# Bilinear Stochastic Elliptic Equations

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Abstract

We study stochastic elliptic PDEs driven by multiplicative Gaussian white noise. Even the simplest equations driven by this noise often do not have a square-integrable solution and must be solved in special weighted spaces. We demonstrate that the Cameron-Martin version of the Wiener chaos decomposition is an effective tool to study such equations and present the corresponding solvability results.

## 1. Introduction

The objective of this paper is to study linear stochastic elliptic equations with multiplicative noise, also known as bi-linear equations. While stochastic elliptic equations, both linear and nonlinear, with additive noise are relatively well-studied (see, for example, [1, 8, 10, 12]), a lot less is known about elliptic equations with multiplicative noise. There are two major difficulties in studying such equations: (a) absence of time evolution complicates a "natural" definition of the stochastic integral; (b) with essentially any definition of the stochastic integral, the solution of the equation is not a square-integrable random field.

In this paper, both difficulties are resolved by considering the equation in a suitable weighted chaos space and defining the integral as an extension of the divergence operator (or Skorokhod integral) from the Malliavin calculus; the resulting stochastic integral is closely connected with the Wick product $\diamond$ and keeps the random perturbation zero on average. The approach also allows us to abandon the usual subordination of the operators in the stochastic part of the equation to the operator in the deterministic part, and to consider equations of full second order, such as the Poisson equation in random medium

$$\sum_{i,j=1}^{d}\left(\frac{\partial}{\partial x_i}\Big(a_{ij}(x)+\varepsilon_{ij}\dot W(x)\Big)\diamond\frac{\partial u(x)}{\partial x_j}\right)=f(x), \tag{1.1}$$

with $\varepsilon_{ij}\dot W$ representing the (small) random variations in the properties of the medium.

## 2. Weighted Chaos Spaces

In this section, we introduce the main notations and tools from the Malliavin calculus.

Let $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathcal{U}$, a real separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{U}}$ and an orthonormal basis $\mathfrak{U} = \{\mathfrak{u}_k,\ k \geq 1\}$. A Gaussian white noise$\dot{W}$ on $\mathcal{U}$ is a collection of zero-mean Gaussian random variables $\{\dot{W}(h),\ h \in \mathcal{U}\}$ such that $\mathbb{E}\big(\dot{W}(h_1)\dot{W}(h_2)\big) = (h_1, h_2)_{\mathcal{U}}$. In particular, $\xi_k = \dot{W}(\mathfrak{u}_k),\ k \geq 1$, are iid standard Gaussian random variables. We assume that $\mathcal{F}$ is the $\sigma$-algebra generated by $\dot{W}(h),\ h \in \mathcal{U}$.

Let $\mathcal{J}$ be the collection of multi-indices $\alpha$ with $\alpha = (\alpha_1, \alpha_2, \ldots)$ so that each $\alpha_k$ is a non-negative integer and $|\alpha| := \sum_{k\geq 1} \alpha_k < \infty$. An alternative way to describe a multi-index $\alpha$ with $|\alpha| = n > 0$ is by its characteristic set $K_\alpha$, that is, an ordered $n$-tuple $K_\alpha = \{k_1, \ldots, k_n\}$, where $k_1 \leq k_2 \leq \ldots \leq k_n$ characterize the locations and the values of the non-zero elements of $\alpha$. More precisely, $k_1$ is the index of the first non-zero element of $\alpha$, followed by $\max(0, \alpha_{k_1} - 1)$ of entries with the same value. The next entry after that is the index of the second non-zero element of $\alpha$, followed by $\max(0, \alpha_{k_2} - 1)$ of entries with the same value, and so on. For example, if $n = 7$ and $\alpha = (1, 0, 2, 0, 0, 1, 0, 3, 0, \ldots)$, then the non-zero elements of $\alpha$ are $\alpha_1 = 1$, $\alpha_3 = 2$, $\alpha_6 = 1$, $\alpha_8 = 3$. As a result, $K_\alpha = \{1, 3, 3, 6, 8, 8, 8\}$, that is, $k_1 = 1$, $k_2 = k_3 = 3$, $k_4 = 6, k_5 = k_6 = k_7 = 8$.

We will use the following notations:

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad \alpha! = \prod_{k\geq 1} \alpha_k!, \quad \mathbb{N}^{q\alpha} = \prod_{k\geq 1} k^{q\alpha_k},\ q \in \mathbb{R}.$$

By $(0)$ we denote the multi-index with all zeroes. By $\varepsilon_i$ we denote the multi-index $\alpha$ with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. With this notation, $n\varepsilon_i$ is the multi-index $\alpha$ with $\alpha_i = n$ and $\alpha_j = 0$ for $j \neq i$. The following two results are often useful:

$$|\alpha|! \leq \alpha!(2\mathbb{N})^{2\alpha}; \tag{2.1}$$

(see [3, page 35]), and

$$\sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{q\alpha} < \infty \ \text{ if and only if } q < -1 \tag{2.2}$$

(see [3, Proposition 2.3.3] or [5, Proposition 7.1]).

Define the collection of random variables $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ as follows:

$$\xi_\alpha = \prod_k \left( \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right), \tag{2.3}$$

where $\xi_k = \dot{W}(\mathfrak{u}_k)$ and

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{2.4}$$

is Hermite polynomial of order $n$.

Given a real separable Hilbert space $X$ and sequence $\mathcal{R} = \{r_\alpha,\ \alpha \in \mathcal{J}\}$ of positive numbers, we define the space $\mathcal{R}L_2(\mathbb{F}; X)$ as the collection of formal series $f = \sum_{\alpha \in \mathcal{J}} f_\alpha \xi_\alpha$, $f_\alpha \in X$, such that $\|f\|^2_{\mathcal{R}L_2(\mathbb{F};X)} := \sum_\alpha \|f_\alpha\|^2_X r^2_\alpha < \infty$. In particular, $\mathcal{R}f = \sum_{\alpha \in \mathcal{J}} r_\alpha f_\alpha \xi_\alpha \in L_2(\mathbb{F}; X)$. Similarly, the space $\mathcal{R}^{-1}L_2(\mathbb{F}; X)$ corresponds to the sequence $\mathcal{R}^{-1} = \{1/r_\alpha,\ \alpha \in \mathcal{J}\}$. For $f \in \mathcal{R}L_2(\mathbb{F}; X)$ and $g \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathbb{R})$ we define

$$\langle\!\langle f, g \rangle\!\rangle := \mathbb{E}\big((\mathcal{R}f)(\mathcal{R}^{-1}g)\big) \in X. \tag{2.5}$$

Important particular cases of the space $\mathcal{R}L_2(\mathbb{F}; X)$ correspond to the following weights: (a) $r^2_\alpha = \prod_{k=1}^\infty q_k^{\alpha_k}$, where $\{q_k,\ k \geq 1\}$ is a non-increasing sequence of positive numbers with $q_1 \leq 1$ (see [6, 9]); (b) Kondratiev's spaces $(\mathcal{S})_{\rho,\ell}(X)$ (see [2, 3]):

$$r^2_\alpha = (\alpha!)^\rho (2\mathbb{N})^{\ell\alpha},\ \rho \leq 0,\ \ell \leq 0. \tag{2.6}$$

The `divergence operator` $\delta$ is defined as a linear operator from $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$ to $\mathcal{R}L_2(\mathbb{F}; X)$ in the same way as in the usual Malliavin calculus. In particular, for $\xi_\alpha \in \Xi$, $h \in X$, and $\mathfrak{u}_k \in \mathfrak{U}$, we have

$$\delta(\xi_\alpha\, h \otimes \mathfrak{u}_k) = h\,\sqrt{\alpha_k + 1}\,\xi_{\alpha+\varepsilon_k}. \tag{2.7}$$

For $\xi_\alpha$, $\xi_\beta$ from $\Xi$, the `Wick product` is defined by

$$\xi_\alpha \diamond \xi_\beta := \sqrt{\left(\frac{(\alpha+\beta)!}{\alpha!\beta!}\right)}\xi_{\alpha+\beta}, \tag{2.8}$$

and then extended by linearity to $\mathcal{R}L_2(\mathbb{F};X)$. It follows from (2.7) and (2.8) that

$$\delta(\xi_\alpha h \otimes \mathfrak{u}_k) = h\xi_\alpha \diamond \xi_k,\ \ h \in X. \tag{2.9}$$

More generally, we have

**Theorem 2.1.** *If $f$ is an element of $\mathcal{R}L_2(\mathbb{F};X\otimes\mathcal{U})$ so that $f=\sum_{k\geq 1} f_k\otimes\mathfrak{u}_k$, with $f_k=\sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\,\xi_\alpha\in\mathcal{R}L_2(\mathbb{F};X)$, then*

$$\delta(f) = \sum_{k\geq 1} f_k \diamond \xi_k := f\diamond \dot{W}, \tag{2.10}$$

$$(\delta(f))_\alpha = \sum_{k\geq 1}\sqrt{\alpha_k}f_{k,\alpha-\varepsilon_k}, \tag{2.11}$$

*and $\delta(f)\in\bar{\mathcal{R}}L_2(\mathbb{F};X)$, where, for $|\alpha|>0$, $\bar{r}_\alpha = r_\alpha/\sqrt{|\alpha|}$.*

Proof – By linearity and (2.9),

$$\delta(f)=\sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}}\delta(\xi_\alpha f_{k,\alpha}\otimes\mathfrak{u}_k) = \sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\xi_\alpha\diamond\xi_k = \sum_{k\geq 1} f_k\diamond\xi_k,$$

which is (2.10). On the other hand, by (2.7),

$$\delta(f)=\sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\sqrt{\alpha_k+1}\,\xi_{\alpha+\varepsilon_k} = \sum_{\alpha\in\mathcal{J}}\sum_{k\geq 1} f_{k,\alpha-\varepsilon_k}\sqrt{\alpha_k}\,\xi_\alpha,$$

and (2.11) follows. □

## 3. Abstract Elliptic Equations

The objective of this section is to study stationary stochastic equation

$$\mathbf{A}u+\delta(\mathbf{M}u)=f \tag{3.1}$$

in a normal triple $(V,H,V')$ of Hilbert spaces.

**Definition 3.1.** *The solution of equation (3.1) with $f\in\mathcal{R}L_2(\mathbb{F};V')$, is a random element $u\in\mathcal{R}L_2(\mathbb{F};V)$ so that, for every $\varphi$ satisfying $\varphi\in\mathcal{R}^{-1}L_2(\mathbb{F};\mathbb{R})$ and $\mathbf{D}\varphi\in\mathcal{R}^{-1}L_2(\mathbb{F};\mathcal{U})$, the equality*

$$\langle\!\langle \mathbf{A}u,\varphi\rangle\!\rangle + \langle\!\langle \delta(\mathbf{M}u),\varphi\rangle\!\rangle = \langle\!\langle f,\varphi\rangle\!\rangle \tag{3.2}$$

*holds in $V'$.*

Fix an orthonormal basis $\mathfrak{U}$ in $\mathcal{U}$ and use (2.10) to rewrite (3.1) as

$$\mathbf{A}u + (\mathbf{M}u) \diamond \dot{W} = f, \tag{3.3}$$

where

$$\mathbf{M}u \diamond \dot{W} := \sum_{k\geq 1} \mathbf{M}_k u \diamond \xi_k. \tag{3.4}$$

Taking $\varphi = \xi_\alpha$ in (3.2) and using relation (2.11) we conclude that

$$u = \sum_{\alpha\in\mathcal{J}} u_\alpha \xi_\alpha$$

is a solution of equation (3.1) if and only if $u_\alpha$ satisfies

$$\mathbf{A}u_\alpha + \sum_{k\geq 1} \sqrt{\alpha_k}\, \mathbf{M}_k u_{\alpha-\varepsilon_k} = f_\alpha \tag{3.5}$$

in the normal triple $(V, H, V')$. This system of equation is lower-triangular and can be solved by induction on $|\alpha|$.

The following example shows the limitations on the "quality" of the solution of equation (3.1).

*Example 3.1* - Consider equation

$$u = 1 + u \diamond \xi. \tag{3.6}$$

Write $u = \sum_{n\geq 0} u_{(n)} H_n(\xi)/\sqrt{n!}$, where $H_n$ is Hermite polynomial of order $n$ (2.4). Then (3.5) implies $u_{(n)} = I_{(n=0)} + \sqrt{n}u_{(n-1)}$ or $u_{(0)} = 1$, $u_{(n)} = \sqrt{n!}$, $n \geq 1$, or $u = 1 + \sum_{n\geq 1} H_n(\xi)$. Clearly, the series does not converge in $L_2(\mathbb{F})$, but does converge in $(\mathcal{S})_{-1,q}$ for every $q < 0$ (see (2.6)). As a result, even a simple stationary equation (3.6) can be solved only in weighted spaces.

**Theorem 3.1.** *Consider equation (3.3) in which $f \in \bar{\mathcal{R}}L_2(\mathbb{F}; V')$ for some $\bar{\mathcal{R}}$.*

*Assume that the deterministic equation $\mathbf{A}U = F$ is uniquely solvable in the normal triple $(V, H, V')$, that is, for every $F \in V'$, there exists a unique solution $U = \mathbf{A}^{-1}F \in V$ so that $\|U\|_V \leq C_A\|F\|_{V'}$. Assume also that each $\mathbf{M}_k$ is a bounded linear operator from $V$ to $V'$ so that, for all $v \in V$*

$$\|\mathbf{A}^{-1}\mathbf{M}_k v\|_V \leq C_k \|v\|_V, \tag{3.7}$$

*with $C_k$ independent of $v$.*

*Then there exists an operator $\mathcal{R}$ and a unique solution $u \in \mathcal{R}L_2(\mathbb{F}; V)$ of (3.1).*

PROOF – By assumption, equation (3.5) has a unique solution $u_\alpha \in V$ for every $\alpha \in \mathcal{J}$. Then direct computations show that one can take

$$r_\alpha = \min\left(\bar{r}_\alpha, \frac{(2\mathbb{N})^{-\kappa\alpha}}{1 + \|u_\alpha\|_V}\right), \quad \kappa > 1/2.$$

□

*Remark 3.1* - The assumption of the theorem about solvability of the deterministic equation holds if the operator $\mathbf{A}$ satisfies $\langle \mathbf{A}v, v\rangle \geq \kappa\|v\|_V^2$ for every $v \in V$, with $\kappa > 0$ independent of $v$.

While Theorem 3.1 establishes that, under very broad assumptions, one can find an operator $\mathcal{R}$ such that equation (3.1) has a unique solution in $\mathcal{R}L_2(\mathbb{F}; V)$, the choice of the operator $\mathcal{R}$ is not sufficiently explicit (because of the presence of $\|u_\alpha\|_V$) and is not necessarily optimal.

Consider equation (3.1) with non-random $f$ and $u_0$. In this situation, it is possible to find more constructive expression for $r_\alpha$ and to derive explicit formulas, both for $\mathcal{R}u$ and for each individual $u_\alpha$, using multiple integrals.

Introduce the following notation to write the multiple integrals:

$$\delta_{\mathbf{B}}^{(0)}(\eta) = \eta, \; \delta_{\mathbf{B}}^{(n)}(\eta) = \delta(\mathbf{B}\delta_{\mathbf{B}}^{(n-1)}(\eta)), \; \eta \in \mathcal{R}L_2(\mathbb{F}; V),$$

where $\mathbf{B}$ is a bounded linear operator from $V$ to $V \otimes \mathcal{U}$.

**Theorem 3.2.** *Under the assumptions of Theorem 3.1, if $f$ is non-random, then the following holds:*

1. *the coefficient $u_\alpha$, corresponding to the multi-index $\alpha$ with $|\alpha| = n \geq 1$ and the characteristic set $K_\alpha = \{k_1, \ldots, k_n\}$, is given by*

$$u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \mathbf{B}_{k_{\sigma(n)}} \cdots \mathbf{B}_{k_{\sigma(1)}} u_{(0)}, \tag{3.8}$$

   *where*

- *$\mathcal{P}_n$ is the permutation group of the set $(1, \ldots, n)$;*
- *$\mathbf{B}_k = -\mathbf{A}^{-1}\mathbf{M}_k$;*
- *$u_{(0)} = \mathbf{A}^{-1} f$.*

2. *the operator $\mathcal{R}$ can be defined by the weights $r_\alpha$ in the form*

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|}\sqrt{|\alpha|!}}, \text{ where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \tag{3.9}$$

*where the numbers $q_k$, $k \geq 1$ are chosen so that $\sum_{k\geq 1} q_k^2 k^2 C_k^2 < 1$, and $C_k$ are defined in (3.7).*

3. *With $r_\alpha$ and $q_k$ defined by (3.9),*

$$\sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha = \delta^{(n)}_{\overline{\mathbf{B}}}(\mathbf{A}^{-1} f), \tag{3.10}$$

*where $\overline{\mathbf{B}} = -(q_1\mathbf{A}^{-1}\mathbf{M}_1, q_2\mathbf{A}^{-1}\mathbf{M}_2, \ldots)$, and*

$$\mathcal{R}u = \mathbf{A}^{-1} f + \sum_{n\geq 1} \frac{1}{2^n\sqrt{n!}}\, \delta^{(n)}_{\overline{\mathbf{B}}}(\mathbf{A}^{-1} f), \tag{3.11}$$

Proof – Define $\widetilde{u}_\alpha = \sqrt{\alpha!}\, u_\alpha$. If $f$ is deterministic, then $\widetilde{u}_{(0)} = \mathbf{A}^{-1} f$ and, for $|\alpha| \geq 1$,

$$\mathbf{A}\widetilde{u}_\alpha + \sum_{k\geq 1} \alpha_k \mathbf{M}_k \widetilde{u}_{\alpha-\varepsilon_k} = 0,$$

or

$$\widetilde{u}_\alpha = \sum_{k\geq 1} \alpha_k \mathbf{B}_k \widetilde{u}_{\alpha-\varepsilon_k} = \sum_{k\in K_\alpha} \mathbf{B}_k \widetilde{u}_{\alpha-\varepsilon_k},$$

where $K_\alpha = \{k_1, \ldots, k_n\}$ is the characteristic set of $\alpha$ and $n = |\alpha|$. By induction on $n$,

$$\widetilde{u}_\alpha = \sum_{\sigma\in\mathcal{P}_n} \mathbf{B}_{k_{\sigma(n)}} \cdots \mathbf{B}_{k_{\sigma(1)}} u_{(0)},$$

and (3.8) follows.

Next, define

$$U_n = \sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha, \ n \geq 0.$$

Let us first show that, for each $n \geq 1$, $U_n \in L_2(\mathbb{F}; V)$. By (3.8) we have

$$\|u_\alpha\|_V^2 \leq C_A^2 \frac{(|\alpha|!)^2}{\alpha!} \|f\|_{V'}^2 \prod_{k \geq 1} C_k^{\alpha_k}. \tag{3.12}$$

By (2.1),

$$\begin{aligned}
\sum_{|\alpha|=n} q^{2\alpha} \|u_\alpha\|_V^2 &\leq C_A^2 2^{2n}\, n! \sum_{|\alpha|=n} \prod_{k \geq 1} (kC_k q_k)^{2\alpha_k} \\
&= C_A^2 2^{2n}\, n! \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,
\end{aligned}$$

because of the selection of $q_k$, and so $U_n \in L_2(\mathbb{F}; V)$. If the weights $r_\alpha$ are defined by (3.9), then

$$\sum_{\alpha \in \mathcal{J}} r_\alpha^2 \|u\|_V^2 = \sum_{n \geq 0} \sum_{|\alpha|=n} r_\alpha^2 \|u\|_V^2 \leq C_A^2 \sum_{n \geq 0} \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,$$

because of the assumption $\sum_{k \geq 1} k^2 C_k^2 q_k^2 < 1$.

Since (3.11) follows directly from (3.10), it remains to establish (3.10), that is,

$$U_n = \delta_{\overline{\mathbf{B}}}(U_{n-1}), \ n \geq 1. \tag{3.13}$$

For $n = 1$ we have

$$U_1 = \sum_{k \geq 1} q_k u_{\varepsilon_k} \xi_k = \sum_{k \geq 1} \overline{\mathbf{B}}_k u_{(0)} \xi_k = \delta_{\overline{\mathbf{B}}}(U_0),$$

where the last equality follows from (2.10). More generally, for $n > 1$ we have by definition of $U_n$ that

$$(U_n)_\alpha = \begin{cases} q^\alpha u_\alpha, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases}$$

From the equation

$$q^\alpha \mathbf{A} u_\alpha + \sum_{k \geq 1} q_k \sqrt{\alpha_k}\, \mathbf{M}_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k} = 0$$

we find

$$(U_n)_\alpha = \begin{cases} \sum_{k\geq 1} \sqrt{\alpha_k}\, q_k \mathbf{B}_k q^{\alpha-\varepsilon_k} u_{\alpha-\varepsilon_k}, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases}$$
$$= \sum_{k\geq 1} \sqrt{\alpha_k}\ \overline{\mathbf{B}}_k (U_{n-1})_{\alpha-\varepsilon_k},$$

and then (3.13) follows from (2.11). Theorem 3.2 is proved. □

Here is another result about solvability of (3.3), this time with random $f$. We use the space $(\mathcal{S})_{\rho,q}$, defined by the weights (2.6).

**Theorem 3.3.** *In addition to the assumptions of Theorem 3.1, let $C_A \leq 1$ and $C_k \leq 1$ for all $k$. If $f \in (\mathcal{S})_{-1,-\ell}(V')$ for some $\ell > 1$, then there exists a unique solution $u \in (\mathcal{S})_{-1,-\ell-4}(V)$ of (3.3) and*

$$\|u\|_{(\mathcal{S})_{-1,-\ell-4}(V)} \leq C(\ell)\|f\|_{(\mathcal{S})_{-1,-\ell}(V')}. \tag{3.14}$$

PROOF – Denote by $u(g;\gamma)$, $\gamma \in \mathcal{J}$, $g \in V'$, the solution of (3.3) with $f_\alpha = gI_{(\alpha=\gamma)}$, and define $\bar{u}_\alpha = (\alpha!)^{-1/2}u_\alpha$. Clearly, $u_\alpha(g,\gamma) = 0$ if $|\alpha| < |\gamma|$ and so

$$\sum_{\alpha\in\mathcal{J}} \|u_\alpha(f_\gamma;\gamma)\|_V^2 r_\alpha^2 = \sum_{\alpha\in\mathcal{J}} \|u_{\alpha+\gamma}(f_\gamma;\gamma)\|_V^2 r_{\alpha+\gamma}^2. \tag{3.15}$$

It follows from (3.5) that

$$\bar{u}_{\alpha+\gamma}(f_\gamma;\gamma) = \bar{u}_\alpha\big(f_\gamma(\gamma!)^{-1/2};(0)\big). \tag{3.16}$$

Now we use (3.12) to conclude that

$$\|\bar{u}_{\alpha+\gamma}(f_\gamma;\gamma)\|_V \leq \frac{|\alpha|!}{\sqrt{\alpha!\gamma!}}\|f\|_{V'}. \tag{3.17}$$

Coming back to (3.15) with $r_\alpha^2 = (\alpha!)^{-1}(2\mathbb{N})^{(-\ell-4)\alpha}$ and using inequality (2.1) we find:

$$\|u(f_\gamma;\gamma)\|_{(\mathcal{S})_{-1,-\ell-4}(V)} \leq C(\ell)(2\mathbb{N})^{-2\gamma}\,\frac{\|f_\gamma\|_{V'}}{(2\mathbb{N})^{(\ell/2)\gamma}\sqrt{\gamma!}},$$

where

$$C(\ell) = \left( \sum_{\alpha \in \mathcal{J}} \left( \frac{|\alpha|!}{\alpha!} \right)^2 (2\mathbb{N})^{(-\ell-4)\alpha} \right)^{1/2};$$

(2.2) and (2.1) imply $C(\ell) < \infty$. Then (3.14) follows by the triangle inequality after summing over all $\gamma$ and using the Cauchy-Schwartz inequality. $\square$

*Remark 3.2* - Example 3.1, in which $f \in (\mathcal{S})_{0,0}$ and $u \in (\mathcal{S})_{-1,q}$, $q < 0$, shows that, while the results of Theorem 3.3 are not sharp, a bound of the type $\|u\|_{(\mathcal{S})_{\rho,q}(V)} \leq C\|f\|_{(\mathcal{S})_{\rho,\ell}(V')}$ is, in general, impossible if $\rho > -1$ or $q \geq \ell$.

## 4. Elliptic SPDEs of the Full Second Order

Let $G$ be a smooth bounded domain in $\mathbb{R}^d$ and $\{h_k,\ k \geq 1\}$, an orthonormal basis in $L_2(G)$. We assume that

(4.1)
$$\sup_{x \in G} |h_k(x)| \leq c_k,\ k \geq 1.$$

A space white noise on $L_2(G)$ is a formal series

(4.2)
$$\dot{W}(x) = \sum_{k \geq 1} h_k(x) \xi_k,$$

where $\xi_k,\ k \geq 1$, are independent standard Gaussian random variables.

Consider the following Dirichlet problem:

(4.3)
$$\begin{gathered} -D_i\Big(a_{ij}(x)\, D_j u(x)\Big) + \\ D_i\Big(\sigma_{ij}(x)\, D_j(u(x))\Big) \diamond \dot{W}(x) = f(x),\ x \in G, \\ u|_{\partial G} = 0, \end{gathered}$$

where $\dot{W}$ is the space white noise (4.2) and $D_i = \partial/\partial x_i$. Assume that the functions $a_{ij}$, $\sigma_{ij}$, $f$, and $g$ are non-random. For brevity, in (4.3) and in similar expressions below we use the summation convention and assume summation over the repeated indices.

We make the following assumptions:

**E1** The functions $a_{ij} = a_{ij}(x)$ and $\sigma_{ij} = \sigma_{ij}(x)$ are measurable and bounded in the closure $\bar{G}$ of $G$.

**E2** There exist positive numbers $A_1, A_2$ so that $A_1|y|^2 \le a_{ij}(x)y_iy_j \le A_2|y|^2$ for all $x \in \bar{G}$ and $y \in \mathbb{R}^d$.

**E3** The functions $h_k$ in (4.2) are bounded and Lipschitz continuous.

Clearly, equation (4.3) is a particular case of equation (3.3) with

$$\mathbf{A}u(x) := -D_i\Big(a_{ij}\,(x)\,D_j u\,(x)\Big) \tag{4.4}$$

and

$$\mathbf{M}_k u(x) := h_k(x)\,D_i\Big(\sigma_{ij}\,(x)\,D_j u\,(x)\Big). \tag{4.5}$$

Assumptions **E1** and **E3** imply that each $\mathbf{M}_k$ is a bounded linear operator from $\overset{\circ}{H}{}_2^1(G)$ to $H_2^{-1}(G)$. Moreover, it is a standard fact that under the assumptions **E1** and **E2** the operator $\mathbf{A}$ is an isomorphism from $V$ onto $V'$ (see e.g. [4]). Therefore, for every $k$ there exists a positive number $C_k$ such that

$$\left\|\mathbf{A}^{-1}M_k v\right\|_V \le C_k \left\|v\right\|_V, \ v \in V. \tag{4.6}$$

**Theorem 4.1.** *Under the assumptions **E1** and **E2**, if $f \in H_2^{-1}(G)$, then there exists a unique solution of the Dirichlet problem (4.3) $u \in \mathcal{R}L_2(\mathbb{F}; \overset{\circ}{H}{}_2^1(G))$ such that*

$$\|u\|_{\mathcal{R}L_2(\mathbb{F};\overset{\circ}{H}{}_2^1(G))} \le C \cdot \|f\|_{H_2^{-1}(G)}. \tag{4.7}$$

*The weights $r_\alpha$ can be taken in the form*

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|}\sqrt{|\alpha|!}}, \text{ where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \tag{4.8}$$

*and the numbers $q_k$, $k \ge 1$ are chosen so that $\sum_{k\ge1} C_k^2 q_k^2 k^2 < 1$, with $C_k$ from (4.6).*

Proof – This follows from Theorem 3.2. □

*Remark 4.1* - With an appropriate change of the boundary conditions, and with extra regularity of the basis functions $h_k$, the results of Theorem 4.1 can be extended to stochastic elliptic equations of order $2m$. The corresponding operators are

$$\mathbf{A}u = (-1)^m D_{i_1} \cdots D_{i_m} \Big( a_{i_1 \dots i_m j_1 \dots j_m} (x) D_{j_1} \cdots D_{j_m} u(x) \Big), \tag{4.9}$$

$$\mathbf{M}_k u = h_k(x) D_{i_1} \cdots D_{i_m} \Big( \sigma_{i_1 \dots i_m j_1 \dots j_m} (x) D_{j_1} \cdots D_{j_m} u(x) \Big). \tag{4.10}$$

Since $G$ is a smooth bounded domain, regularity of $h_k$ is not a problem: we can take $h_k$ as the eigenfunctions of the Dirichlet Laplacian in $G$. Equation (1.1) is also covered, with $\mathbf{A} = D_i(a_{ij}D_j u)$ and $\mathbf{M}_k u = h_k \varepsilon_{ij} D_{ij} u + \varepsilon_{ij}(D_i h_k)(D_j u)$.

Some related results and examples could be found in [7, 11]

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