Estimating Speed and Damping in the Stochastic Wave Equation

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Contents

Abstract

A parameter estimation problem is considered for a one-dimensional stochastic wave equation driven by additive space-time Gaussian white noise. The estimator is of spectral type and utilizes a finite number of the spatial Fourier coefficients of the solution. The asymptotic properties of the estimator are studied as the number of the Fourier coefficients increases, while the observation time and the noise intensity are fixed.

1. Introduction

Consider the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \theta_1 \frac{\partial^2 u}{\partial x^2} + \theta_2 \frac{\partial u}{\partial t} + \dot{W}(t),\ 0 < t < T,\ 0 < x < \pi, \tag{1.1}$$

with zero initial boundary conditions, driven by space-time white noise $\dot{W}$. The solution of this equation can be written as a Fourier series

$$u(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} u_k(t) \sin(kx).$$

The objective is to construct and investigate the maximum likelihood estimators of the unknown numbers $\theta_1 > 0$ and $\theta_2 \in \mathbb{R}$, given $\{u_1(t), \dots, u_N(t)\}$, $t \in [0,T]$, the first N Fourier coefficients of the solution.

A similar problem for stochastic parabolic equations is relatively well studied, with the first result announced in the paper by Huebner, Khasminskii, and Rozovskii [3]. While most of the existing work in the parabolic setting has been about estimating either a single parameter [3, 6, 13] or a function of time [4, 5], estimation of several parameters in parabolic equations has also been studied [2, 11]. A more detailed survey of the existing results is in [12].

In the parabolic setting, the observations, being Fourier coefficients of the solution, are essentially discrete in space, but are usually continuous in time. While such observations might not always be available in reality, absence of time discretization makes it possible to isolate and study the infinite-dimensional effects of the model. Continuous in time observations ensure that all the estimators are available in closed form and the only asymptotic parameter is the number of the Fourier coefficients. Estimation from discrete-time observations is a somewhat different problem and so far has only being studied in [14].

Accordingly, in this paper we adopt the traditional approach from the parabolic setting and assume that the spatial Fourier coefficients of the solution of (1.1) are observed in continuous time. The main result of the paper, which can be viewed as an extension of [2] to hyperbolic equations, is as follows.

Theorem 1.1. *The (joint) maximum likelihood estimator of the parameters θ_1, θ_2 is strongly consistent and asymptotically normal as $N \to \infty$. The normalizing matrix is diagonal, with the diagonal elements $N^{3/2}$ and $N^{1/2}$; these elements specify the rate of convergence of the estimator to θ_1 and θ_2, respectively.*

This theorem is proved in Section 3. In Section 2, we establish existence, uniqueness, and regularity of the solution of (1.1).

Throughout the presentation below, we fix a stochastic basis

$$\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$$

with the usual assumptions (completeness of $\mathcal{F}_0$ and right-continuity of $\mathcal{F}_t$). We also assume that $\mathbb{F}$ is large enough to support countably many independent standard Brownian motions. For a random variable ξ, $\mathbb{E}\xi$ denotes the expectation. $\mathbb{R}^n$ is an n-dimensional Euclidean space; $\mathcal{C}(A; B)$ is the space of continuous functions from A to B; $\mathcal{N}(m, \sigma^2)$ is a Gaussian random variable with mean m and variance σ^2.

Finally, for the convenience of the reader, we recall that a cylindrical Brownian motion $W = W(t)$, $t \geq 1$, over (or on) a Hilbert space H is a linear mapping

$$W : f \mapsto W_f(\cdot)$$

from H to the space of zero-mean Gaussian processes such that, for every $f, g \in H$ and $t, s > 0$,

$$\mathbb{E}\big(W_f(t)W_g(s)\big) = \min(t, s)(f, g)_H. \tag{1.2}$$

If $\{h_k,\ k \geq 1\}$ is an orthonormal basis in H and $w_k,\ k \geq 1$, are independent standard Brownian motions, then

$$f \mapsto \sum_{k\geq 1} (f, h_k)_H w_k(t) \tag{1.3}$$

is a cylindrical Brownian motion. Thus, a cylindrical Brownian motion W is often represented by a generalized Fourier series

$$W(t) = \sum_{k\geq 1} w_k(t) h_k, \tag{1.4}$$

where $w_k = W_{h_k}$. The corresponding space-time white noise is then

$$\dot{W}(t) = \sum_{k\geq 1} \dot{w}_k(t) h_k.$$

2. Stochastic Wave Equation

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - 2b \frac{\partial u}{\partial t} + \dot{W}(t),\ 0 < t < T,\ 0 < x < \pi, \tag{2.1}$$

where W is a cylindrical Brownian motion over $L_2((0, \pi))$. For simplicity, we assume

$$a \geq 1, \quad 2|b| \leq 1; \tag{2.2}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0; \tag{2.3}$$

see Remark 2.1 below about relaxing these assumptions. In physical models, $a > 0$ represents the speed of the wave and b characterizes damping (amplification, if $b < 0$).

For $\gamma \in \mathbb{R}$, define the Hilbert space H^γ as the closure of the set of smooth compactly supported functions on $(0, \pi)$ with respect to the norm

$$\|f\|_\gamma = \left(\sum_{k\geq 1} k^{2\gamma} f_k^2\right)^{1/2}, \text{ where } f_k = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin(kx)\, dx. \tag{2.4}$$

Note that each of the functions $\sin(kx)$ belongs to every H^γ, and if f is twice continuously-differentiable on $(0,\pi)$ with $f(0)=f(\pi)=0$, then, after two integrations by parts, $|f_k| \leq k^{-2}\sup_{x\in(0,\pi)}|f''(x)|$, so that, in particular, $f \in H^1$. More generally, every $f \in H^\gamma$ can be identified with a sequence $\{f_k,\ k\geq 1\}$ of real numbers such that $\sum_{k\geq 1} k^{2\gamma} f_k^2 < \infty$. Even though f is a generalized function when $\gamma < 0$, we will still occasionally write $f = f(x)$, keeping in mind a generalized Fourier series representation $f(x) = \sqrt{2/\pi}\sum_{k\geq 1} f_k \sin(kx)$.

Given $\gamma > 0$, $f \in H^{-\gamma}$ and $g \in H^\gamma$, we define

$$(f,g) = \sum_{k\geq 1} f_k g_k;$$

if $f, g \in L_2((0,\pi))$, then

$$(f,g) = \int_0^\pi f(x)g(x)dx.$$

In other words, $(\cdot,\cdot)$ is the duality between H^γ and $H^{-\gamma}$ relative to the inner product in $H^0 = L_2((0,\pi))$; see [8, Section IV.1.10].

Equation (2.1) is interpreted as a system of two first-order Itô equations

$$du = vdt,\ dv = (a^2 u_{xx} - 2bv)dt + dW(t). \tag{2.5}$$

More precisely, we have the following definition.

Definition 2.1. *An adapted process $u \in L_2\big(\Omega\times(0,T)\times(0,\pi)\big)$ is called a solution of (2.1) if there exists an adapted process v such that*

1. $v \in L_2\big(\Omega; L_2((0,T);H^{-1})\big)$;
2. *For every twice continuously-differentiable on $(0,\pi)$ function $f = f(x)$ with $f(0)=f(\pi)=0$, the equalities*

$$\begin{aligned}(u(t,\cdot),f) &= \int_0^t (v(t,\cdot),f)(s)ds,\\ (v(t,\cdot),f) &= \int_0^t \big(a^2(u(t,\cdot),f'') - 2b(v(t,\cdot),f)\big)ds + W_f(t)\end{aligned} \tag{2.6}$$

hold for all $t\in[0,T]$ on the same set of probability one.

Here is the main result about existence and uniqueness of solution of (2.1).

Theorem 2.1. *Under assumptions* (2.2) *and* (2.3), *equation* (2.1) *has a unique solution and, for every* $\gamma < 1/2$,

$$u \in L_2\big(\Omega; L_2((0,T); H^{\gamma})\big); \quad v \in L_2\big(\Omega; L_2((0,T); H^{\gamma-1})\big). \tag{2.7}$$

Proof – While the result can be derived from the general theory of stochastic hyperbolic equations (see, for example, Chow [1, Theorem 6.8.4]), we present a different, and a more direct, proof. This proof will also help in the construction and analysis of the estimators.

Take in (2.6) $f(x) = \sqrt{2/\pi}\sin(kx)$ and write $u_k(t) = (u(t,\cdot), f)$, $v_k(t) = (v(t,\cdot), f)$, $w_k = W_f$. Then

$$u_k(t) = \int_0^t v_k(s)ds, \ v_k(t) = -a^2k^2\int_0^t u_k(s)ds - 2b\int_0^t v_k(s)ds + w_k(t), \tag{2.8}$$

or

$$\ddot{u}_k(t) + 2b\dot{u}_k(t) + a^2k^2u_k(t) = \dot{w}_k(t), \quad u_k(0) = \dot{u}_k(0) = 0. \tag{2.9}$$

By assumption (2.2),

$$a^2k^2 > b^2 \tag{2.10}$$

for all $k \geq 1$. Define

$$\ell_k = \sqrt{a^2k^2 - b^2}. \tag{2.11}$$

Using the variation of parameters formula for the linear second-order equation with constant coefficients, we conclude that the solution of (2.8) is

$$\begin{aligned} u_k(t) &= \frac{1}{\ell_k}\int_0^t e^{-b(t-s)}\sin\big(\ell_k(t-s)\big)dw_k(s), \\ v_k(t) &= \frac{1}{\ell_k}\int_0^t e^{-b(t-s)}\Big(\ell_k\cos\big(\ell_k(t-s)\big) - b\sin\big(\ell_k(t-s)\big)\Big)dw_k(s). \end{aligned} \tag{2.12}$$

By direct computation, there exists a number $C = C(T,a,b)$ such that, for all $t, s \in [0,T]$,

$$\mathbb{E}u_k^2(t) \leq \ell_k^{-2}C(T) = \frac{C(T)}{a^2k^2 - b^2}, \quad \mathbb{E}v_k^2(t) \leq C(T). \tag{2.13}$$

Then the Gaussian processes

(2.14) $$u(t,x)=\sqrt{\frac{2}{\pi}}\sum_{k\geq 1}u_k(t)\sin(kx),\quad v(t,x)=\sqrt{\frac{2}{\pi}}\sum_{k\geq 1}v_k(t)\sin(kx)$$

satisfy (2.6) and (2.7). Uniqueness of the solution follows from the completeness of the system $\{\sqrt{2/\pi}\sin(kx),\ k\geq 1\}$ in $L_2((0,\pi))$. □

Remark 2.1 - We can now comment on the significance of assumptions (2.2) and (2.3). Assumption (2.2) can be relaxed to $a>0$, because we will still have $a^2k^2>b^2$ for all sufficiently large k, and so representation formulas (2.12) for the solution of equation (2.9) will continue to hold for all sufficiently large k. In other words, if $a>0$, then the free motion (any solution of the homogeneous version of (2.9)) is oscillatory for all sufficiently large $k\geq 1$; the oscillations are damped if $b>0$, harmonic if $b=0$, and amplified if $b>0$. This is also the reason to call b the `damping coefficient`, with an understanding that negative damping means amplification. Thus, (2.2) is only needed to simplify the computations by ensuring that equalities (2.12) hold *for all* $k\geq 1$.

Non-zero initial conditions, if sufficiently regular, will not affect existence and regularity of the solution. Similarly, the analysis will not change much for zero Neumann or other homogeneous boundary conditions.

3. Estimating the Coefficients

In this section, we study the question of estimating the numbers a^2, b from the observations of the solution $u=u(t,x),\ v=v(t,x)$ of equation (2.1). It will be convenient to introduce the notations

(3.1) $$\theta_1=a^2,\ \theta_2=-2b,$$

so that (2.1) becomes

(3.2) $$\frac{\partial^2 u}{\partial t^2}=\theta_1\frac{\partial^2 u}{\partial x^2}+\theta_2\frac{\partial u}{\partial t}+\dot{W}(t),\ t<0<T,\ 0<x<\pi.$$

To simplify the presentation, we keep the assumptions (2.2) and (2.3). By Theorem 2.1, the solution of (3.2) has a Fourier series expansion (2.12). We will construct the maximum likelihood estimators of θ_1 and θ_2 using the observations of the $2N$-dimensional process $\{u_k(t),\ v_k(t),\ k=1,\dots,N,\ t\in[0,T]\}$ and study the asymptotic properties of the estimators in the limit $N\to\infty$. Note that both the amplitude of noise and the observation time are fixed.

By (2.8),

$$u_k(t)=\int_0^t v_k(s)ds,\ v_k(t)=-\theta_1 k^2\int_0^t u_k(s)ds+\theta_2\int_0^t v_k(s)ds+w_k(t). \tag{3.3}$$

For each $k\geq 1$, the processes u_k, v_k, and w_k generate measures $\mathbf{P}_k^u$, $\mathbf{P}_k^v$, $\mathbf{P}_k^w$ in the space $\mathcal{C}((0,T);\mathbb{R})$ of continuous, real-valued functions on $[0,T]$. Since u_k is a continuously-differentiable function, the measures $\mathbf{P}_k^u$ and $\mathbf{P}_k^w$ are mutually singular. On the other hand, we can write

$$dv_k(t)=F_k(v)dt+dw_k, \tag{3.4}$$

where $F_k(v)=-\theta_1 k^2\int_0^t v_k(s)ds+\theta_2 v_k(t)$ is a non-anticipating functional of v. Thus, the process v is a process of diffusion type in the sense of Liptser and Shiryaev [10, Definition 4.2.7]. Further analysis shows that the measure $\mathbf{P}_k^v$ is absolutely continuous with respect to the measure $\mathbf{P}_k^w$, and

$$\begin{aligned}\frac{d\mathbf{P}_k^v}{d\mathbf{P}_k^w}(v_k)=\exp\Bigg(&\int_0^T\big(-\theta_1 k^2 u_k(t)+\theta_2 v_k(t)\big)dv_k(t)\\ &-\frac{1}{2}\int_0^T\big(-\theta_1 k^2 u_k(t)+\theta_2 v_k(t)\big)^2 dt\Bigg);\end{aligned} \tag{3.5}$$

see [10, Theorem 7.6]. Since the processes w_k are independent for different k, so are the processes v_k. Therefore, the measure $\mathbf{P}^{v,N}$ generated in $\mathcal{C}((0,T);\mathbb{R}^N)$ by the vector process $\{v_k,\ k=1,\dots,N\}$ is absolutely continuous with respect to the measure $\mathbf{P}^{w,N}$ generated in $\mathcal{C}((0,T);\mathbb{R}^N)$ by the vector process $\{w_k,\ k=1,\dots,N\}$, and the density is

$$\begin{aligned}\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)=\exp\Bigg(&\sum_{k=1}^N\int_0^T\big(-\theta_1 k^2 u_k(t)+\theta_2 v_k(t)\big)dv_k(t)\\ &-\frac{1}{2}\sum_{k=1}^N\int_0^T\big(-\theta_1 k^2 u_k(t)+\theta_2 v_k(t)\big)^2 dt\Bigg);\end{aligned} \tag{3.6}$$

the corresponding log-likelihood ratio is

$$Z_N(\theta_1,\theta_2)=\sum_{k=1}^{N}\left(\int_0^T\big(-\theta_1k^2u_k(t)+\theta_2v_k(t)\big)dv_k(t)-\frac{1}{2}\int_0^T\big(-\theta_1k^2u_k(t)+\theta_2v_k(t)\big)^2dt\right). \tag{3.7}$$

Introduce the following notations:

$$\begin{aligned} J_{1,N}&=\sum_{k=1}^{N}k^4\int_0^T u_k^2(t)dt,\quad J_{2,N}=\sum_{k=1}^{N}\int_0^T v_k^2(t)dt,\\ J_{12,N}&=\sum_{k=1}^{N}k^2\int_0^T u_k(t)v_k(t)dt;\\ B_{1,N}&=-\sum_{k=1}^{N}k^2\int_0^T u_k(t)dv_k(t),\ \xi_{1,N}=\sum_{k=1}^{N}k^2\int_0^T u_k(t)dw_k(t);\\ B_{2,N}&=\sum_{k=1}^{N}\int_0^T v_k(t)dv_k(t),\ \xi_{2,N}=\sum_{k=1}^{N}\int_0^T v_k(t)dw_k(t). \end{aligned} \tag{3.8}$$

Note that the numbers J and B are computable from the observations of u_k and v_k, $k=1,\dots,N$, and also

$$B_{1,N}=\theta_1J_{1,N}-\theta_2J_{12,N}-\xi_{1,N},\quad B_{2,N}=-\theta_1J_{12,N}+\theta_2J_{2,N}+\xi_{2,N}, \tag{3.9}$$

$$J_{12,N}=\frac{1}{2}\sum_{k=1}^{N}k^2u_k^2(T). \tag{3.10}$$

We consider the problem of estimating simultaneously both θ_1 and θ_2 from the observations

$$\Big\{u_k(t),\ v_k(t),\ k=1,\dots,N,\ t\in[0,T]\Big\}.$$

The maximum likelihood estimators $\hat\theta_{1,N}$, $\hat\theta_{2,N}$ satisfy

$$\left.\frac{\partial Z_N(\theta_1,\theta_2)}{\partial\theta_1}\right|_{\theta_1=\hat\theta_{1,N},\theta_2=\hat\theta_{2,N}}=0\quad\text{and}\quad\left.\frac{\partial Z_N(\theta_1,\theta_2)}{\partial\theta_2}\right|_{\theta_1=\hat\theta_{1,N},\theta_2=\hat\theta_{2,N}}=0,$$

or, after solving the system of equations,

$$\hat\theta_{1,N}=\frac{B_{1,N}J_{2,N}+B_{2,N}J_{12,N}}{J_{1,N}J_{2,N}-J_{12,N}^2},\quad \hat\theta_{2,N}=\frac{B_{1,N}J_{12,N}+B_{2,N}J_{1,N}}{J_{1,N}J_{2,N}-J_{12,N}^2}. \tag{3.11}$$

For $T > 0$ and $\theta_2 \in \mathbb{R}$, define

$$C(\theta_2, T) = \begin{cases} \dfrac{e^{\theta_2 T} - \theta_2 T - 1}{2\theta_2^2}, & \text{if } \theta_2 \neq 0; \\ \dfrac{T^2}{4}, & \text{if } \theta_2 = 0. \end{cases} \tag{3.12}$$

Note that $C(\theta_2, T) > 0$ for all $T > 0$ and $\theta_2 \in \mathbb{R}$.

The following theorem describes the asymptotic behavior of the estimators (3.11).

Theorem 3.1. *Under assumptions* (2.2) *and* (2.3),

$$\lim_{N\to\infty} \hat{\theta}_{1,N} = \theta_1, \quad \lim_{N\to\infty} \hat{\theta}_{2,N} = \theta_2$$

with probability one and

$$\lim_{N\to\infty} N^{3/2}(\hat{\theta}_{1,N} - \theta_1) = \mathcal{N}\left(0, \frac{3\theta_1}{C(\theta_2, T)}\right),$$
$$\lim_{N\to\infty} N^{1/2}(\hat{\theta}_{2,N} - \theta_2) = \mathcal{N}\left(0, \frac{1}{C(\theta_2, T)}\right)$$

in distribution.

Proof – Define

$$D_N = \frac{J_{12,N}^2}{J_{1,N}J_{2,N}}.$$

It follows from (3.9) and (3.11) that

$$\begin{aligned} \hat{\theta}_{1,N} &= \theta_1 + \frac{1}{1 - D_N}\left(\frac{\xi_{1,N}}{J_{1,N}} + \xi_{2,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right), \\ \hat{\theta}_{2,N} &= \theta_2 + \frac{1}{1 - D_N}\left(\frac{\xi_{2,N}}{J_{2,N}} + \xi_{1,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right). \end{aligned} \tag{3.13}$$

By direct computations using (2.12) (and keeping in mind (3.1)),

$$\lim_{k\to\infty} k^2 \mathbb{E}\int_0^T u_k^2(t)dt = \frac{C(\theta_2, T)}{\theta_1}, \tag{3.14}$$

and

$$\lim_{N\to\infty} N^{-3}\mathbb{E}J_{1,N} = \frac{C(\theta_2, T)}{3\theta_1}. \tag{3.15}$$

Since each u_k is a Gaussian process,

$$\sup_k k^4 \mathbb{E} \int_0^T u_k^4(t)dt < \infty,$$

and then the strong law of large numbers implies

$$\lim_{N\to\infty} \frac{J_{1,N}}{\mathbb{E}J_{1,N}} = 1, \ \lim_{N\to\infty} \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} = 0,$$

both with probability one [apply the first theorem in Appendix, taking $\xi_k = k^4 \int_0^T u_k^2 dt$ and then, $\xi_k = k^2 \int_0^T u_k(t)dw_k(t)$]. The central limit theorem implies

$$\lim_{N\to\infty} \frac{\xi_{1,N}}{\sqrt{\mathbb{E}J_{1,N}}} = \mathcal{N}(0,1)$$

in distribution [apply the second theorem in Appendix, taking $f_k(t) = k^2 u_k(t)$]. Similarly,

$$\lim_{k\to\infty} \mathbb{E} \int_0^T v_k^2(t)dt = C(\theta_2, T), \tag{3.16}$$

and

$$\lim_{N\to\infty} N^{-1}\mathbb{E}J_{2,N} = C(\theta_2, T). \tag{3.17}$$

Since each v_k is a Gaussian process,

$$\sup_k \mathbb{E} \int_0^T v_k^4(t)dt < \infty,$$

and then the strong law of large numbers implies

$$\lim_{N\to\infty} \frac{J_{2,N}}{\mathbb{E}J_{2,N}} = 1, \ \lim_{N\to\infty} \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} = 0,$$

both with probability one. The central limit theorem implies

$$\lim_{N\to\infty} \frac{\xi_{2,N}}{\sqrt{\mathbb{E}J_{2,N}}} = \mathcal{N}(0,1)$$

in distribution. Finally, define

$$\tilde{C}(\theta_2, T) = \begin{cases} \dfrac{e^{\theta_2 T} - 1}{2\theta_2}, & \text{if } \theta_2 \neq 0; \\ \dfrac{T}{2}, & \text{if } \theta_2 = 0. \end{cases}$$

Then (3.10) and (2.12) imply

$$\lim_{N\to\infty} N^{-1}\mathbb{E}J_{12,N} = \frac{\tilde{C}(\theta_2,T)}{2\theta_1},$$

and, by the strong law of large numbers,

$$\lim_{N\to\infty} \frac{J_{12,N}}{\mathbb{E}J_{12,N}} = 1$$

with probability one. Then (3.15) and (3.17) imply

$$\lim_{N\to\infty} D_N = 0, \quad \lim_{N\to\infty} \frac{J_{12,N}}{J_{2,N}} = \frac{\tilde{C}(\theta_2,T)}{2\theta_1 C(\theta_2,T)},$$

both with probability one. The conclusions of the theorem now follow. □

Unlike the iid case, where the rate of convergence is always $N^{1/2}$, the convergence rates in the above theorem, $N^{3/2}$ for $\hat{\theta}_{1,N}$ and $N^{1/2}$ for $\hat{\theta}_{2,N}$, are hard, if not impossible, to guess without going through the proof. Still, it is not surprising that the estimator of θ_1 converges faster than the estimator of θ_2: by Theorem 2.1, the term u_{xx} is less regular than u_t,

$$u_{xx} \in L_2\big(\Omega; L_2((0,T); H^{\gamma-2})\big); \quad u_t \in L_2\big(\Omega; L_2((0,T); H^{\gamma-1})\big),$$

which makes the term u_{xx}, together with the coefficient θ_1, more "visible" in the noise.

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Appendix

Below, we formulate the strong law of large numbers and the central limit theorem used in the proof of Theorem 3.1.

Theorem 4.1 (Strong Law of Large Numbers). *Let $\xi_k,\ k \geq 1$, be independent random variables with the following properties:*

- $\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 > 0$,
- *There exist real numbers $c > 0$ and $\alpha \geq -1$ such that*

$$\lim_{k\to\infty} k^{-\alpha}\mathbb{E}\xi_k^2 = c.$$

Then, with probability one,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N \xi_k}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 0.$$

If, in addition, $\mathbb{E}\xi_k^4 \leq c_1\left(\mathbb{E}\xi_k^2\right)^2$ for all $k \geq 1$, with $c_1 > 0$ independent of k, then, also with probability one,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N \xi_k^2}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 1.$$

PROOF – This is a particular case of Kolmogorov's strong law of large numbers; see, for example, Shiryaev [15, Theorem IV.3.2]. □

Theorem 4.2 (Central Limit Theorem). *Let $w_k = w_k(t)$ be independent standard Brownian motions and let $f_k = f_k(t)$ be adapted, continuous, square-integrable processes such that*

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N \int_0^T f_k^2(t)dt}{\sum_{k=1}^N \mathbb{E}\int_0^T f_k^2(t)dt} = 1$$

in probability. Then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N \int_0^T f_k(t)dw_k(t)}{\left(\sum_{k=1}^N \mathbb{E}\int_0^T f_k^2(t)dt\right)^{1/2}} = \mathcal{N}(0,1)$$

in distribution.

PROOF – This is a particular case of a martingale limit theorem; see, for example Jacod and Shiryaev [7, Theorem VIII.4.17] or Liptser and Shiryaev [9, Theorem 5.5.4(II)]. □

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