SIAM J. MATH. ANAL.
Vol. 41, No. 4, pp. 1295–1322

STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY PURELY SPATIAL NOISE*

SERGEY V. LOTOTSKY† AND BORIS L. ROZOVSKII‡

Abstract. We study bilinear stochastic parabolic and elliptic PDEs driven by purely spatial white noise. Even the simplest equations driven by this noise often do not have a square-integrable solution and must be solved in special weighted spaces. We demonstrate that the Cameron–Martin version of the Wiener chaos decomposition is an effective tool to study both stationary and evolution equations driven by space-only noise. The paper presents results about solvability of such equations in weighted Wiener chaos spaces and studies the long-time behavior of the solutions of evolution equations with space-only noise.

Key words. generalized random elements, Malliavin calculus, Skorokhod integral, Wiener chaos, weighted spaces

AMS subject classifications. 60H15, 35R60, 60H40

DOI. 10.1137/070698440

1. Introduction.

Let us consider a stochastic PDE of the form

$$u(t,x) = u(0,x) + \int_0^t \mathbf{A}u(s,x)ds + \int_0^t \mathbf{M}u(s,x)\,dW(s,x), \tag{1.1}$$

where $\mathbf{A}$ and $\mathbf{M}$ are linear partial differential operators, $\dot{W}(t,x)$ is space-time white noise, and the last term is understood as an Itô integral, and are usually referred to as bilinear evolution stochastic partial differential equations (SPDEs).[1] Alternatively (1.1) could be written in the form

$$\dot{u}(t,x) = \mathbf{A}u(t,x) + \mathbf{M}u(t,x) \diamond \dot{W}(t,x), \tag{1.2}$$

where $\diamond$ stands for Wick product (see Definition 2.6 and Appendix A).

Bilinear SPDEs are of interest in various applications, e.g., nonlinear filtering of diffusion processes [33], propagation of magnetic field in random flow [2], stochastic transport [6, 7, 20], porous media [3], and others.

The theory and the applications of bilinear SPDEs have been actively investigated for a few decades now; see, for example, [4, 15, 28, 29, 32]. In contrast, very little is known about bilinear parabolic and elliptic equations driven by purely spatial Gaussian white noise $\dot{W}(x)$. Important examples of these equations include the following.

*Received by the editors July 26, 2007; accepted for publication (in revised form) May 14, 2009; published electronically August 21, 2009.
http://www.siam.org/journals/sima/41-4/69844.html

†Department of Mathematics, USC, Los Angeles, CA 90089 (lototsky@math.usc.edu). This author was supported by Sloan Research Fellowship, the NSF CAREER award DMS-0237724, and NSF grant DMS-0803378.

‡Division of Applied Mathematics, Brown University, Providence, RI 02912 (rozovsky@dam.brown.edu). This author was supported by NSF grant DMS 0604863, ARO grant W911NF-07-1-0044, and ONR grant N00014-07-1-0044.

[1] Bilinear SPDEs differ from linear by the term including multiplicative noise. Bilinear SPDEs are technically more difficult than linear. On the other hand, multiplicative models preserve many features of the unperturbed equation, such as positivity of the solution and conservation of mass, and are often more "physical".

1. Heat equation with random potential modeled by spatial white noise:

$$\dot{u}(t,x) = \Delta u(t,x) + u(t,x) \diamond \dot{W}(x), \tag{1.3}$$

where $\diamond$ denotes the Wick product, which, in this case, coincides with the Skorohod integral in the sense of Malliavin calculus. A surprising discovery made in [10] was that the spatial regularity of the solution of (1.3) is better than in the case of a similar equation driven by the space-time white noise.

2. Stochastic Poisson equations in random medium [30, 31]:

$$\nabla(A_\varepsilon(x) \diamond \nabla u(x)) = f(x), \tag{1.4}$$

where $A_\varepsilon(x) := (a(x) + \varepsilon \dot{W}(x))$, $a(x)$ is a deterministic positive-definite matrix, and ε is a positive number. (Note that $a(x) \diamond \nabla u(x) = a(x)\nabla u(x)$.)

3. Heat equation in random medium:

$$\dot{v}(t,x) = \nabla(A_\varepsilon(x) \diamond \nabla v(t,x)) + g(t,x). \tag{1.5}$$

Note that the matrix A_ε in (1.4) and (1.5) is not necessarily positive definite; only its expectation $a(x)$ is.

Equations (1.4) and (1.5) are random perturbation of the deterministic Poisson and heat equations. In the instance of positive (lognormal) noise, these equations were extensively studied in [9]; see also the references therein.

The objective of this paper is to develop a systematic approach to bilinear SPDEs driven by purely spatial Gaussian noise. More specifically, we will investigate bilinear parabolic equations,

$$\frac{\partial v(t,x)}{\partial t} = \mathbf{A}v(t,x) + \mathbf{M}v(t,x) \diamond \dot{W}(x) - f(x), \tag{1.6}$$

and elliptic equations

$$\mathbf{A}u(x) + \mathbf{M}u(x) \diamond \dot{W}(x) = f(x), \tag{1.7}$$

for a wide range of operators $\mathbf{A}$ and $\mathbf{M}$.

Purely spatial white noise is an important type of stationary perturbation. However, except for elliptic equations with additive random forcing [5, 23, 27], SPDEs driven by spatial noise have not been investigated nearly as extensively as those driven by strictly temporal or space-time noise.

In the case of spatial white noise, there is no natural and convenient filtration, especially in the dimension $d > 2$. Therefore, it makes sense to consider anticipative solutions. This rules out Itô calculus and makes it necessary to rely on Skorohod integrals and Malliavin calculus.

In particular, everywhere below in this paper the expression $\mathbf{M}u \diamond \dot{W}$ refers to the Skorohod integral or Malliavin divergence operator with respect to white noise $\dot{W}(x)$.

An interesting feature of linear equations perturbed by $\mathbf{M}u \diamond \dot{W}$ is that the resulting SPDEs are *unbiased* in that they preserve the mean dynamics. For example, the functions $u_0(x) := \mathbb{E}u(x)$ and $v_0 := \mathbb{E}v(t,x)$ solve the deterministic Poisson equation,

$$\nabla(a(x)\nabla u_0(t,x)) = \mathbb{E}f(x)$$

and the deterministic heat equation

$$\dot{v}_0(t,x) = \nabla(a(x) v_0(t,x)) + \mathbb{E}g(t,x),$$

respectively. However, we stress that, in general, this property applies only to *linear and bilinear equations.*

In this paper, we deal with broad classes of operators $\mathbf{A}$ and $\mathbf{M}$ that were investigated previously for nonanticipating solutions of (1.1) driven by space-time white noise.

The notion of ellipticity for SPDEs is more restrictive than in deterministic theory. Traditionally, nonanticipating solutions of (1.1) were studied under the following assumptions:

(i) *The operator* $\mathbf{A} - \frac{1}{2}\mathbf{M}\mathbf{M}^{\star}$ *is an "elliptic" (possibly degenerate coercive) operator.*

Of course, this assumption does not hold for (1.5) and other equations in which the operators $\mathbf{A}$ and $\mathbf{M}$ have the same order. Therefore, it is important to study (1.7) and (1.6) under weaker assumptions, for example.

(ii) *The operator* $\mathbf{A}$ *is coercive and* $ord(\mathbf{M}) \le ord(\mathbf{A})$.

In 1981, it was shown by Krylov and Rozovskii [16] that, unless assumption (i) holds, (1.1) has no solutions in the space $L_2(\Omega; X)$ of square-integrable (in probability) solutions in any reasonable functional space X. The same effect holds for bilinear SPDEs driven by space-only white noise.

Numerous attempts to investigate solutions of stochastic PDEs violating the stochastic ellipticity conditions have been made since then. In particular it was shown in [22, 24, 26] that if the operator $\mathbf{A}$ is coercive ("elliptic") and $ord(\mathbf{M})$ is *strictly* less then $ord(\mathbf{A})$, then there there exists a unique generalized (Wiener chaos) nonanticipative solution of (1.1). This generalized solution is a formal Wiener chaos series $u = \sum_{|\alpha|<\infty} u_\alpha \xi_\alpha$, where $\{\xi_\alpha\}_{|\alpha|<\infty}$ is the Cameron–Martin orthonormal basis in the space $L_2(\Omega)$. Regularity of this solution is determined by a system of positive weights $\{r_\alpha\}_{|\alpha|<\infty}$ and a function space X such that

$$\|u\|^2_{\mathcal{R},X} := \sum_{|\alpha|<\infty} r_\alpha^2 \, \|u_\alpha\|^2_{L_2((0,T);X)} < \infty. \tag{1.8}$$

The stochastic Fourier coefficients u_α satisfy a lower-triangular system of deterministic PDEs. This system, called `propagator`, is uniquely determined by the underlying (1.1).

Stochastic spaces equipped with the norms similar to (1.8) have been known for quite some time; see, e.g., [12, 13, 26]. For historical remarks regarding other types of generalized solutions and applications to SPDEs, see the review paper [21] and the references therein.

The Wiener chaos solution is a bona fide generalization of the classical Itô solution: if it exists, a nonanticipating square integrable Itô solution coincides with the Wiener chaos solution.

In this paper, we establish the existence and uniqueness of Wiener chaos solutions for stationary (elliptic) equations of the type (1.7) and evolution (parabolic) equations of the type (1.6). These results are proved under assumption (ii) that allows us in particular to deal with equations like (1.4) and (1.5). In many cases we are able to find optimal or near-optimal systems of weights $\{r_\alpha\}_{|\alpha|<\infty}$ that guarantee (1.8).

Finally, we establish the convergence, as $t \to +\infty$, of the solution of the evolution equation to the solution of the related stationary equation.

The structure of the paper is as follows. Section 2 reviews the definition of the Skorohod integral in the framework of the Malliavin calculus and shows how to compute the integral using Wiener chaos. Sections 3 and 4 deal with existence and uniqueness of solutions to abstract evolution and stationary equations, respectively, driven by a general (not necessarily white) spatial Gaussian noise; section 4 describes also the limiting behavior of the solution of the evolution equation; section 5 illustrates the general results for bilinear SPDEs driven by purely spatial white noise.

2. Weighted Wiener chaos and Malliavin calculus. Let $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathcal{U}$, a real separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{U}}$. On $\mathbb{F}$, consider a zero-mean Gaussian family

$$\dot{W} = \left\{ \dot{W}(h),\ h \in \mathcal{U} \right\}$$

so that

$$\mathbb{E}\left(\dot{W}(h_1)\, \dot{W}(h_2) \right) = (h_1, h_2)_{\mathcal{U}}.$$

It suffices, for our purposes, to assume that $\mathcal{F}$ is the σ-algebra generated by $\dot{W}$. Given a real separable Hilbert space X, we denote by $L_2(\mathbb{F}; X)$ the Hilbert space of square-integrable $\mathcal{F}$-measurable X-valued random elements f. In particular,

$$(f,g)^2_{L_2(\mathbb{F};X)} := \mathbb{E}(f,g)^2_X.$$

When $X = \mathbb{R}$, we write $L_2(\mathbb{F})$ instead of $L_2(\mathbb{F};\mathbb{R})$.

DEFINITION 2.1. *A formal series*

$$\dot{W} = \sum_k \dot{W}(\mathfrak{u}_k)\, \mathfrak{u}_k, \tag{2.1}$$

where $\{\mathfrak{u}_k, k \geq 1\}$ *is a complete orthonormal basis in* $\mathcal{U}$, *is called (Gaussian)* white noise *on* $\mathcal{U}$.

The white noise on $\mathcal{U} = L_2(G)$, where G is a domain in $\mathbb{R}^d$, is usually referred to as a *spatial or space white noise* (on $L_2(G)$). The space white noise is of central importance for this paper.

Below, we will introduce a class of spaces that are convenient for treating nonlinear functionals of white noise, in particular, solutions of SPDEs driven by white noise.

Given an orthonormal basis $\mathfrak{U} = \{\mathfrak{u}_k, k \geq 1\}$ in $\mathcal{U}$, define a collection $\{\xi_k, k \geq 1\}$ of independent standard Gaussian random variables so that $\xi_k = \dot{W}(\mathfrak{u}_k)$. Denote by $\mathcal{J}$ the collection of multi-indices α with $\alpha = (\alpha_1, \alpha_2, \dots)$ so that each α_k is a nonnegative integer and $|\alpha| := \sum_{k\geq 1} \alpha_k < \infty$. For $\alpha, \beta \in \mathcal{J}$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots), \quad \alpha! = \prod_{k \geq 1} \alpha_k!.$$

By (0) we denote the multi-index with all zeroes. By ε_i we denote the multi-index α with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. With this notation, $n\varepsilon_i$ is the multi-index α with $\alpha_i = n$ and $\alpha_j = 0$ for $j \neq i$. The following inequality holds (see Appendix B for the proof):

$$|\alpha|! \leq \alpha!(2\mathbb{N})^{2\alpha}, \quad \text{where } \ (2\mathbb{N})^{2\alpha} = \prod_{k\geq 1} (2k)^{2\alpha_k}. \tag{2.2}$$

Define the collection of random variables $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ as follows:

$$\xi_\alpha = \prod_k \left(\frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right), \tag{2.3}$$

where

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{2.4}$$

is an Hermite polynomial of order n.

THEOREM 2.2 (Cameron and Martin [1]). *The collection* $\Xi = \{\xi_\alpha,\ \alpha \in \mathcal{J}\}$ *is an orthonormal basis in* $L_2(\mathbb{F})$*: if* $\eta \in L_2(\mathbb{F})$ *and* $\eta_\alpha = \mathbb{E}(\eta\xi_\alpha)$*, then* $\eta = \sum_{\alpha\in\mathcal{J}} \eta_\alpha \xi_\alpha$ *and* $\mathbb{E}|\eta|^2 = \sum_{\alpha\in\mathcal{J}} \eta_\alpha^2$.

Expansions with respect to the Cameron–Martin basis Ξ are usually referred to as Wiener chaos. Next, we introduce a modification of the Wiener chaos expansion which will be called *weighted Wiener chaos.*

Let $\mathcal{R}$ be a bounded linear operator on $L_2(\mathbb{F})$ defined by $\mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha$ for every $\alpha \in \mathcal{J}$, where the *weights* $\{r_\alpha,\ \alpha \in \mathcal{J}\}$ are positive numbers. By Theorem 2.2, $\mathcal{R}$ is bounded if and only if the weights r_α are uniformly bounded from above: $r_\alpha < C$ for all $\alpha \in \mathcal{J}$, with C independent of α. The inverse operator $\mathcal{R}^{-1}$ is defined by $\mathcal{R}^{-1}\xi_\alpha = r_\alpha^{-1}\xi_\alpha$.

We extend $\mathcal{R}$ to an operator on $L_2(\mathbb{F};X)$ by defining $\mathcal{R}f$ as the unique element of $L_2(\mathbb{F};X)$ so that, for all $g \in L_2(\mathbb{F};X)$,

$$\mathbb{E}(\mathcal{R}f, g)_X = \sum_{\alpha\in\mathcal{J}} r_\alpha \mathbb{E}\big((f,g)_X \xi_\alpha\big).$$

Denote by $\mathcal{R}L_2(\mathbb{F};X)$ the closure of $L_2(\mathbb{F};X)$ with respect to the norm

$$\|f\|^2_{\mathcal{R}L_2(\mathbb{F};X)} := \|\mathcal{R}f\|^2_{L_2(\mathbb{F};X)}.$$

Then the elements of $\mathcal{R}L_2(\mathbb{F};X)$ can be identified with a formal series $\sum_{\alpha\in\mathcal{J}} f_\alpha \xi_\alpha$, where $f_\alpha \in X$ and $\sum_{\alpha\in\mathcal{J}} \|f_\alpha\|_X^2 r_\alpha^2 < \infty$.

We define the space $\mathcal{R}^{-1}L_2(\mathbb{F};X)$ as the dual of $\mathcal{R}L_2(\mathbb{F};X)$ relative to the inner product in the space $L_2(\mathbb{R};X)$:

$$\mathcal{R}^{-1}L_2(\mathbb{F};X) = \left\{g \in L_2(\mathbb{F};X) : \mathcal{R}^{-1}g \in L_2(\mathbb{F};X)\right\}.$$

For $f \in \mathcal{R}L_2(\mathbb{F};X)$ and $g \in \mathcal{R}^{-1}L_2(\mathbb{F}) := \mathcal{R}^{-1}L_2(\mathbb{F};\mathbb{R})$ we define the scalar product

$$\langle\!\langle f, g\rangle\!\rangle := \mathbb{E}\big((\mathcal{R}f)(\mathcal{R}^{-1}g)\big) \in X. \tag{2.5}$$

In what follows, we will identify the operator $\mathcal{R}$ with the corresponding collection $(r_\alpha,\ \alpha \in \mathcal{J})$. Note that if $u \in \mathcal{R}_1 L_2(\mathbb{F};X)$ and $v \in \mathcal{R}_2 L_2(\mathbb{F};X)$, then both u and v belong to $\mathcal{R}L_2(\mathbb{F};X)$, where $r_\alpha = \min(r_{1,\alpha}, r_{2,\alpha})$. As usual, the argument X will be omitted if $X = \mathbb{R}$.

Important particular cases of the space $\mathcal{R}L_2(\mathbb{F};X)$ correspond to the following weights:

1. $$r_\alpha^2 = \prod_{k=1}^{\infty} q_k^{\alpha_k},$$

   where $\{q_k,\ k \geq 1\}$ is a nonincreasing sequence of positive numbers with $q_1 \leq 1$ (see [22, 26]);

2.

$$r_\alpha^2 = (\alpha!)^\rho (2\mathbb{N})^{\ell\alpha}, \quad \rho \le 0,\ \ell \le 0, \quad \text{where } (2\mathbb{N})^{\ell\alpha} = \prod_{k\ge 1} (2k)^{\ell\alpha_k}. \tag{2.6}$$

This set of weights defines Kondratiev's spaces $(\mathcal{S})_{\rho,\ell}(X)$.

Now we will sketch the basics of Malliavin calculus on $\mathcal{R}L_2(\mathbb{F}; X)$.

Denote by $\mathbf{D}$ the *Malliavin derivative* on $L_2(\mathbb{F})$ (see, e.g., [25]). In particular, if $F : \mathbb{R}^N \to \mathbb{R}$ is a smooth function and $h_i \in \mathcal{U},\ i = 1, \dots N$, then

$$\mathbf{D}F(\dot W(h_1), \dots \dot W(h_N)) = \sum_{i=1}^{N} \frac{\partial F}{\partial x_i}(\dot W(h_1), \dots, \dot W(h_N)) h_i \in L_2(\mathbb{F}; \mathcal{U}). \tag{2.7}$$

It is known [25] that the domain $\mathbb{D}^{1,2}(\mathbb{F})$ of the operator $\mathbf{D}$ is a dense linear subspace of $L_2(\mathbb{F})$.

The adjoint of the Malliavin derivative on $L_2(\mathbb{F})$ is the Itô–Skorohod integral and is traditionally denoted by δ [25]. We will keep this notation for the extension of this operator to $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$.

For $f \in \mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$, we define $\delta(f)$ as the unique element of $\mathcal{R}L_2(\mathbb{F}; X)$ with the property

$$\langle\!\langle \delta(f), \varphi \rangle\!\rangle = \mathbb{E}(\mathcal{R}f, \mathcal{R}^{-1}\mathbf{D}\varphi)_{\mathcal{U}} \tag{2.8}$$

for every φ satisfying $\varphi \in \mathcal{R}^{-1} L_2(\mathbb{F})$ and $\mathbf{D}\varphi \in \mathcal{R}^{-1} L_2(\mathbb{F}; \mathcal{U})$.

Next, we derive the expressions for the Malliavin derivative $\mathbf{D}$ and its adjoint δ in the basis Ξ. To begin, we compute $\mathbf{D}(\xi_\alpha)$.

PROPOSITION 2.3. *For each $\alpha \in \mathcal{J}$, we have*

$$\mathbf{D}(\xi_\alpha) = \sum_{k\ge 1} \sqrt{\alpha_k}\, \xi_{\alpha - \varepsilon_k} \mathfrak{u}_k. \tag{2.9}$$

Proof. The result follows by direct computation using the property (2.7) of the Malliavin derivative and the relation $H_n'(x) = nH_{n-1}(x)$ for the Hermite polynomials (cf. [25]). $\square$

Obviously, the set $\mathcal{J}$ is not invariant with respect to subtraction. In particular, the expression $\alpha - \varepsilon_k$ is undefined if $\alpha_k = 0$. In (2.9) and everywhere below in this paper where undefined expressions of this type appear, we use the following convention: if $\alpha_k = 0$, then $\sqrt{\alpha_k}\, \xi_{\alpha-\varepsilon_k} = 0$.

PROPOSITION 2.4. *For $\xi_\alpha \in \Xi$, $h \in X$, and $\mathfrak{u}_k \in \mathfrak{U}$, we have*

$$\delta(\xi_\alpha\, h \otimes \mathfrak{u}_k) = h\sqrt{\alpha_k + 1}\, \xi_{\alpha+\varepsilon_k}. \tag{2.10}$$

Proof. It is enough to verify (2.8) with $f = h \otimes \mathfrak{u}_k\, \xi_\alpha$ and $\varphi = \xi_\beta$, where $h \in X$. By (2.9),

$$\mathbb{E}(f, \mathbf{D}\varphi)_{\mathcal{U}} = \sqrt{\beta_k}\, h\, \mathbb{E}(\xi_\alpha \xi_{\beta-\varepsilon_k}) = \begin{cases} \sqrt{\alpha_k+1}\, h & \text{if } \alpha = \beta - \varepsilon_k, \\ 0 & \text{if } \alpha \ne \beta - \varepsilon_k. \end{cases}$$

In other words,

$$\mathbb{E}(\xi_\alpha\, h \otimes \mathfrak{u}_k, \mathbf{D}\xi_\beta)_{\mathcal{U}} = h\, \mathbb{E}\left(\sqrt{\alpha_k+1}\xi_{\alpha+\varepsilon_k}\xi_\beta\right)$$

for all $\beta \in \mathcal{J}$. $\square$

Remark 2.5. The operator $\delta\mathbf{D}$ is linear and unbounded on $L_2(\mathbb{F})$; it follows from Propositions 2.3 and 2.4 that the random variables ξ_α are eigenfunctions of this operator:

$$\delta(\mathbf{D}(\xi_\alpha)) = |\alpha|\xi_\alpha. \tag{2.11}$$

To give an alternative characterization of the operator δ, we define a new operation on the elements of Ξ.

Definition 2.6. *For ξ_α, ξ_β from Ξ, define the Wick product*

$$\xi_\alpha \diamond \xi_\beta := \sqrt{\left(\frac{(\alpha+\beta)!}{\alpha!\beta!}\right)}\xi_{\alpha+\beta}. \tag{2.12}$$

In particular, taking in (2.6) $\alpha = k\varepsilon_i$ and $\beta = n\varepsilon_i$, and using (2.3), we get

$$H_k(\xi_i) \diamond H_n(\xi_i) = H_{k+n}(\xi_i). \tag{2.13}$$

By linearity, we define the Wick product $f \diamond \eta$ for $f \in \mathcal{R}L_2(\mathbb{F}; X)$ and $\eta \in \mathcal{R}L_2(\mathbb{F})$: if $f = \sum_{\alpha\in\mathcal{J}} f_\alpha\xi_\alpha$, $f_\alpha \in X$, and $\eta = \sum_{\alpha\in\mathcal{J}} \eta_\alpha\xi_\alpha$, $\eta_\alpha \in \mathbb{R}$, then

$$f \diamond \eta = \sum_{\alpha,\beta} f_\alpha\eta_\beta\xi_\alpha \diamond \xi_\beta.$$

Proposition 2.7. *If $f \in \mathcal{R}L_2(\mathbb{F}; X)$ and $\eta \in \mathcal{R}L_2(\mathbb{F})$, then $f \diamond \eta$ is an element of $\bar{\mathcal{R}}L_2(\mathbb{F}; X)$ for a suitable operator $\bar{\mathcal{R}}$.*

Proof. It follows from (2.6) that $f \diamond \eta = \sum_{\alpha\in\mathcal{J}} F_\alpha\xi_\alpha$ and

$$F_\alpha = \sum_{\beta,\gamma\in\mathcal{J}:\beta+\gamma=\alpha} \sqrt{\left(\frac{(\beta+\gamma)!}{\beta!\gamma!}\right)} f_\beta\eta_\gamma.$$

Therefore, each $F_\alpha X$ is an element of X, because, for every $\alpha \in \mathcal{J}$, there are only finitely many multi-indices β, γ satisfying $\beta+\gamma = \alpha$. It is known [21, Proposition 7.1] that

$$\sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{q\alpha} < \infty \text{ if and only if } q < -1. \tag{2.14}$$

Therefore, $f \diamond \eta \in \bar{\mathcal{R}}L_2(\mathbb{F}; X)$, where the operator $\bar{\mathcal{R}}$ can be defined using the weights $\bar{r}_\alpha^2 = (2\mathbb{N})^{-2\alpha}/(1 + \|F_\alpha\|_X^2)$. ∎

An immediate consequence of Proposition 2.4 and Definition 2.6 is the following identity:

$$\delta(\xi_\alpha h \otimes \mathfrak{u}_k) = h\xi_\alpha \diamond \xi_k,\ h \in X. \tag{2.15}$$

Below we summarize the properties of the operator δ.

Theorem 2.8. *If f is an element of $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$ so that $f = \sum_{k\geq 1} f_k \otimes \mathfrak{u}_k$, with $f_k = \sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\,\xi_\alpha \in \mathcal{R}L_2(\mathbb{F}; X)$, then*

$$\delta(f) = \sum_{k\geq 1} f_k \diamond \xi_k \tag{2.16}$$

and

$$(\delta(f))_\alpha = \sum_{k\geq 1} \sqrt{\alpha_k} f_{k,\alpha-\varepsilon_k}. \tag{2.17}$$

Proof. By linearity and (2.15),

$$\delta(f) = \sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} \delta(\xi_\alpha f_{k,\alpha}\otimes \mathfrak{u}_k) = \sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\xi_\alpha \diamond \xi_k = \sum_{k\geq 1} f_k\diamond\xi_k,$$

which is (2.16). On the other hand, by (2.10),

$$\delta(f) = \sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} f_{k,\alpha}\sqrt{\alpha_k+1}\,\xi_{\alpha+\varepsilon_k} = \sum_{k\geq 1}\sum_{\alpha\in\mathcal{J}} f_{k,\alpha-\varepsilon_k}\sqrt{\alpha_k}\,\xi_\alpha,$$

and (2.17) follows. □

Remark 2.9. It is not difficult to show that the operator δ can be considered as an extension of the Skorohod integral to the weighted spaces $\mathcal{R}L_2(\mathbb{F}; X\otimes\mathcal{U})$.

One way to describe a multi-index α with $|\alpha| = n > 0$ is by its characteristic set K_α; that is, an ordered n-tuple $K_\alpha = \{k_1, \dots, k_n\}$, where $k_1 \leq k_2 \leq \cdots \leq k_n$ characterize the locations and the values of the nonzero elements of α. More precisely, k_1 is the index of the first nonzero element of α, followed by $\max(0, \alpha_{k_1}-1)$ of entries with the same value. The next entry after that is the index of the second nonzero element of α, followed by $\max(0, \alpha_{k_2}-1)$ of entries with the same value, and so on. For example, if $n = 7$ and $\alpha = (1, 0, 2, 0, 0, 1, 0, 3, 0, \dots)$, then the nonzero elements of α are $\alpha_1 = 1$, $\alpha_3 = 2$, $\alpha_6 = 1$, and $\alpha_8 = 3$. As a result, $K_\alpha = \{1, 3, 3, 6, 8, 8, 8\}$; that is, $k_1 = 1$, $k_2 = k_3 = 3$, $k_4 = 6$, and $k_5 = k_6 = k_7 = 8$.

Using the notion of the characteristic set, we now state the following analogue of the well-known result of Itô [11] connecting multiple Wiener integrals and Hermite polynomials.

PROPOSITION 2.10. *Let $\alpha\in\mathcal{J}$ be a multi-index with $|\alpha| = n \geq 1$ and characteristic set $K_\alpha = \{k_1, \dots, k_n\}$. Then*

$$\xi_\alpha = \frac{\xi_{k_1}\diamond\xi_{k_2}\diamond\cdots\diamond\xi_{k_n}}{\sqrt{\alpha!}}. \tag{2.18}$$

Proof. This follows from (2.3) and (2.13), because by (2.13), for every i and k,

$$H_k(\xi_i) = \underbrace{\xi_i\diamond\cdots\diamond\xi_i}_{k\text{ times}}. \qquad \square$$

3. Evolution equations driven by white noise.

3.1. The setting. In this section we study anticipating solutions of stochastic evolution equations driven by Gaussian white noise on a Hilbert space $\mathcal{U}$.

DEFINITION 3.1. *The triple (V, H, V') of Hilbert spaces is called normal if the following holds.*

1. *$V \hookrightarrow H \hookrightarrow V'$ and both embeddings $V\hookrightarrow H$ and $H\hookrightarrow V'$ are dense and continuous;*
2. *the space V' is the dual of V relative to the inner product in H;*
3. *there exists a constant $C > 0$ so that $|(h, v)_H| \leq C\|v\|_V\|h\|_{V'}$ for all $v\in V$ and $h\in H$.*

For example, the Sobolev spaces $(H_2^{\ell+\gamma}(\mathbb{R}^d), H_2^{\ell}(\mathbb{R}^d), H_2^{\ell-\gamma}(\mathbb{R}^d))$, $\gamma > 0$, $\ell \in \mathbb{R}$, form a normal triple.

Denote by $\langle v', v\rangle$, $v' \in V'$, $v \in V$, the duality between V and V' relative to the inner product in H. The properties of the normal triple imply that $|\langle v', v\rangle| \leq C\|v\|_V \|v'\|_{V'}$, and, if $v' \in H$ and $v \in V$, then $\langle v', v\rangle = (v', v)_H$.

We will also use the following notation:

$$\mathcal{V} = L_2((0,T); V),\ \mathcal{H} = L_2((0,T); H),\ \mathcal{V}' = L_2((0,T); V'). \tag{3.1}$$

Given a normal triple (V, H, V'), let $\mathbf{A} : V \to V'$ and $\mathbf{M} : V \to V' \otimes \mathcal{U}$ be bounded linear operators.

DEFINITION 3.2. *The solution of the stochastic evolution equation*

$$\dot{u} = \mathbf{A}u + f + \delta(\mathbf{M}u),\ 0 < t \leq T, \tag{3.2}$$

with $f \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V}')$ *and* $u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H)$, *is a process* $u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ *so that, for every* φ *satisfying* $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ *and* $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$, *the equality*

$$\langle\!\langle u(t), \varphi\rangle\!\rangle = \langle\!\langle u_0, \varphi\rangle\!\rangle + \int_0^t \langle\!\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \varphi\rangle\!\rangle ds \tag{3.3}$$

holds in $\mathcal{V}'$; *see* (2.5) *for the definition of* $\langle\!\langle \cdot, \cdot\rangle\!\rangle$.

Remark 3.3. (a) The solutions described by Definitions 3.2 and 4.1 belong to the class of "variational solutions," which is quite typical for PDEs (see [14, 18, 19, 29], etc.).

(b) Since $\langle\!\langle u(t), \varphi\rangle\!\rangle \in \mathcal{V}$ and $\langle\!\langle u(t), \varphi\rangle\!\rangle_t \in \mathcal{V}'$, by the standard embedding theorem (see, e.g., [19, Section 1.2.2]) there exists a version of $\langle\!\langle u(t), \varphi\rangle\!\rangle \in \mathbf{C}([0,T]; H)$. Clearly, one could also select a version of $u(t)$ such that $\langle\!\langle u(t), \varphi\rangle\!\rangle \in \mathbf{C}([0,T]; H)$. In the future, we will consider only this version of the solution. By doing this we ensure that formula (3.3), which is understood as an equality in $\mathcal{V}'$, yields $u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H)$.

Remark 3.4. To simplify the notations and the overall presentation, we assume that $\mathbf{A}$ and $\mathbf{M}$ do not depend on time, even though many of the results in this paper can easily be extended to time-dependent operators.

Fix an orthonormal basis $\mathfrak{U}$ in $\mathcal{U}$. Then, for every $v \in V$, there exists a collection $v_k \in V'$, $k \geq 1$, so that

$$\mathbf{M}v = \sum_{k\geq 1} v_k \otimes \mathfrak{u}_k.$$

We therefore define the operators $\mathbf{M}_k : V \to V'$ by setting $\mathbf{M}_k v = v_k$ and write

$$\mathbf{M}v = \sum_{k\geq 1} (\mathbf{M}_k v) \otimes \mathfrak{u}_k.$$

By (2.16), equation (3.2) becomes

$$\dot{u}(t) = \mathbf{A}u(t) + f(t) + \mathbf{M}u\,(t) \diamond \dot{W}, \tag{3.4}$$

where

$$\mathbf{M}v \diamond \dot{W} := \sum_{k\geq 1} (\mathbf{M}_k v) \diamond \xi_k. \tag{3.5}$$

3.2. Equivalence theorem. In this section we investigate a stochastic Fourier representation of (3.4).

Recall that every process $u = u(t)$ from $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$ is represented by a formal series $u(t) = \sum_{\alpha\in\mathcal{J}} u_\alpha(t)\xi_\alpha$, with $u_\alpha \in \mathcal{V}$ and

$$\sum_{\alpha} r_\alpha^2 \, \|u_\alpha\|_{\mathcal{V}}^2 < \infty. \tag{3.6}$$

THEOREM 3.5. *Let $u = \sum_{\alpha\in\mathcal{J}} u_\alpha\xi_\alpha$ be an element of $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$. The process u is a solution of* (3.2) *if and only if the functions u_α have the following properties:*

1. *Every u_α is an element of $\mathbf{C}([0,T];H)$);*
2. *the system of equalities*

$$u_\alpha(t) = u_{0,\alpha} + \int_0^t \left(\mathbf{A}u_\alpha(s) + f_\alpha(s) + \sum_{k\geq 1} \sqrt{\alpha_k}\, \mathbf{M}_k u_{\alpha-\varepsilon_k}(s) \right) ds \tag{3.7}$$

   holds in V' for all $t \in [0,T]$ and $\alpha \in \mathcal{J}$.

Proof. Let u be a solution of (3.2) in $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$. Taking $\varphi = \xi_\alpha$ in (3.3) and using relation (2.17), we obtain (3.7). By Remark 3.3 $u_\alpha \in \mathcal{V}\bigcap \mathbf{C}([0,T];H)$.

Conversely, let $\{u_\alpha, \alpha \in \mathcal{J}\}$ be a collection of functions from $\mathcal{V}\bigcap\mathbf{C}([0,T];H)$ satisfying (3.6) and (3.7). Set $u(t) := \sum_{\alpha\in\mathcal{J}} u_\alpha(t)\xi_\alpha$. Then, by Theorem 2.8, (3.7) yields that, for every $\alpha \in \mathcal{J}$,

$$\langle\!\langle u(t), \xi_\alpha\rangle\!\rangle = \langle\!\langle u_0, \xi_\alpha\rangle\!\rangle + \int_0^t \langle\!\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \xi_\alpha\rangle\!\rangle ds.$$

By continuity, we conclude that for any $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ such that $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F};\mathcal{U})$, the equality

$$\langle\!\langle u(t), \varphi\rangle\!\rangle = \langle\!\langle u_0, \varphi\rangle\!\rangle + \int_0^t \langle\!\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \varphi\rangle\!\rangle ds$$

holds in $\mathcal{V}'$. By Remark 3.3 $\langle\!\langle u(t), \varphi\rangle\!\rangle \in \mathbf{C}([0,T];H)$. $\square$

This simple but very helpful result establishes the equivalence of the "physical" (3.4) and the (stochastic) Fourier (3.7) forms of (3.2). The system of equations (3.7) is often referred to in the literature as the *propagator* of (3.4). Note that the propagator is lower-triangular and can be solved by induction on $|\alpha|$.

3.3. Existence and uniqueness. Below, we will present several results on the existence and uniqueness of evolution equations driven by Gaussian white noise.

Before proceeding with general existence-uniqueness problems, we will introduce two simple examples that indicate the limits of the "quality" of solutions of bilinear SPDEs driven by general Gaussian white noise.

Example 3.6. Consider the equation

$$u(t) = \phi + \int_0^t (b\, u(s) \diamond \xi - \lambda u(s)) ds, \tag{3.8}$$

where ϕ, λ are real numbers, b is a complex number, and ξ is a standard Gaussian random variable. In other words ξ is Gaussian white noise on $\mathcal{U} = \mathbb{R}$. With only one Gaussian random variable ξ, the set $\mathcal{J}$ becomes $\{0, 1, 2, \dots\}$ so that

$u(t) = \sum_{n\geq 0} u_{(n)}(t)H_n(\xi)/\sqrt{n!}$, where H_n is an Hermite polynomial of order n (2.4). According to (3.7),

$$u_{(n)}(t) = \phi I_{(n=0)} - \int_0^t \lambda u_{(n)}(s)ds + \int_0^t b\sqrt{n}u_{(n-1)}(s)ds.$$

It follows that $u_{(0)}(t) = \phi e^{-\lambda t}$ and then, by induction, $u_{(n)}(t) = \phi \frac{(b\,t)^n}{\sqrt{n!}} e^{-\lambda t}$. As a result,

$$u(t) = e^{-\lambda t}\left(\phi + \sum_{n\geq 1} \frac{(b\,t)^n}{n!} H_n(\xi)\right) = \phi e^{-\lambda t + \left(b\,t\xi - |b|^2 t^2/2\right)}.$$

Obviously, the solution of the equation is square-integrable on any fixed time interval. However, as the next example indicates, the solutions of SPDEs driven by stationary noise are much more intricate than the nonanticipating, or adapted, solutions of SPDEs driven by space-time white noise.

Example 3.7. With ξ as in the previous examples, consider a PDE

$$u_t(t,x) = au_{xx}(t,x) + (\beta u(t,x) + \sigma u_x(t,x)) \diamond \xi, \quad t > 0, \quad x \in \mathbb{R}, \tag{3.9}$$

with some initial condition $u_0 \in L_2(\mathbb{R})$. By taking the Fourier transform and using the results of Example 3.6 with $\phi = \hat{u}_0(y)$, $\lambda = -ay^2$, $b = \beta + \sqrt{-1}y\sigma$, we find

$$\hat{u}_t(t,y) = -y^2 a\hat{u}(t) + \left(\beta + \sqrt{-1}y\sigma\right)\hat{u}(t,y) \diamond \xi;$$
$$\hat{u}(t,y) = \hat{u}_0(y)\exp\left(-tay^2 + \left(\sigma^2 y^2 - \beta^2\right)t^2/2 + \sqrt{-1}\beta\sigma y t^2 + \left(\sqrt{-1}\sigma y + \beta\right)t\xi\right).$$

If $\sigma = 0$, i.e., the "diffusion" operator in (3.9) is of order zero, then the solution belongs to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ for all t. However, if $\sigma > 0$, then the solution $u(t,\cdot)$ will, in general, belong to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ only for $t \leq 2a/\sigma^2$. This blow-up in finite time is in sharp contrast with the solution of the equation

$$u_t = au_{xx} + \sigma u_x \diamond \dot{w}, \tag{3.10}$$

driven by the standard one-dimensional white noise $\dot{w}(t) = \partial_t W(t)$, where $W(t)$ is the one-dimensional Brownian motion; a more familiar way of writing (3.10) is in the Itô form

$$du = au_{xx}dt + \sigma u_x dW(t). \tag{3.11}$$

It is well known that the solution of (3.11) belongs to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ for every $t > 0$ as long as $u_0 \in L_2(\mathbb{R})$ and

$$a - \sigma^2/2 \geq 0; \tag{3.12}$$

see, for example, [29].

The existence of a square-integrable (global) solution of an Itô's SPDE with square-integrable initial condition hinges on the parabolic condition, which in the case of (3.10) is given by (3.12). Example 3.7 shows that this condition is not in any way sufficient for SPDEs involving a Skorohod-type integral. The next theorem provides sufficient conditions for the existence and uniqueness of a solution to (3.4) in the space $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$, which appears to be a reasonable extension of the class of square-integrable solutions.

First, we introduce an additional assumption on the operator $\mathbf{A}$ that will be used throughout this section:

(A): *For every* $U_0 \in H$ *and* $F \in \mathcal{V}' := L_2((0,T);V')$, *there exists a function* $U \in \mathcal{V}$ *that solves the deterministic equation*

$$\partial_t U(t) = \mathbf{A}U(t) + F(t),\ U(0) = U_0, \tag{3.13}$$

and there exists a constant $C = C(\mathbf{A},T)$ *so that*

$$\|U\|_{\mathcal{V}} \le C(\mathbf{A},T)\big(\|U_0\|_H + \|F\|_{\mathcal{V}'}\big). \tag{3.14}$$

Remark 3.8. Assumption (A) implies that a solution of (3.13) is unique and belongs to $\mathbf{C}((0,T);H)$ (cf. Remark 3.3). The assumption also implies that the operator $\mathbf{A}$ generates a semigroup $\Phi = \Phi_t$, $t \ge 0$, and, for every $v \in \mathcal{V}$,

$$\int_0^T \left\| \int_0^t \ \Phi_{t-s}\mathbf{M}_k v\,(s)\,ds \right\|_V^2 dt \le C_k^2\,\|v\|_{\mathcal{V}}^2\,, \tag{3.15}$$

with numbers C_k independent of v.

Remark 3.9. There are various types of assumptions on the operator $\mathbf{A}$ that yield the statement of the assumption (A). In particular, (A) holds if the operator $\mathbf{A}$ is coercive in (V,H,V'):

$$\langle \mathbf{A}v, v\rangle + \gamma\|v\|_V^2 \le C\|v\|_H^2$$

for every $v \in V$, where $\gamma > 0$ and $C \in \mathbb{R}$ are both independent of v.

THEOREM 3.10. *Assume(A). Consider* (3.4) *in which* $u_0 \in \bar{\mathcal{R}}L_2(\mathbb{F};H)$, $f \in \bar{\mathcal{R}}L_2(\mathbb{F};\mathcal{V}')$ *for some operator* $\bar{\mathcal{R}}$, *and each* $\mathbf{M}_k$ *is a bounded linear operator from* V *to* V'.

Then there exist an operator $\mathcal{R}$ *and a unique solution* $u \in \mathcal{R}L_2(\mathbb{F};\mathcal{V})$ *of* (3.4).

Proof. By Theorem 3.5, it suffices to prove that the propagator (3.7) has a unique solution $(u_\alpha(t))_{\alpha\in\mathcal{J}}$ such that for each $\alpha, u_\alpha \in \mathcal{V}\bigcap\mathbf{C}([0,T];H)$ and $u := \sum_{\alpha\in\mathcal{J}} u_\alpha\xi_\alpha \in \mathcal{R}L_2(\mathbb{F};\mathcal{V})$.

For $\alpha = (0)$, that is, when $|\alpha| = 0$, (3.7) reduces to

$$u_{(0)}(t) = u_{0,(0)} + \int_0^t \big(\mathbf{A}u_{(0)} + f_{(0)}\big)\,(s)ds.$$

By (A), this equation has a unique solution and

$$\|u_{(0)}\|_{\mathcal{V}} \le C(\mathbf{A},T)\left(\|u_{0,(0)}\|_H + \|f_{(0)}\|_{\mathcal{V}'}\right).$$

Using assumption (A), it follows by induction on $|\alpha|$ that, for every $\alpha \in \mathcal{J}$, equation

$$\partial_t u_\alpha(t) = \mathbf{A}u_\alpha(t) + f_\alpha(t) + \sum_{k\ge1}\sqrt{\alpha_k}\,\mathbf{M}_k u_{\alpha-\varepsilon_k}(t),\ u_\alpha(0) = u_{0,\alpha} \tag{3.16}$$

has a unique solution in $\mathcal{V}\bigcap\mathbf{C}([0,T];H)$. Moreover, by (3.14),

$$\|u_\alpha\|_{\mathcal{V}} \le \overline{C}(\mathbf{A},\mathbf{M},T)\left(\|u_{0,\alpha}\|_H + \|f_\alpha\|_{\mathcal{V}'} + \sum_{k\ge1}\sqrt{\alpha_k}\|u_{\alpha-\varepsilon_k}\|_{\mathcal{V}}\right).$$

Since only finitely many of α_k are different from 0, we conclude that $\|u_\alpha\|_{\mathcal{V}} < \infty$ for all $\alpha \in \mathcal{J}$.

Define the operator $\mathcal{R}$ on $L_2(\mathbb{F})$ using the weights

$$r_\alpha = \min\left(\bar{r}_\alpha, \frac{(2\mathbb{N})^{-\kappa\alpha}}{1+\|u_\alpha\|_{\mathcal{V}}}\right),$$

where $\kappa > 1/2$ (cf. (2.6)). Then $u(t) := \sum_{\alpha\in\mathcal{J}} u_\alpha(t)\xi_\alpha$ is a solution of (3.4) and, by (2.14), belongs to $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$. $\square$

While Theorem 3.10 establishes that under very broad assumptions one can find an operator $\mathcal{R}$ such that (3.4) has a unique solution in $\mathcal{R}L_2(\mathbb{F};\mathcal{V})$; the choice of the operator $\mathcal{R}$ is not sufficiently explicit (because of the presence of $\|u_\alpha\|_{\mathcal{V}}$) and is not necessarily optimal.

Consider (3.4) with nonrandom f and u_0. In this situation, it is possible to find a more constructive expression for r_α and to derive explicit formulas, both for $\mathcal{R}u$ and for each individual u_α.

THEOREM 3.11. *If u_0 and f are nonrandom, then the following holds:*

1. *The coefficient u_α, corresponding to the multi-index α with $|\alpha| = n \geq 1$ and characteristic set $K_\alpha = \{k_1, \dots, k_n\}$, is given by*

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \Phi_{t-s_n}\mathbf{M}_{k_{\sigma(n)}} \cdots \Phi_{s_2-s_1}\mathbf{M}_{k_{\sigma(1)}} u_{(0)}(s_1)\, ds_1 \dots ds_n, \tag{3.17}$$

   where
   - *$\mathcal{P}_n$ is the permutation group of the set $(1, \dots, n)$;*
   - *Φ_t is the semigroup generated by $\mathbf{A}$;*
   - *$u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) ds$.*
2. *The weights r_α can be taken in the form*

$$r_\alpha = \frac{q^\alpha}{\sqrt{|\alpha|!}}, \text{ where } q^\alpha = \prod_{k=1}^\infty q_k^{\alpha_k}, \tag{3.18}$$

   and the numbers q_k, $k \geq 1$ are chosen so that $\sum_{k\geq1} q_k^2 C_k^2 < 1$, with C_k from (3.15).
3. *With q_k and r_α from* (3.18), *we have*

$$\sum_{|\alpha|=n} q^\alpha u_\alpha(t)\xi_\alpha = \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \Phi_{t-s_n}\delta(\overline{\mathbf{M}}\Phi_{s_n-s_{n-1}}\delta(\dots\delta(\overline{\mathbf{M}}u_{(0)}))\dots)ds_1\dots ds_{n-1}ds_n, \tag{3.19}$$

   where $\overline{\mathbf{M}} = (q_1\mathbf{M}_1, q_2\mathbf{M}_2, \dots)$, and

$$\mathcal{R}u(t) = u_{(0)}(t) + \sum_{n=1}^\infty \frac{1}{2^n\sqrt{n!}} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \times \Phi_{t-s_n}\delta(\overline{\mathbf{M}}\Phi_{s_n-s_{n-1}}\delta(\dots\delta(\overline{\mathbf{M}}u_0(s_1)))\dots)ds_1\dots ds_{n-1}ds_n. \tag{3.20}$$

Proof. If u_0 and f are deterministic, then (3.7) becomes

$$u_{(0)}(t) = u_0 + \int_0^t \mathbf{A}u_{(0)}(s)ds + \int_0^t f(s)ds,\ |\alpha| = 0; \tag{3.21}$$

$$u_\alpha(t) = \int_0^t \mathbf{A}u_\alpha(s)ds + \sum_{k\geq 1} \sqrt{\alpha_k} \int_0^t \mathbf{M}_k u_{\alpha-\varepsilon_k}(s)ds,\ |\alpha| > 0. \tag{3.22}$$

Define $\widetilde{u}_\alpha = \sqrt{\alpha!}\, u_\alpha$. Then $\widetilde{u}_{(0)} = u_{(0)}$ and, for $|\alpha| > 0$, (3.22) implies

$$\widetilde{u}_\alpha(t) = \int_0^t \mathbf{A}\widetilde{u}_\alpha(s)ds + \sum_{k\geq 1} \int_0^t \alpha_k\, \mathbf{M}_k \widetilde{u}_{\alpha-\varepsilon_k}(s)ds$$

or

$$\widetilde{u}_\alpha(t) = \sum_{k\geq 1} \alpha_k \int_0^t \Phi_{t-s}\mathbf{M}_k \tilde{u}_{\alpha-\varepsilon_k}(s)ds = \sum_{k\in K_\alpha} \int_0^t \Phi_{t-s}\mathbf{M}_k \tilde{u}_{\alpha-\varepsilon_k}(s)ds.$$

By induction on n,

$$\widetilde{u}_\alpha(t) = \sum_{\sigma\in\mathcal{P}_n} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \Phi_{t-s_n}\mathbf{M}_{k_{\sigma(n)}} \cdots \Phi_{s_2-s_1}\mathbf{M}_{k_{\sigma(1)}} u_{(0)} ds_1 \ldots ds_n,$$

and (3.17) follows.

Since (3.20) follows directly from (3.19), it remains to establish (3.19). To this end, define

$$U_n(t) = \sum_{|\alpha|=n} q^\alpha u_\alpha(t)\xi_\alpha,\ n \geq 0.$$

Let us first show that, for each $n \geq 1$, $U_n \in L_2(\mathbb{F}; \mathcal{V})$. Indeed, for $\alpha = (0)$, $u_\alpha(0) = u_0$, $f_\alpha = f$, and

$$u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s)ds.$$

By (3.14), we have

$$\|u_{(0)}\|_{\mathcal{V}} \leq C(\mathbf{A}, T)\left(\|u_0\|_H + \|f\|_{\mathcal{V}'}\right). \tag{3.23}$$

When $|\alpha| \geq 1$, $f_\alpha = 0$ and the solution of (3.22) is given by

$$u_\alpha(t) = \sum_{k\geq 1} \sqrt{\alpha_k} \int_0^t \Phi_{t-s}\mathbf{M}_k u_{\alpha-\varepsilon_k}(s)ds. \tag{3.24}$$

By (3.17), together with (3.14), (3.23), and (3.15), we have

$$\|u_\alpha\|^2_{\mathcal{V}} \leq C^2(\mathbf{A}, T)\frac{(|\alpha|!)^2}{\alpha!}\left(\|u_0\|^2_H + \|f\|^2_{\mathcal{V}'}\right) \prod_{k\geq 1} C_k^{2\alpha_k}. \tag{3.25}$$

By the multinomial formula,

$$\left(\sum_{k\geq 1} x_k\right)^n = \sum_{|\alpha|=n} \left(\frac{n!}{\alpha!} \prod_{k\geq 1} x_k^{\alpha_k}\right). \tag{3.26}$$

Then

$$\sum_{|\alpha|=n} q^{2\alpha}\|u_\alpha\|^2_{\mathcal{V}} \le C^2(\mathbf{A},T)\big(\|u_0\|^2_H+\|f\|^2_{\mathcal{V}'}\big)\, n!\sum_{|\alpha|=n}\left(\frac{n!}{\alpha!}\prod_{k\ge1}(C_kq_k)^{2\alpha_k}\right)$$
$$= C^2(\mathbf{A},T)\big(\|u_0\|^2_H+\|f\|^2_{\mathcal{V}'}\big)\, n!\left(\sum_{k\ge1}C_k^2q_k^2\right)^n<\infty,$$

because of the selection of q_k, and so $U_n\in L_2(\mathbb{F};\mathcal{V})$. Moreover, if the weights r_α are defined by (3.18), then

$$\sum_{\alpha\in\mathcal{J}} r_\alpha^2\|u_\alpha\|^2_{\mathcal{V}} = \sum_{n\ge0}\sum_{|\alpha|=n} r_\alpha^2\|u_\alpha\|^2_{\mathcal{V}}$$
$$\le C^2(\mathbf{A},T)\big(\|u_0\|^2_H+\|f\|^2_{\mathcal{V}'}\big)\sum_{n\ge1}\left(\sum_{k\ge1}C_k^2q_k^2\right)^n<\infty$$

because of the assumption $\sum_{k\ge1}C_k^2q_k^2<1$.

Next, the definition of $U_n(t)$ and (3.24) imply that (3.19) is equivalent to

$$U_n(t)=\int_0^t\Phi_{t-s}\delta(\overline{\mathbf{M}}U_{n-1}(s))ds,\ \ n\ge1. \tag{3.27}$$

Accordingly, we will prove (3.27). For $n=1$, we have

$$U_1(s)=\sum_{k\ge1}q_ku_{\varepsilon_k}(t)\xi_k=\sum_{k\ge1}\int_0^t q_k\Phi_{t-s}\mathbf{M}_ku_{(0)}\xi_kdt=\int_0^t\Phi_{t-s}\delta(\overline{\mathbf{M}}U_0(s))ds,$$

where the last equality follows from (2.16). More generally, for $n>1$ we have by definition of U_n that

$$(U_n)_\alpha(t)=\begin{cases}q^\alpha u_\alpha(t) & \text{if } |\alpha|=n,\\ 0 & \text{otherwise.}\end{cases}$$

From the equation

$$q^\alpha u_\alpha(t)=\int_0^t\mathbf{A}q^\alpha u_\alpha(s)ds+\sum_{k\ge1}\int_0^t q_k\sqrt{\alpha_k}\,\mathbf{M}_kq^{\alpha-\varepsilon_k}u_{\alpha-\varepsilon_k}(s)ds,$$

we find

$$(U_n(t))_\alpha=\begin{cases}\displaystyle\sum_{k\ge1}\sqrt{\alpha_k}q_k\int_0^t\Phi_{t-s}\mathbf{M}_kq^{\alpha-\varepsilon_k}u_{\alpha-\varepsilon_k}(s)ds & \text{if } |\alpha|=n,\\ 0 & \text{otherwise}\end{cases}$$
$$=\sum_{k\ge1}\sqrt{\alpha_k}\int_0^t\Phi_{t-s}\overline{\mathbf{M}}_k(U_{n-1}(s))_{\alpha-\varepsilon_k}ds,$$

and then (3.27) follows from (2.17). Theorem 4.5 is proved. ∎

Formula (3.19) is similar to the multiple Wiener integral representation of the solution of a stochastic parabolic equation driven by the Wiener process; see [22, Theorem 3.8].

Example 3.12. Consider the equation

$$u(t,x) = u_0(x) + \int_0^t u_{xx}(s,x)ds + \sum_{k\geq 1}\int_0^t \sigma_k u_{xx}(s,x) \diamond \xi_k ds. \tag{3.28}$$

With no loss of generality assume that $\sigma_k \neq 0$ for all k. Standard properties of the heat kernel imply assumption (A) and inequality (3.15) with $C_k = \sigma_k^2$. Then the conclusions of Theorem 3.11 hold, and we can take $q_k^2 = k^{-2}4^{-k}(1+\sigma_k^2)^{-k}$. Note that Theorem 3.11 covers (3.28) with no restrictions on the numbers σ_k.

In the existing literature on the subject, equations of the type (3.4) are considered only under the following assumption:

(H): Each $\mathbf{M}_k$ *is a bounded linear operator from* V *to* H.

Obviously this assumption rules out (3.28) but still covers (3.9).

Of course, Theorem 3.11 does not rule out a possibility of a better-behaving solution under additional assumptions on the operators $\mathbf{M}_k$. Indeed, it was shown in [21] that if (H) is assumed and the space-only Gaussian noise in (3.4) is replaced by the space-time white noise, then a more delicate analysis of (3.4) is possible. In particular, the solution can belong to a much smaller Wiener chaos space even if u_0 and f are not deterministic.

If the operators $\mathbf{M}_k$ are bounded in H (see, e.g., (3.9) with $\sigma = 0$), then, as the following theorem shows, the solutions can be square integrable (cf. [10]).

THEOREM 3.13. *Assume that the operator* $\mathbf{A}$ *satisfies*

$$\langle \mathbf{A}v, v\rangle + \kappa\|v\|_V^2 \leq C_A\|v\|_H^2 \tag{3.29}$$

for every $v \in V$, *with* $\kappa > 0$, $C_A \in \mathbb{R}$ *independent of* v, *and assume that each* $\mathbf{M}_k$ *is a bounded operator on* H *so that* $\|\mathbf{M}_k\|_{H\to H} \leq c_k$ *and*

$$C_M := \sum_{k\geq 1} c_k^2 < \infty. \tag{3.30}$$

If $f \in \mathcal{V}'$ *and* $u_0 \in H$ *are nonrandom, then there exists a unique solution* u *of* (3.4) *so that* $u(t) \in L_2(\mathbb{F}; H)$ *for every* t *and*

$$\mathbb{E}\|u(t)\|_H^2 \leq C(C_A, C_M, \kappa, t)\left(\int_0^t \|f(s)\|_{V'}^2 ds + \|u_0\|_H^2\right). \tag{3.31}$$

Proof. Existence and uniqueness of the solution follow from Theorem 3.10 and Remark 3.9, and it remains to establish (3.31).

It follows from (3.7) that

$$u_\alpha = \frac{1}{\sqrt{\alpha!}}\sum_{k\in K_\alpha}^{|\alpha|}\int_0^t \Phi_{t-s}\mathbf{M}_k u_{\alpha-\varepsilon_k}(s)ds, \tag{3.32}$$

where Φ is the semigroup generated by $\mathbf{A}$ and K_α is the characteristic set of α. Assumption (3.29) implies that $\|\Phi_t\|_{H\to H} \leq e^{pt}$ for some $p \in \mathbb{R}$. A straightforward calculation using relation (3.32) and induction on $|\alpha|$ shows that

$$\|u_\alpha(t)\|_H \leq e^{pt}\frac{t^{|\alpha|}c^\alpha}{\sqrt{\alpha!}}\|u_{(0)}\|_H, \tag{3.33}$$

where $c^\alpha = \prod_k c_k^{\alpha_k}$ and $u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s)ds$. Assumption (3.29) implies that $\|u_{(0)}\|_H^2 \le C(C_A, \kappa, t)(\int_0^t \|f(s)\|_{V'}^2 ds + \|u_0\|_H^2)$. To establish (3.31), it remains to observe that

$$\sum_{\alpha\in\mathcal{J}} \frac{c^{2\alpha} t^{2|\alpha|}}{\alpha!} = e^{C_M t^2}.$$

Theorem 3.13 is proved. □

Remark 3.14. Taking $\mathbf{M}_k u = c_k u$ shows that, in general, bound (3.33) cannot be improved. When condition (3.30) does not hold, a bound similar to (3.31) can be established in a weighted space $\mathcal{R}L_2(\mathbb{F}; H)$, for example with $r_\alpha = q^\alpha$, where $q_k = 1/(2^k(1 + c_k))$. For special operators $\mathbf{M}_k$, a more delicate analysis might be possible; see, for example, [10].

If f and u_0 are not deterministic, then the solution of (3.4) might not satisfy

$$\mathbb{E}\|u(t)\|_H^2 \le C(C_A, C_M, \kappa, t)\left(\int_0^t \mathbb{E}\|f(s)\|_{V'}^2 ds + \mathbb{E}\|u_0\|_H^2\right)$$

even if all other conditions of Theorem 3.13 are fulfilled. An example can be constructed similar to Example 9.7 in [21]: an interested reader can verify that the solution of the equation $u(t) = u_0 + \int_0^t u(s) \diamond \xi \, ds$, where ξ is a standard Gaussian random variable and $u_0 = \sum_{n\ge 0} a_n \frac{H_n(\xi)}{\sqrt{n!}}$, satisfies $\mathbb{E}u^2(1) \ge \frac{1}{10}\sum_{n\ge 1} a_n^2 e^{\sqrt{n}}$. For equations with random input, one possibility is to use the spaces $(\mathcal{S})_{-1,q}$; see (2.6). Examples of the corresponding results are Theorems 4.6 and 5.1 below and Theorem 9.8 in [21].

4. Stationary equations.

4.1. Definitions and analysis.

The objective of this section is to study stationary stochastic equation

$$\mathbf{A}u + \delta(\mathbf{M}u) = f. \tag{4.1}$$

DEFINITION 4.1. *The solution of* (4.1) *with* $f \in \mathcal{R}L_2(\mathbb{F}; V')$ *is a random element* $u \in \mathcal{R}L_2(\mathbb{F}; V)$ *so that, for every* φ *satisfying* $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ *and* $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$, *the equality*

$$\langle\!\langle \mathbf{A}u, \varphi\rangle\!\rangle + \langle\!\langle \delta(\mathbf{M}u), \varphi\rangle\!\rangle = \langle\!\langle f, \varphi\rangle\!\rangle \tag{4.2}$$

holds in V'.

As with evolution equations, we fix an orthonormal basis $\mathfrak{U}$ in $\mathcal{U}$ and use (2.16) to rewrite (4.1) as

$$\mathbf{A}u + (\mathbf{M}u) \diamond \dot{W} = f, \tag{4.3}$$

where

$$\mathbf{M}u \diamond \dot{W} := \sum_{k\ge 1} \mathbf{M}_k u \diamond \xi_k. \tag{4.4}$$

Taking $\varphi = \xi_\alpha$ in (4.2) and using relation (2.17) we conclude, as in Theorem 3.5, that $u = \sum_{\alpha\in\mathcal{J}} u_\alpha \xi_\alpha$ is a solution of (4.1) if and only if u_α satisfies

$$\mathbf{A}u_\alpha + \sum_{k\ge 1} \sqrt{\alpha_k}\, \mathbf{M}_k u_{\alpha-\varepsilon_k} = f_\alpha \tag{4.5}$$

in the normal triple (V, H, V'). This system of equation is lower-triangular and can be solved by induction on $|\alpha|$.

The following example elucidates the limitations on the "quality" of the solution of (4.1).

Example 4.2. Consider equation

$$u = 1 + u \diamond \xi. \tag{4.6}$$

Similar to Example 3.6, we write $u = \sum_{n\geq 0} u_{(n)} H_n(\xi)/\sqrt{n!}$, where H_n is an Hermite polynomial of order n (2.4). Then (4.5) implies $u_{(n)} = I_{(n=0)} + \sqrt{n} u_{(n-1)}$ or $u_{(0)} = 1$, $u_{(n)} = \sqrt{n!}$, $n \geq 1$, or $u = 1 + \sum_{n\geq 1} H_n(\xi)$. Clearly, the series does not converge in $L_2(\mathbb{F})$, but does converge in $(\mathcal{S})_{-1,q}$ for every $q < 0$ (see (2.6)). As a result, even a simple stationary equation (4.6) can be solved only in weighted spaces.

THEOREM 4.3. *Consider* (4.3) *in which* $f \in \bar{\mathcal{R}} L_2(\mathbb{F}; V')$ *for some* $\bar{\mathcal{R}}$.

Assume that the deterministic equation $\mathbf{A}U = F$ *is uniquely solvable in the normal triple* (V, H, V')*; that is, for every* $F \in V'$*, there exists a unique solution* $U = \mathbf{A}^{-1}F \in V$ *so that* $\|U\|_V \leq C_A \|F\|_{V'}$*. Assume also that each* $\mathbf{M}_k$ *is a bounded linear operator from* V *to* V' *so that, for all* $v \in V$

$$\left\|\mathbf{A}^{-1}\mathbf{M}_k v\right\|_V \leq C_k \|v\|_V, \tag{4.7}$$

with C_k *independent of* v.

Then there exists an operator $\mathcal{R}$ *and a unique solution* $u \in \mathcal{R}L_2(\mathbb{F}; V)$ *of* (3.4).

Proof. The argument is identical to the proof of Theorem 3.10. ∎

Remark 4.4. The assumption of the theorem about solvability of the deterministic equation holds if the operator $\mathbf{A}$ satisfies $\langle \mathbf{A}v, v\rangle \geq \kappa \|v\|_V^2$ for every $v \in V$, with $\kappa > 0$ independent of v.

An analogue of Theorem 3.11 exists if f is nonrandom. With no time variable, we introduce the following notation to write multiple integrals in the time-independent setting:

$$\delta_{\mathbf{B}}^{(0)}(\eta) = \eta, \quad \delta_{\mathbf{B}}^{(n)}(\eta) = \delta\left(\mathbf{B}\delta_{\mathbf{B}}^{(n-1)}(\eta)\right), \quad \eta \in \mathcal{R}L_2(\mathbb{F}; V),$$

where $\mathbf{B}$ is a bounded linear operator from V to $V \otimes \mathcal{U}$.

THEOREM 4.5. *Under the assumptions of Theorem* 4.3, *if* f *is nonrandom, then the following holds:*

1. *The coefficient* u_α*, corresponding to the multi-index* α *with* $|\alpha| = n \geq 1$ *and the characteristic set* $K_\alpha = \{k_1, \ldots, k_n\}$*, is given by*

$$u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \mathbf{B}_{k_{\sigma(n)}} \cdots \mathbf{B}_{k_{\sigma(1)}} u_{(0)}, \tag{4.8}$$

   where
   - $\mathcal{P}_n$ *is the permutation group of the set* $(1, \ldots, n)$*;*
   - $\mathbf{B}_k = -\mathbf{A}^{-1}\mathbf{M}_k$*;*
   - $u_{(0)} = \mathbf{A}^{-1}f$.

2. *The operator* $\mathcal{R}$ *can be defined by the weights* r_α *in the form*

$$r_\alpha = \frac{q^\alpha}{\sqrt{|\alpha|!}}, \text{ where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \tag{4.9}$$

   where the numbers q_k, $k \geq 1$ *are chosen so that* $\sum_{k\geq 1} q_k^2 C_k^2 < 1$, *and* C_k *is defined in* (4.7).

3. *With r_α and q_k defined by* (4.9),

$$\sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha = \delta^{(n)}_{\overline{\mathbf{B}}}\left(\mathbf{A}^{-1}f\right), \tag{4.10}$$

where $\overline{\mathbf{B}} = -(q_1\mathbf{A}^{-1}\mathbf{M}_1, q_2\mathbf{A}^{-1}\mathbf{M}_2, \dots)$, *and*

$$\mathcal{R}u = \mathbf{A}^{-1}f + \sum_{n\ge 1}\frac{1}{\sqrt{n!}}\,\delta^{(n)}_{\overline{\mathbf{B}}}\left(\mathbf{A}^{-1}f\right). \tag{4.11}$$

Proof. While the proofs of Theorems 3.11 and 4.5 are similar, the complete absence of time makes (4.3) different from either (3.4) or anything considered in [22]. Accordingly, we present a complete proof.

Define $\widetilde{u}_\alpha = \sqrt{\alpha!}\,u_\alpha$. If f is deterministic, then $\widetilde{u}_{(0)} = \mathbf{A}^{-1}f$ and, for $|\alpha|\ge 1$,

$$\mathbf{A}\widetilde{u}_\alpha + \sum_{k\ge 1}\alpha_k \mathbf{M}_k \widetilde{u}_{\alpha-\varepsilon_k} = 0$$

or

$$\widetilde{u}_\alpha = \sum_{k\ge 1}\alpha_k \mathbf{B}_k \widetilde{u}_{\alpha-\varepsilon_k} = \sum_{k\in K_\alpha}\mathbf{B}_k \widetilde{u}_{\alpha-\varepsilon_k},$$

where $K_\alpha = \{k_1, \dots, k_n\}$ is the characteristic set of α and $n = |\alpha|$. By induction on n,

$$\widetilde{u}_\alpha = \sum_{\sigma\in\mathcal{P}_n}\mathbf{B}_{k_{\sigma(n)}}\cdots\mathbf{B}_{k_{\sigma(1)}}u_{(0)},$$

and (4.8) follows.

Next, define

$$U_n = \sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha,\ n\ge 0.$$

Let us first show that, for each $n\ge 1$, $U_n \in L_2(\mathbb{F}; V)$. By (4.8) we have

$$\|u_\alpha\|_V^2 \le C_A^2 \frac{(|\alpha|!)^2}{\alpha!}\|f\|_{V'}^2 \prod_{k\ge 1} C_k^{\alpha_k}. \tag{4.12}$$

By (3.26),

$$\begin{aligned}\sum_{|\alpha|=n} q^{2\alpha}\|u_\alpha\|_V^2 &\le C_A^2\|f\|_{V'}^2\, n! \sum_{|\alpha|=n}\left(\frac{n!}{\alpha!}\prod_{k\ge 1}(C_k q_k)^{2\alpha_k}\right)\\ &= C_A^2\|f\|_{V'}^2\, n!\left(\sum_{k\ge 1} C_k^2 q_k^2\right)^n < \infty,\end{aligned}$$

because of the selection of q_k, and so $U_n \in L_2(\mathbb{F}; V)$. If the weights r_α are defined by (4.9), then

$$\sum_{\alpha\in\mathcal{J}} r_\alpha^2\|u\|_V^2 = \sum_{n\ge 0}\sum_{|\alpha|=n} r_\alpha^2\|u\|_V^2 \le C_A^2\|f\|_{V'}^2 \sum_{n\ge 0}\left(\sum_{k\ge 1} C_k^2 q_k^2\right)^n < \infty,$$

because of the assumption $\sum_{k\ge 1} C_k^2 q_k^2 < 1$.

Since (4.11) follows directly from (4.10), it remains to establish (4.10); that is,

$$U_n = \delta_{\overline{\mathbf{B}}}(U_{n-1}),\ n \geq 1. \tag{4.13}$$

For $n=1$ we have

$$U_1 = \sum_{k\geq 1} q_k u_{\varepsilon_k}\xi_k = \sum_{k\geq 1} \overline{\mathbf{B}}_k u_{(0)}\xi_k = \delta_{\overline{\mathbf{B}}}(U_0),$$

where the last equality follows from (2.16). More generally, for $n>1$ we have by definition of U_n that

$$(U_n)_\alpha = \begin{cases} q^\alpha u_\alpha & \text{if } |\alpha| = n,\\ 0 & \text{otherwise.}\end{cases}$$

From the equation

$$q^\alpha \mathbf{A}u_\alpha + \sum_{k\geq 1} q_k\sqrt{\alpha_k}\,\mathbf{M}_k q^{\alpha-\varepsilon_k}u_{\alpha-\varepsilon_k} = 0$$

we find

$$\begin{aligned}(U_n)_\alpha &= \begin{cases} \sum_{k\geq 1}\sqrt{\alpha_k}\,q_k\mathbf{B}_k q^{\alpha-\varepsilon_k}u_{\alpha-\varepsilon_k} & \text{if } |\alpha| = n,\\ 0 & \text{otherwise}\end{cases}\\ &= \sum_{k\geq 1}\sqrt{\alpha_k}\,\overline{\mathbf{B}}_k(U_{n-1})_{\alpha-\varepsilon_k},\end{aligned}$$

and then (4.13) follows from (2.17). Theorem 4.5 is proved. $\square$

Here is another result about solvability of (4.3), this time with random f. We use the space $(\mathcal{S})_{\rho,q}$, defined by the weights (2.6).

THEOREM 4.6. *In addition to the assumptions of Theorem* 4.3, *let* $C_A \leq 1$ *and* $C_k \leq 1$ *for all* k. *If* $f \in (\mathcal{S})_{-1,-\ell}(V')$ *for some* $\ell > 1$, *then there exists a unique solution* $u \in (\mathcal{S})_{-1,-\ell-4}(V)$ *of* (4.3) *and*

$$\|u\|_{(\mathcal{S})_{-1,-\ell-4}(V)} \leq C(\ell)\|f\|_{(\mathcal{S})_{-1,-\ell}(V')}. \tag{4.14}$$

Proof. Denote by $u(g;\gamma)$, $\gamma\in\mathcal{J}$, $g\in V'$, the solution of (4.3) with $f_\alpha = gI_{(\alpha=\gamma)}$, and define $\bar{u}_\alpha = (\alpha!)^{-1/2}u_\alpha$. Clearly, $u_\alpha(g,\gamma) = 0$ if $|\alpha| < |\gamma|$ and so

$$\sum_{\alpha\in\mathcal{J}} \|u_\alpha(f_\gamma;\gamma)\|_V^2 r_\alpha^2 = \sum_{\alpha\in\mathcal{J}} \|u_{\alpha+\gamma}(f_\gamma;\gamma)\|_V^2 r_{\alpha+\gamma}^2. \tag{4.15}$$

It follows from (4.5) that

$$\bar{u}_{\alpha+\gamma}(f_\gamma;\gamma) = \bar{u}_\alpha\big(f_\gamma(\gamma!)^{-1/2};(0)\big). \tag{4.16}$$

Now we use (4.12) to conclude that

$$\|\bar{u}_{\alpha+\gamma}(f_\gamma;\gamma)\|_V \leq \frac{|\alpha|!}{\sqrt{\alpha!\gamma!}}\|f\|_{V'}. \tag{4.17}$$

Coming back to (4.15) with $r_\alpha^2 = (\alpha!)^{-1}(2\mathbb{N})^{(-\ell-4)\alpha}$ and using inequality (2.2) we find

$$\|u(f_\gamma;\gamma)\|_{(\mathcal{S})_{-1,-\ell-4}(V)} \le C(\ell)(2\mathbb{N})^{-2\gamma}\,\frac{\|f_\gamma\|_{V'}}{(2\mathbb{N})^{(\ell/2)\gamma}\sqrt{\gamma!}},$$

where

$$C(\ell) = \left(\sum_{\alpha\in\mathcal{J}}\left(\frac{|\alpha|!}{\alpha!}\right)^2 (2\mathbb{N})^{(-\ell-4)\alpha}\right)^{1/2};$$

(2.14) and (2.2) imply $C(\ell) < \infty$. Then (4.14) follows by the triangle inequality after summing over all γ and using the Cauchy–Schwarz inequality. ∎

Remark 4.7. Example 4.2, in which $f \in (\mathcal{S})_{0,0}$ and $u \in (\mathcal{S})_{-1,q}$, $q < 0$, shows that, while the results of Theorem 4.6 are not sharp, a bound of the type $\|u\|_{(\mathcal{S})_{\rho,q}(V)} \le C\|f\|_{(\mathcal{S})_{\rho,\ell}(V')}$ is, in general, impossible if $\rho > -1$ or $q \ge \ell$.

4.2. Convergence to stationary solution. Let (V, H, V') be a normal triple of Hilbert spaces. Consider equation

$$\dot{u}(t) = (\mathbf{A}u(t) + f(t)) + \mathbf{M}_k u(t) \diamond \xi_k, \tag{4.18}$$

where the operators $\mathbf{A}$ and $\mathbf{M}_k$ do not depend on time, and assume that there exists an $f^* \in \mathcal{R}L_2(\mathbb{F}; H)$ such that $\lim_{t\to\infty}\|f(t) - f^*\|_{\mathcal{R}L_2(\mathbb{F};H)} = 0$. The objective of this section is to study convergence, as $t \to +\infty$, of the solution of (4.18) to the solution u^* of the stationary equation

$$-\mathbf{A}u^* = f^* + \mathbf{M}_k u^* \diamond \xi_k. \tag{4.19}$$

THEOREM 4.8. *Assume the following.*

(C1) *Each $\mathbf{M}_k$ is a bounded linear operator from H to H, and $\mathbf{A}$ is a bounded linear operator from V to V' with the property*

$$\langle \mathbf{A}v, v\rangle + \kappa\|v\|_V^2 \le -c\|v\|_H^2 \tag{4.20}$$

for every $v \in V$, with $\kappa > 0$ and $c > 0$ both independent of v.

(C2) *$f \in \bar{\mathcal{R}}L_2(\mathbb{F}; \mathcal{H})$ and there exists an $f^* \in \bar{\mathcal{R}}L_2(\mathbb{F}; H)$ such that $\lim_{t\to+\infty}\|f(t) - f^*\|_{\bar{\mathcal{R}}L_2(\mathbb{F};H)}$.*

Then, for every $u_0 \in \bar{\mathcal{R}}L_2(\mathbb{F}; H)$, there exists an operator $\mathcal{R}$ so that

1. *there exists a unique solution $u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ of* (4.18),
2. *there exists a unique solution $u^* \in \mathcal{R}L_2(\mathbb{F}; V)$ of* (4.19), *and*
3. *the following convergence holds:*

$$\lim_{t\to+\infty}\|u(t) - u^*\|_{\mathcal{R}L_2(\mathbb{F};H)} = 0. \tag{4.21}$$

Proof. 1. Existence and uniqueness of the solution of (4.18) follow from Theorem 3.10 and Remark 3.9.

2. Existence and uniqueness of the solution of (4.19) follow from Theorem 4.3 and Remark 4.4.

3. The proof of (4.21) is based on the following result.

LEMMA 4.9. *Assume that the operator $\mathbf{A}$ satisfies* (4.20) *and $F = F(t)$ is a deterministic function such that $\lim_{t\to+\infty}\|F(t)\|_H = 0$. Then, for every $U_0 \in H$,* *the solution $U = U(t)$ of the equation $U(t) = U_0 + \int_0^t \mathbf{A}U(s)ds + \int_0^t F(s)ds$ satisfies* $\lim_{t\to+\infty} \|U(t)\|_H = 0$.

Proof. If $\Phi = \Phi_t$ is the semigroup generated by the operator $\mathbf{A}$ (which exists because of (4.20)), then

$$U(t) = \Phi_t U_0 + \int_0^t \Phi_{t-s} F(s)ds.$$

Condition (4.20) implies $\|\Phi_t U_0\|_H \le e^{-ct}\|U_0\|_H$, and then

$$\|U(t)\|_H \le e^{-ct}\|U_0\|_H + \int_0^t e^{-c(t-s)}\|F(s)\|_H ds.$$

The convergence of $\|U(t)\|_H$ to zero now follows from the Toeplitz lemma (see Lemma C.1). Lemma 4.9 is proved. $\square$

To complete the proof of Theorem 4.8, we define $v_\alpha(t) = u_\alpha(t) - u_\alpha^*$ and note that

$$\dot{v}_\alpha(t) = \mathbf{A}v_\alpha(t) + (f_\alpha(t) - f_\alpha^*) + \sum_k \sqrt{\alpha_k}\, \mathbf{M}_k v_{\alpha-\varepsilon_k}.$$

By Theorem 4.3, $u_\alpha^* \in V$ and so $v_\alpha(0) \in H$ for every $\alpha \in \mathcal{J}$. By Lemma 4.9, $\lim_{t\to+\infty} \|v_{(0)}(t)\|_H = 0$. Using induction on $|\alpha|$ and the inequality $\|\mathbf{M}_k v_{\alpha-\varepsilon_k}(t)\|_H \le c_k\|v_{\alpha-\varepsilon_k}(t)\|_H$, we conclude that $\lim_{t\to+\infty} \|v_\alpha(t)\|_H = 0$ for every $\alpha \in \mathcal{J}$. Since $v_\alpha \in \mathbf{C}((0,T);H)$ for every T, it follows that $\sup_{t\ge 0} \|v_\alpha(t)\|_H < \infty$. Define the operator $\mathcal{R}$ on $L_2(\mathbb{F})$ so that $\mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha$, where

$$r_\alpha = \frac{(2\mathbb{N})^{-\alpha}}{1 + \sup\limits_{t\ge 0} \|v_\alpha(t)\|_H}.$$

Then (4.21) follows by the dominated convergence theorem.

Theorem 4.8 is proved. $\square$

5. Bilinear parabolic and elliptic SPDEs. Let G be a smooth bounded domain in $\mathbb{R}^d$ and $\{h_k,\ k \ge 1\}$ be an orthonormal basis in $L_2(G)$. We assume that

$$\sup_{x\in G} |h_k(x)| \le c_k,\ k \ge 1. \tag{5.1}$$

A space white noise on $L_2(G)$ is a formal series

$$\dot{W}(x) = \sum_{k\ge 1} h_k(x)\xi_k, \tag{5.2}$$

where $\xi_k,\ k \ge 1$, are independent standard Gaussian random variables.

5.1. Dirichlet problem for parabolic SPDE of the second order. Consider the following equation:

(5.3)
$$\begin{aligned} u_t(t,x) &= a_{ij}(x)D_iD_j u(t,x) + b_i(x)D_i u(t,x) + c(x)u(t,x) + f(t,x) \\ &\quad + (\sigma_i(x)D_i u(t,x) + \nu(x)u(t,x) + g(t,x)) \diamond \dot{W}(x),\ 0 < t \le T,\ x \in G, \end{aligned}$$

with zero boundary conditions and some initial condition $u(0,x)=u_0(x)$; the functions a_{ij}, b_i, c, f, σ_i, ν, g, and u_0 are nonrandom. In (5.3) and in similar expressions below we assume summation over the repeated indices. Let (V,H,V') be the normal triple with $V=\overset{\circ}{H}{}^1_2(G)$, $H=L_2(G)$, and $V'=H_2^{-1}(G)$. In view of equation (5.2), (5.3) is a particular case of (3.4) so that

$$\mathbf{A}u=a_{ij}(x)D_iD_ju+b_i(x)D_iu+c(x)u,\ \mathbf{M}_ku=(\sigma_i(x)D_iu+\nu(x)u)h_k(x), \tag{5.4}$$

and $f(t,x)+g(t,x)\diamond\dot{W}(x)$ is the free term.

We make the following assumptions about the coefficients:

D1. The functions a_{ij} are Lipschitz continuous in the closure $\bar{G}$ of G, and the functions b_i, c, σ_i, and ν are bounded and measurable in $\bar{G}$.

D2. There exist positive numbers A_1,A_2 so that $A_1|y|^2\le a_{ij}(x)y_iy_j\le A_2|y|^2$ for all $x\in\bar{G}$ and $y\in\mathbb{R}^d$.

Given a $T>0$, recall the notation $\mathcal{V}=L_2((0,T);V)$ and similarly for $\mathcal{H}$ and $\mathcal{V}'$ (see (3.1)).

THEOREM 5.1. *Under the assumptions* D1 *and* D2, *if* $f\in\mathcal{V}'$, $g\in\mathcal{H}$, *and* $u_0\in H$, *then there exists an* $\ell>1$ *and a number* $C>0$, *both independent of* u_0,f *and* g, *so that* $u\in\mathcal{R}L_2(\mathbb{F};\mathcal{V})$ *and*

$$\|u\|_{\mathcal{R}L_2(\mathbb{F};\mathcal{V})}\le C\cdot\big(\|u_0\|_H+\|f\|_{\mathcal{V}'}+\|g\|_{\mathcal{H}}\big), \tag{5.5}$$

where the operator $\mathcal{R}$ *is defined by the weights*

$$r_\alpha^2=c^{-2\alpha}(|\alpha|!)^{-1}(2\mathbb{N})^{-2\ell\alpha} \tag{5.6}$$

and $c^\alpha=\prod_k c_k^{\alpha_k}$, *with* c_k *from* (5.1); *the number* ℓ *in general depends on* T.

Proof. We derive the result from Theorem 3.11. Consider the deterministic equation $\dot{U}(t)=\mathbf{A}U(t)+F$. Assumptions D1 and D2 imply that there exists a unique solution of this equation in the normal triple (V,H,V'), and the solution satisfies

$$\sup_{0<t<T}\|U(t)\|_H+\|U\|_{\mathcal{V}}\le C\cdot\big(\|U(0)\|_H+\|F\|_{\mathcal{V}'}\big), \tag{5.7}$$

where the number C depends on T and the operator $\mathbf{A}$. Moreover, (5.1) implies that (3.15) holds with $C_k=C_0c_k$ for some positive number C_0 independent of k, but possibly depending on T.

To proceed, let us assume first that $g=0$. Then the statement of the theorem follows directly from Theorem 3.11 if we take in (3.18) $q_k={c_k}^{-1}(2k)^{-\ell}$ with sufficiently large ℓ.

It now remains to consider the case $g\ne0$ and $f=u_0=0$. Even though g is nonrandom, $g\xi_k$ is, and therefore a direct application of Theorem 3.11 is not possible. Instead, let us look more closely at the corresponding equations for u_α. For $\alpha=(0)$,

$$u_{(0)}(t)=\int_0^t\mathbf{A}u_{(0)}(s)ds,$$

which implies $u_{(0)}(t)=0$ for all t. For $\alpha=\varepsilon_k$,

$$u_{\varepsilon_k}(t)=\int_0^t\mathbf{A}u_{\varepsilon_k}(s)ds+h_k\int_0^t g(s)ds$$

or

$$u_{\varepsilon_k}(t) = \int_0^t \Phi_{t-s} h_k g(s) ds,$$

so that

$$\|u_{\varepsilon_k}\|_{\mathcal{V}} \leq C_0 c_k \|g\|_{\mathcal{H}}. \tag{5.8}$$

If $|\alpha| > 1$, then

$$u_\alpha(t) = \int_0^t \mathbf{A} u_\alpha(s) ds + \sum_{k\geq 1} \sqrt{\alpha_k}\, \mathbf{M}_k u_{\alpha-\varepsilon_k},$$

which is the same as (3.22). In particular, if $|\alpha| = 2$ and $\{i, j\}$ is the characteristic set of α, then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \int_0^t \Phi_{t-s} \left(\mathbf{M}_i u_{\varepsilon_j}(s) + \mathbf{M}_j u_{\varepsilon_i}(s)\right) ds.$$

More generally, by analogy with (3.25), if $|\alpha| = n > 2$ and $\{k_1, \ldots, k_n\}$ is the characteristic set of α, then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_0^t \int_0^{s_n} \ldots \int_0^{s_3} \Phi_{t-s_n} \mathbf{M}_{k_{\sigma(n)}} \cdots \Phi_{s_3-s_2} \mathbf{M}_{k_{\sigma(2)}} u_{\varepsilon_{\sigma(1)}}(s_2) ds_2 \ldots ds_n.$$

By the triangle inequality and (5.8),

$$\|u_\alpha\|_{\mathcal{V}} \leq \frac{|\alpha|! C_0^{|\alpha|} c^\alpha}{\sqrt{\alpha!}} \|g\|_{\mathcal{H}},$$

and then (5.5) follows from (2.2) if ℓ is sufficiently large.

This completes the proof of Theorem 5.1. □

THEOREM 5.2. *In addition to* D1 *and* D2, *assume that*

1. $\sigma_i = 0$ *for all* i;
2. *the operator* $\mathbf{A}$ *in* G *with zero boundary conditions satisfies* (4.20).

If there exist functions f^* *and* g^* *from* H *so that*

$$\lim_{t\to+\infty} (\|f(t) - f^*\|_H + \|g(t) - g^*\|_H) = 0, \tag{5.9}$$

then the solution u *of* (5.3) *satisfies*

$$\lim_{t\to+\infty} \|u(t) - u^*\|_{\mathcal{R}L_2(\mathbb{F};H)} = 0, \tag{5.10}$$

where the operator $\mathcal{R}$ *is defined by the weights* (5.6) *and* u^* *is the solution of the stationary equation*

$$\begin{aligned} a_{ij}(x) D_i D_j u^*(x) + b_i(x) D_i u^*(x) + c(x) u^*(x) + f^*(x) \\ + (\nu(x) u^*(x) + g^*(x)) \diamond \dot{W}(x) = 0,\ x \in G;\ u|_{\partial G} = 0. \end{aligned} \tag{5.11}$$

Proof. This follows from Theorem 4.8. □

Remark 5.3. The operator $\mathbf{A}$ satisfies (4.20) if, for example, each a_{ij} is twice continuously differentiable in $\bar{G}$, each b_i continuously differentiable in $\bar{G}$, and

$$\inf_{x\in\bar{G}} c(x) - \sup_{x\in\bar{G}} (|D_i D_j a_{ij}(x)| + |D_i b_i(x)|) \geq \varepsilon > 0; \tag{5.12}$$

this is verified directly using integration by parts.

5.2. Elliptic SPDEs of full second order. Consider the following Dirichlet problem:

$$\begin{array}{c} -D_i\left(a_{ij}(x)\,D_j u(x)\right) + D_i\left(\sigma_{ij}(x)\,D_j\left(u(x)\right)\right)\diamond \dot{W}(x) = f(x),\ x\in G,\\ u_{|\partial G} = 0, \end{array} \tag{5.13}$$

where $\dot{W}$ is the space white noise (5.2). Assume that the functions a_{ij}, σ_{ij}, f, and g are nonrandom. Recall that according to our summation convention, in (5.13) and in similar expressions below we assume summation over the repeated indices.

We make the following assumptions:

E1. The functions $a_{ij} = a_{ij}(x)$ and $\sigma_{ij} = \sigma_{ij}(x)$ are measurable and bounded in the closure $\bar{G}$ of G.

E2. There exist positive numbers A_1, A_2 so that $A_1|y|^2 \le a_{ij}(x)y_iy_j \le A_2|y|^2$ for all $x \in \bar{G}$ and $y \in \mathbb{R}^d$.

E3. The functions h_k in (5.2) are bounded and Lipschitz continuous.

Clearly, (5.13) is a particular case of (4.3) with

$$\mathbf{A}u(x) := -D_i\Big(a_{ij}(x)\,D_j u(x)\Big) \tag{5.14}$$

and

$$\mathbf{M}_k u(x) := h_k(x)\,D_i\Big(\sigma_{ij}(x)\,D_j u(x)\Big). \tag{5.15}$$

Assumptions E1 and E3 imply that each $\mathbf{M}_k$ is a bounded linear operator from $\overset{\circ}{H}{}_2^1(G)$ to $H_2^{-1}(G)$. Moreover, it is a standard fact that under the assumptions E1 and E2 the operator $\mathbf{A}$ is an isomorphism from V onto V' (see, e.g., [19]). Therefore, for every k there exists a positive number C_k such that

$$\left\|\mathbf{A}^{-1}M_k v\right\|_V \le C_k\,\|v\|_V\,,\ v\in V. \tag{5.16}$$

THEOREM 5.4. *Under the assumptions* E1 *and* E2, *if* $f \in H_2^{-1}(G)$, *then there exists a unique solution of the Dirichlet problem* (5.13) $u \in \mathcal{R}L_2(\mathbb{F}; \overset{\circ}{H}{}_2^1(G))$ *such that*

$$\|u\|_{\mathcal{R}L_2(\mathbb{F};\overset{\circ}{H}{}_2^1(G))} \le C\cdot\|f\|_{H_2^{-1}(G)}. \tag{5.17}$$

The weights r_α can be taken in the form

$$r_\alpha = \frac{q^\alpha}{\sqrt{|\alpha|!}},\ \textit{where}\ q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \tag{5.18}$$

and the numbers q_k, $k \ge 1$ are chosen so that $\sum_{k\ge1} C_k^2 q_k^2 < 1$, with C_k from (5.16).

Proof. This follows from Theorem 4.5. ◻

Remark 5.5. With an appropriate change of the boundary conditions, and with extra regularity of the basis functions h_k, the results of Theorem 5.4 can be extended to stochastic elliptic equations of order $2m$. The corresponding operators are

$$\mathbf{A}u = (-1)^m D_{i_1}\cdots D_{i_m}\left(a_{i_1\ldots i_m j_1\ldots j_m}(x)\,D_{j_1}\cdots D_{j_m}u(x)\right) \tag{5.19}$$

and

$$\mathbf{M}_k u = h_k(x)\,D_{i_1}\cdots D_{i_m}\left(\sigma_{i_1\ldots i_m j_1\ldots j_m}(x)\,D_{j_1}\cdots D_{j_m}u(x)\right). \tag{5.20}$$

Since G is a smooth bounded domain, regularity of h_k is not a problem: we can take h_k as the eigenfunctions of the Dirichlet Laplacian in G.

Appendix A. Two ways of multiplying random variables.

It might be instructive to examine the differences and similarities between the Wick product models

$$\dot{u}(t,x) = \mathbf{A}u(t,x) + \mathbf{M}u\,(t,x) \diamond \dot{W}\,(t,x)\,, \tag{A.1}$$

$$\mathbf{A}u(t,x) = \mathbf{M}u\,(t,x) \diamond \dot{W}\,(x) \tag{A.2}$$

and their more "intuitive" counterparts with the usual product

$$\dot{u}(t,x) = \mathbf{A}u(t,x) + \mathbf{M}u\,(t,x) \cdot \dot{W}\,(t,x)\,, \tag{A.3}$$

$$\mathbf{A}u(t,x) = \mathbf{M}u\,(t,x) \cdot \dot{W}\,(x)\,. \tag{A.4}$$

Obviously, if interpreted literally, the "dot" product models (A.3) and (A.4) are not well-defined. Historically, Wick product models were introduced to bypass the exceeding singularity of the "intuitive" models. In fact, the idea of reduction to Wick product could be traced to the pioneering work of Itô on stochastic calculus and stochastic ODEs.

Itô–Skorohod integrals can be interpreted in terms of Wick product: if $f = f(t)$ is an adapted square-integrable process and W is a standard Brownian motion, then $\int_0^T f(t) \diamond \dot{W}(t)dt$ is equivalent to the Itô integral $\int_0^T f(t)dW(t)$ (see [8, 9, 17]). More generally, in contrast to the "dot" product, Wick product (Itô–Skorohod integral) is a stochastic convolution, and one could argue that, from the physics standpoint, convolution is a more realistic model of the system's response to a perturbation.

In spite of the aforementioned differences between the Wick product and "dot" product models, they are closely related. In fact, the Wick product models could be viewed as the highest stochastic order approximations to "dot" product models. Indeed, if H_m and H_n are Hermite polynomials and ξ is a standard normal random variable, then $H_m(\xi) \diamond H_n(\xi) = H_{m+n}(\xi)$, while $H_m(\xi) \cdot H_n(\xi) = H_{m+n}(\xi) + R_{n,m}(\xi)$, where R_{m+n} is a linear combination of Hermite polynomials of orders lesser than $m+n$.

We are not aware of any systematic efforts to investigate SPDEs (A.3) and (A.4). It appears that the general methodology developed in this paper could be extended to address (A.3) and (A.4). However, in the "dot" product setting, the propagator is not lower-triangular and the solution spaces are expected to be much larger than in the setting of this paper.

Appendix B. A factorial inequality.

LEMMA B.1. *For every multi-index* $\alpha \in \mathcal{J}$,

$$|\alpha|! \le \alpha!(2\mathbb{N})^{2\alpha}. \tag{B.1}$$

Proof. Recall that, for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{J}$,

$$|\alpha| = \sum_{\ell=1}^{k} \alpha_\ell,\ \ \alpha! = \prod_{\ell=1}^{k} \alpha_\ell!,\ \ \mathbb{N}^\alpha = \prod_{\ell=1}^{k} \ell^{\alpha_\ell}. \tag{B.2}$$

It is therefore clear that if $|\alpha| = n$, then it is enough to establish (A1) for α with $\alpha_k = 0$ for $k \geq n+1$, because a shift of a multi-index entry to the right increases the right-hand side of (A5) but does not change the left-hand side. For example, if $\alpha = (1,3,2,0,\dots)$ and $\beta = (1,3,0,2,0,\dots)$, then $|\alpha| = |\beta|$, $\alpha! = \beta!$, but $\mathbb{N}^{\alpha} < \mathbb{N}^{\beta}$. Then

$$4^n \geq \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)^n = \sum_{\alpha_1+\ldots+\alpha_n=n} \frac{|\alpha|!}{\alpha!}\frac{1}{\mathbb{N}^{2\alpha}}, \tag{B.3}$$

where the equality follows by the multinomial formula. Since all the terms in the sum are nonnegative, we get (B.1). $\square$

The proof shows that inequality (B.1) can be improved by observing that

$$\sum_{k\geq 1} k^{-2} = \pi^2/6 < 2.$$

One can also consider $\sum_{k\geq1} k^{-q}$ for some $1 < q < 2$. A different proof was given in [9].

Appendix C. A version of the Toeplitz lemma.

LEMMA C.1. *Assume that $f = f(t)$ is an integrable function and $\lim_{t\to+\infty} |f(t)| = 0$. Then, for every $c > 0$, $\lim_{t\to+\infty} \int_0^t e^{-c(t-s)} f(s)ds = 0$.*

Proof. Given $\varepsilon > 0$, choose T so that $|f(t)| < \varepsilon$ for all $t > T$. Then

$$\left|\int_0^t e^{-c(t-s)} f(s)ds\right| \leq e^{-ct}\int_0^T e^{cs}|f(s)|ds + \varepsilon\int_T^t e^{-c(t-s)}ds.$$

Passing to the limit as $t \to +\infty$, we find $\lim_{t\to+\infty} |\int_0^t e^{-c(t-s)} f(s)ds| \leq \varepsilon/c$, which completes the proof. $\square$

REFERENCES

[1] R. H. CAMERON AND W. T. MARTIN, *The orthogonal development of nonlinear functionals in a series of Fourier-Hermite functions*, Ann. Math., 48 (1947), pp. 385–392.

[2] R. A. CARMONA AND S. A. MOLCHANOV, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc., 108 (1994), pp. viii+125.

[3] R. A. CARMONA AND B. L. ROZOVSKII, eds., *Stochastic partial differential equations: Six perspectives*, Mathematical Surveys and Monographs 64, American Mathematical Society, Providence, RI, 1999.

[4] G. DA PRATO AND J. ZABCZYK, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[5] C. DONATI-MARTIN, *Quasi-linear elliptic stochastic partial differential equation: Markov property*, Stoch. Stoch. Rep., 41 (1992), pp. 219–240.

[6] K. GAWĘDZKI AND M. VERGASSOLA, *Phase transition in the passive scalar advection*, Phys. D, 138 (2000), pp. 63–90.

[7] K. GAWEDZKI AND A. KUPIAINEN, *Universality in turbulence: An exactly solvable model*, in Low-dimensional Models in Statistical Physics and Quantum Field Theory, Springer, Berlin, 1996, pp. 71–105.

[8] T. HIDA, H-H. KUO, J. POTTHOFF, AND L. SREIT, *White Noise*, Kluwer Academic Publishers, Boston, 1993.

[9] H. HOLDEN, B. ØKSENDAL, J. UBØE, AND T. ZHANG, *Stochastic Partial Differential Equations*, Birkhäuser, Boston, 1996.

[10] Y. HU, *Chaos expansion of heat equations with white noise potentials*, Potential Anal., 16 (2002), pp. 45–66.

[11] K. ITO, *Multiple Wiener integral*, J. Math. Soc. Japan, 3 (1951), pp. 157–169.

[12] Yu. G. Kondratiev, P. Leukert, J. Potthoff, L. Streit, and W. Westerkamp, *Generalized functionals in Gaussian spaces: The characterization theorem revisited*, J. Funct. Anal., 141 (1996), pp. 301–318.

[13] Yu. G. Kondratiev and Yu. S. Samoylenko, *The spaces of trial and generalized functions of infinite number of variables*, Rep. Math. Phys., 14 (1978), pp. 325–350.

[14] N. V. Krylov, *Introduction to the Theory of Diffusion Processes*, American Mathematical Society, Providence, RI, 1995.

[15] N. V. Krylov, *An analytic approach to SPDEs*, in Stochastic Partial Differential Equations. Six Perspectives, Mathematical Surveys and Monographs, B. L. Rozovskii and R. Carmona, eds., AMS, Providence, RI, 1999, pp. 185–242.

[16] N. V. Krylov and B. L. Rozovskii, *Stochastic evolution equations*, in Stochastic Differential Equations: Theory and Applications, Interdiscip. Math. Sci. 2, World Scientific Publishers, Hackensack, NJ, 2007, pp. 1–69.

[17] H.-H. Kuo, *White noise distribution theory*, Probability and Stochastics Series, CRC Press, Boca Raton, FL, 1996.

[18] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[19] J.-L. Lions and E. Magenes, *Problémes aux Limites Non Homogènes et Applications,* Vol. 1, Dunod, Paris, 1968.

[20] S. V. Lototsky and B. L. Rozovskii, *Passive scalar equation in a turbulent incompressible Gaussian velocity field*, Russian Math. Surveys, 59 (2004), pp. 297–312.

[21] S. V. Lototsky and B. L. Rozovskii, *Stochastic differential equations: A Wiener chaos approach*, in From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift, Yu. Kabanov, R. Liptser, and J. Stoyanov, eds., Springer, Berlin, 2006, pp. 433–507.

[22] S. V. Lototsky and B. L. Rozovskii, *Wiener chaos solutions of linear stochastic evolution equations*, Ann. Probab., 34 (2006), pp. 638–662.

[23] J. C. Matingly and Ya. G. Sinai, *An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations*, Comm. Control Math., 1 (1999), pp. 497–516.

[24] R. Mikulevicius and B. L. Rozovskii, *Linear parabolic stochastic PDE's and Wiener chaos*, SIAM J. Math. Anal., 29 (1998), pp. 452–480.

[25] D. Nualart, *Malliavin Calculus and Related Topics*, 2nd ed., Springer, New York, 2006.

[26] D. Nualart and B. L. Rozovskii, *Weighted stochastic Sobolev spaces and bilinear SPDE's driven by space-time white noise*, J. Funct. Anal., 149 (1997), pp. 200–225.

[27] D. Nualart and S. Tindel, *Quasilinear stochastic elliptic equations with reflection*, Stochastic Process. Appl., 57 (1995), pp. 73–82.

[28] E. Pardoux, *Equations aux derivéespartielles stochastiques non linearies monotones. Etude de solutions fortes de type Itô*, Ph.D. thesis, Univ. Paris Sud, 1975. Thése Doct. Sci. Math.

[29] B. L. Rozovskii, *Stochastic Evolution Systems*, Kluwer Academic Publishers, Dordrecht, 1990.

[30] T. Gorm Theting, *Solving Wick-stochastic boundary value problems using a finite element method*, Stoch. Stoch. Rep., 70 (2000), pp. 241–270.

[31] G. Våge, *Variational methods for PDEs applied to stochastic partial differential equations*, Math. Scand., 82 (1998), pp. 113–137.

[32] J. B. Walsh, *An introduction to stochastic partial differential equations*, in École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math. 1180, Springer, Berlin, 1986, pp. 265–439.

[33] M. Zakai, *On the optimal filtering of diffusion processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 11 (1969), pp. 230–243.