boldsymbol{\beta}}$, $|\boldsymbol{\beta}| > 1$, be defined through the recursive equation

$$M^{(d)}_{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma}+\boldsymbol{\delta}=\boldsymbol{\beta}, |\boldsymbol{\gamma}|,|\boldsymbol{\delta}|>0} M^{(d)}_{\boldsymbol{\gamma}} M^{(d)}_{\boldsymbol{\delta}}.$$

Consider

$$F_d(z_1, z_2, ..., z_d) = \sum_{\boldsymbol{\beta}} M^{(d)}_{\boldsymbol{\beta}} z_1^{\beta_1} z_2^{\beta_2} \ldots z_d^{\beta_d}.$$

Then

$$\begin{aligned} F_d^2(z_1, z_2, ..., z_d) &= \sum_{|\boldsymbol{\beta}|>1} \left(\sum_{\boldsymbol{\gamma}+\boldsymbol{\delta}=\boldsymbol{\beta}, |\boldsymbol{\gamma}|,|\boldsymbol{\delta}|>0} M^{(d)}_{\boldsymbol{\gamma}} M^{(d)}_{\boldsymbol{\delta}} \right) z_1^{\beta_1} z_2^{\beta_2} \ldots z_d^{\beta_d} \\ &= \sum_{|\boldsymbol{\beta}|>1} M^{(d)}_{\boldsymbol{\beta}} z_1^{\beta_1} z_2^{\beta_2} \ldots z_d^{\beta_d} \\ &= F_d(z_1, z_2, ..., z_d) - \sum_i M^{(d)}_{\boldsymbol{\epsilon_i}} z_i, \end{aligned}$$

where $\boldsymbol{\epsilon_i}$ is the d-dimensional vector with 1 at the i^{th} coordinate and 0 everywhere else. Solving for F_d we get,

$$F_d(z_1, z_2, ..., z_d) = \frac{1 \pm \sqrt{1 - 4 \sum_i M^{(d)}_{\boldsymbol{\epsilon_i}} z_i}}{2}.$$

By comparing the coefficients of similar powers we obtain,

$$M^{(d)}_{\boldsymbol{\beta}} = \frac{1}{|\boldsymbol{\beta}|} \binom{2|\boldsymbol{\beta}| - 2}{|\boldsymbol{\beta}| - 1} \binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} \prod_i (M^{(d)}_{\boldsymbol{\epsilon_i}})^{\beta_i}; \tag{5.4}$$

here $\binom{|\boldsymbol{\beta}|}{\boldsymbol{\beta}} = |\boldsymbol{\beta}|!/\boldsymbol{\beta}!$. Let $\Gamma_d = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J} : \alpha_i = 0 \quad \text{for all} \quad i > d\}$. If $\boldsymbol{\alpha} \in \Gamma_d$ then, by (5.2), $A_{\boldsymbol{\alpha}}$ is uniquely determined by $\{A_{\boldsymbol{\beta}} : \boldsymbol{\beta} < \boldsymbol{\alpha}\}$. Also, if $\boldsymbol{\alpha} \in \Gamma_d$ and $\boldsymbol{\beta} < \boldsymbol{\alpha}$, then $\boldsymbol{\beta} \in \Gamma_d$ as well. Set $M^{(d)}_{\boldsymbol{\epsilon_i}} = A_{\boldsymbol{\epsilon_i}}$, for each $\epsilon_i \in \Gamma_d$. Then

$$M^{(d)}_{\boldsymbol{\alpha^d}} = A_{\boldsymbol{\alpha}}, \tag{5.5}$$

where $\boldsymbol{\alpha^d} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, which follows by induction on $|\boldsymbol{\alpha}|$. Indeed, equality (5.5) is true if $\boldsymbol{\alpha} \in \Gamma_d$ such that $|\boldsymbol{\alpha}| = 1$. Assume (5.5) is true for $\boldsymbol{\alpha} \in \Gamma_d$ such that $|\boldsymbol{\alpha}| < N$.

For $\boldsymbol{\alpha} \in \Gamma_d$ such that $|\boldsymbol{\alpha}| = N$,

$$\begin{aligned} A_{\boldsymbol{\alpha}} &= \sum_{\boldsymbol{\gamma}+\boldsymbol{\delta}=\boldsymbol{\alpha},|\boldsymbol{\gamma}|,|\boldsymbol{\delta}|>0} A_{\boldsymbol{\gamma}} A_{\boldsymbol{\delta}} \\ &= \sum_{\boldsymbol{\gamma}^{\boldsymbol{d}}+\boldsymbol{\delta}^{\boldsymbol{d}}=\boldsymbol{\alpha}^{\boldsymbol{d}},|\boldsymbol{\gamma}^{\boldsymbol{d}}|,|\boldsymbol{\delta}^{\boldsymbol{d}}|>0} M^{(d)}_{\boldsymbol{\gamma}^{\boldsymbol{d}}} M^{(d)}_{\boldsymbol{\delta}^{\boldsymbol{d}}} \\ &= M^{(d)}_{\boldsymbol{\alpha}^{\boldsymbol{d}}}, \end{aligned}$$

and the general equality (5.5) follows.

Given any $\boldsymbol{\alpha} \in \mathcal{J}$ there exists a $d \in \mathbb{N}$ such that $\boldsymbol{\alpha} \in \Gamma_d$. Then for such a d, $M^{(d)}_{\boldsymbol{\alpha}^{\boldsymbol{d}}} = A_{\boldsymbol{\alpha}}$. Using (5.4),

$$A_{\boldsymbol{\alpha}} = \frac{1}{|\boldsymbol{\alpha}|}\binom{2|\boldsymbol{\alpha}|-2}{|\boldsymbol{\alpha}|-1}\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}}\prod_i A^{\alpha_i}_{\boldsymbol{\epsilon_i}}. \tag{5.6}$$

We now notice that

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \ n \geq 0,$$

is the nth Catalan number, a very popular combinatorial object (see, for example, Stanley [26]). Hence, retracing our arguments back to $L_{\boldsymbol{\alpha}}$, we conclude that

$$L_{\boldsymbol{\alpha}} \leq (2\lambda)^{2|\boldsymbol{\alpha}|} C_{|\boldsymbol{\alpha}|-1}\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}}\mathfrak{q}^{m\boldsymbol{\alpha}}. \tag{5.7}$$

The final two observations are

- $C_n \leq 4^n$, which follows by the Stirling formula;
- $\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \leq \mathfrak{q}^{\boldsymbol{\alpha}}$, which we proved in Proposition 2.3.

Therefore, $L_{\boldsymbol{\alpha}} \leq (8\lambda)^{|\boldsymbol{\alpha}|}\mathfrak{q}^{(m+1)\boldsymbol{\alpha}}$, which establishes (4.10) and completes the proof of Theorem 3.3.

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