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ASYMPTOTIC PROPERTIES OF THE MAXIMUM LIKELIHOOD ESTIMATOR FOR STOCHASTIC PARABOLIC EQUATIONS WITH ADDITIVE FRACTIONAL BROWNIAN MOTION

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A parameter estimation problem is considered for a diagonalizable stochastic evolution equation using a finite number of the Fourier coefficients of the solution. The equation is driven by additive noise that is white in space and fractional in time with the Hurst parameter $H \geq 1/2$. The objective is to study asymptotic properties of the maximum likelihood estimator as the number of the Fourier coefficients increases. A necessary and sufficient condition for consistency and asymptotic normality is presented in terms of the eigenvalues of the operators in the equation.

Keywords: Asymptotic normality; ergodicity; parameter estimation; stochastic evolution equations.

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1. Introduction

In the classical statistical estimation problem, the starting point is a family $\mathbf{P}^\theta$ of probability measures depending on the parameter θ in some subset Θ of a finite-dimensional Euclidean space. Each $\mathbf{P}^\theta$ is the distribution of a random element. It is assumed that a realization of one random element corresponding to one value $\theta = \theta_0$ of the parameter is observed, and the objective is to estimate the values of this parameter from the observations.

The intuition is to select the value θ corresponding to the random element that is *most likely* to produce the observations. A rigorous mathematical implementation of this idea leads to the notion of the regular statistical model [4]: the statistical model (or estimation problem) $\mathbf{P}^\theta,\ \theta \in \Theta$, is called regular, if there exists a probability measure $\mathbf{Q}$ such that all the measures $\mathbf{P}^\theta$ are absolutely continuous with respect to $\mathbf{Q}$ and the density $d\mathbf{P}^\theta/d\mathbf{Q}$, called the likelihood ratio, has a certain regularity.

In regular models, the estimator $\widehat{\theta}$ of the unknown parameter is constructed by maximizing the likelihood ratio and is called the maximum likelihood estimator (MLE). Since, as a rule, $\widehat{\theta} \neq \theta_0$, the consistency of the estimator is studied, that is, the convergence of $\widehat{\theta}$ to θ_0 as more and more information becomes available. In all known regular statistical problems, the amount of information can be increased in one of two ways: (a) increasing the sample size, for example, the observation time interval (large sample asymptotic); (b) reducing the amplitude of noise (small noise asymptotic).

In finite-dimensional models, the only way to increase the sample size is to increase the observation time. In infinite-dimensional models, in particular, those provided by stochastic partial differential equations (SPDEs), another possibility is to increase the dimension of the spatial projection of the observations. Thus, a consistent estimator can be possible on a finite time interval with fixed noise intensity. This possibility was first suggested by Huebner *et al.* [2] for parabolic equations driven by additive spacetime white noise, and was further investigated by Huebner and Rozovskii [3], where a necessary and sufficient condition for the existence of a consistent estimator was stated in terms of the orders of the operators in the equation.

The objective of this paper is to extend the model from [3] to parabolic equations in which the time component of the noise is fractional with the Hurst parameter $H \geq 1/2$. More specifically, we consider an abstract evolution equation

$$u(t) + \int_0^t (\mathcal{A}_0 + \theta\mathcal{A}_1)u(s)ds = W^H(t), \tag{1.1}$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ are known linear operators and $\theta \in \Theta \subseteq \mathbb{R}$ is the unknown parameter; the zero initial condition is taken to simplify the presentation. The noise $W^H(t)$ is the cylindrical fractional Brownian motion on a Hilbert space $\mathbf{H}$ and can be written as

$$W^H(t) = \sum_{j=1}^{\infty} w_j^H(t)h_j, \tag{1.2}$$

where $\{w_j^H,\ j \geq 1\}$ are independent fractional Brownian motions with the same Hurst parameter $H \geq 1/2$ and $\{h_j,\ j \geq 1\}$ is an orthonormal basis in a Hilbert space $\mathbf{H}$; $H = 1/2$ corresponds to the cylindrical Brownian motion. It can be shown (see Proposition 2.1 below) that (1.2) defines a continuous Gaussian process with values in some larger Hilbert space $\mathbf{X}$. Existence and uniqueness of the solution under certain regularity conditions on $\mathcal{A}_0$ and $\mathcal{A}_1$ for such equations are well known for all $H \in (0,1)$; see, for example, Tindel *et al.* [17, Theorem 1].

The main additional assumption about (1.1), both in [3] and in this paper, is that the equation is *diagonalizable*: $\{h_j,\ j \geq 1\}$ from (1.2) is a common system of eigenfunction of the operators $\mathcal{A}_0$ and $\mathcal{A}_1$:

$$\mathcal{A}_0 h_j = \rho_j h_j, \quad \mathcal{A}_1 h_j = \nu_j h_j. \tag{1.3}$$

Under certain conditions on the numbers $\rho_j,\ \nu_j$, the solution of (1.1) is a convergent Fourier series $u(t) = \sum_{j\geq 1} u_j(t)h_j$, and each $u_j(t)$ is a fractional Ornstein–Uhlenbeck (OU) process. An N-dimensional projection of the solution is then an N-dimensional fractional OU process with independent components. A Girsanov-type formula (for example, from Kleptsyna *et al.* [8, Theorem 3]) leads to a maximum likelihood estimator $\hat{\theta}_N$ of θ based on the first N Fourier coefficients $u_1, \ldots, u_N$ of the solution of (1.1). An explicit expression for this estimator exists but requires a number of additional notations; see formula (3.8) below. A particular case of Eq. (1.1) was studied by Prakasa Rao [14].

If the solution of (1.1) is completely observable at every $t \in [0,T]$, then each Fourier coefficient u_k, being a linear functional of the solution, can be computed. For example, if u is regular enough so that $u(t) \in L_2(\Omega; H)$ for every t, then $u_k(t) = (u(t), h_k)_H$.

The following is the main result of the paper.

Theorem 1.1. *Define $\mu_j = \theta\nu_j + \rho_j$. Then the maximum likelihood estimator $\hat{\theta}_N$ of θ is strongly consistent and asymptotically normal, as $N \to \infty$, if and only if the series $\sum_j \nu_j^2\mu_j^{-1}$ diverges; the rate of convergence of the estimator is given by the square root of the partial sums of this series: as $N \to \infty$, the sequence $\big(\sum_{j\leq N} \nu_j^2\mu_j^{-1}\big)^{1/2}(\hat{\theta}_N - \theta)$ converges in distribution to a Gaussian random variable.*

If the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are elliptic of orders m_0 and m_1 in $L_2(M)$, where M is a d-dimensional manifold, and $2m = \max(m_0, m_1)$, then the asymptotic of the eigenvalues of elliptic operators implies that the divergence of $\sum_j \nu_j^2\mu_j^{-1}$ is equivalent to $m_1 \geq m - (d/2)$; in the case $H = 1/2$ this condition on the orders was derived in [3]. Thus, beside extending the results of [3] to fractional-in-time noise, we also generalize the necessary and sufficient condition for consistency of the estimator.

While parameter estimation for the finite-dimensional fractional OU and similar processes has been recently investigated by Tudor and Viens [18] for all $H \in (0,1)$, our analysis in infinite dimensions requires more delicate results: an explicit expression for the Laplace transform of a certain functional of the fractional OU process, as obtained by Kleptsyna and Le Breton [7], and for now this expression exists only for $H \geq 1/2$.

2. Stochastic Parabolic Equations with Additive FBM

In this section we introduce a diagonalizable stochastic parabolic equation driven by a cylindrical fractional Brownian motion and study the main properties of the solution.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbf{H}$ be a separable Hilbert space with an orthonormal basis $\{h_j,\ j \geq 1\}$.

Definition 2.1. (a) A fractional Brownian motion with a Hurst parameter $H \in (0,1)$ is a Gaussian process w^H with zero mean and covariance

$$\mathbb{E}w^H(t)w^H(s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geq 0.$$

(b) The *cylindrical fractional Brownian motion* W^H on a Hilbert space $\mathbf{H}$ is a collection of zero-mean Gaussian processes $W_f^H = W_f^H(t)$ indexed by the elements f of the space $\mathbf{H}$ such that, for every $f, g \in \mathbf{H}$ and $t, s \geq 0$,

$$\mathbb{E}(W_f^H(t)W_g^H(s)) = \frac{(f,g)_{\mathbf{H}}}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.1}$$

Proposition 2.1. *Let* $\mathbf{H}$ *be a separable Hilbert space with an orthonormal basis* $\{h_j,\ j \geq 1\}$.

(a) *If* W^H *is a cylindrical Brownian motion on* $\mathbf{H}$, *then* $\{W_{h_j}^H(t),\ j \geq 1\}$ *is a collection of independent fractional Brownian motions with the same Hurst parameter* H.
(b) *Let* $\mathbf{X}$ *is a Hilbert space such that* $\mathbf{H}$ *is a dense subset of* $\mathbf{X}$ *and the inclusion operator* $\mathfrak{j} : \mathbf{H} \to \mathbf{X}$ *is Hilbert–Schmidt. If* $\{w_j^H(t),\ j \geq 1\}$ *is a collection of independent fractional Brownian motions with the same Hurst parameter* H, *then the process* $W^H(t) = \sum_{j\geq 1} w_j^H(t)h_j$ *is a continuous* $\mathbf{X}$*-valued Gaussian process and is a cylindrical fractional Brownian motion on* $\mathbf{H}$.

Proof. (a) This follows directly from (2.1).

(b) Since the inclusion $\mathfrak{j} : \mathbf{H} \hookrightarrow \mathbf{X}$ is Hilbert–Schmidt, we have

$$\sum_{j\geq 1} \|h_j\|_{\mathbf{X}}^2 = C_{\mathfrak{j}} < \infty.$$

Therefore, since

$$\mathbb{E}w_j^H(t)w_m^H(t) = \begin{cases} t^{2H}, & \text{if } k = m \\ 0, & \text{if } k \neq m, \end{cases}$$

we find

$$\mathbb{E}\|W^H(t)\|_{\mathbf{X}}^2 = t^{2H}C_{\mathfrak{j}} < \infty.$$

Similarly,

$$\mathbb{E}\|W^H(t) - W^H(s)\|_{\mathbf{X}}^2 = |t-s|^{2H}C_{\mathfrak{j}},$$

which, for $H > 1/2$, implies continuity by the Kolmogorov criterion. For $H \le 1/2$, we consider $\mathbb{E}\|W^H(t) - W^H(s)\|_{\mathbf{X}}^{2n}$ for sufficiently large integer n. Next, setting

$$W_f^H(t) = \sum_{j\ge 1}(f, h_j)_{\mathbf{H}}^2,$$

we see that W_f^H is a zero-mean Gaussian process and (2.1) holds. □

Consider the following equation:

$$du(t) + (\mathcal{A}_0 + \theta\mathcal{A}_1)u(t)dt = dW^H(t), \quad 0 < t \le T, \tag{2.2}$$

where $\mathcal{A}_0$, $\mathcal{A}_1$ are linear operators, θ is a scalar parameter belonging to an open set $\Theta \subset \mathbb{R}$, and, for simplicity, $u(0) = 0$.

Definition 2.2. Equation (2.2) is called diagonalizable if the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ have point spectrum and a common system of eigenfunctions $\{h_j,\ j \ge 1\}$.

Denote by ρ_j, ν_j, and $\mu_j(\theta)$ the eigenvalues of the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_0 + \theta\mathcal{A}_1$:

$$\mathcal{A}_0 h_j = \rho_j h_j, \quad \mathcal{A}_1 h_j = \nu_j h_j, \quad \mu_j(\theta) = \rho_j + \theta\nu_j, \qquad j \ge 1. \tag{2.3}$$

Definition 2.3. A diagonalizable equation (2.2) is called parabolic if there exist positive numbers C^*, c_1, c_2 such that, after possible re-arrangement, $\{\mu_j(\theta) + C^*,\ j \ge 1\}$ is a positive, non-decreasing, and unbounded sequence for all $\theta \in \Theta$ and

$$c_1 \le \frac{\mu_j(\theta_1) + C^*}{\mu_j(\theta_2) + C^*} \le c_2 \tag{2.4}$$

for all $\theta_1, \theta_2 \in \Theta$.

In particular, if Eq. (2.2) is parabolic, then $\lim_{j\to\infty} \mu_j(\theta) = +\infty$ for all $\theta \in \Theta$, and there exists an index $J \ge 1$ such that, for all $\theta \in \Theta$ and $j \ge J$,

$$\mu_j(\theta) > 0. \tag{2.5}$$

Indeed, if such index does not exist, then there exists a sequence of $\theta_j \in \Theta$, such that $\mu_j(\theta_j) \le 0$, and hence $\mu_j(\theta_j) + C^* \le C^*$. By (2.4), for a fixed θ, we have

$$\lim_{j\to\infty} \frac{\mu_j(\theta_j) + C^*}{\mu_j(\theta) + C^*} = 0,$$

that contradicts the assumption that $c_1 > 0$.

Example 2.1. Let G be a smooth bounded domain in $\mathbb{R}^d$ or a smooth compact d-dimensional manifold with a smooth measure, $\mathbf{H} = L_2(G)$, and let $\boldsymbol{\Delta}$ be the Laplace operator on G (with zero boundary conditions if G is a domain). It is known (see, for example, Safarov and Vassiliev [15] or Shubin [16]) that $\boldsymbol{\Delta}$ has a complete orthonormal system of eigenfunctions in $\mathbf{H}$, and the corresponding eigenvalues λ_j

are negative, can be arranged in decreasing order, and there is a positive number $c_\circ$ such that

$$\lim_{j\to\infty} j^{-2/d}|\lambda_j| = c_\circ. \tag{2.6}$$

The reader can verify that each of the following equations is diagonalizable and parabolic:

$$du - \theta\boldsymbol{\Delta} u dt = dW^H,\ \theta \geq a > 0, \tag{2.7}$$

$$du - (\boldsymbol{\Delta} u + \theta u)dt = dW^H, \tag{2.8}$$

$$du + (\boldsymbol{\Delta}^2 u + \theta\boldsymbol{\Delta} u)dt = dW^H. \tag{2.9}$$

The following result both defines the solution of (2.2) and establishes existence and uniqueness of the solution.

Theorem 2.1. *Assume that Eq.* (2.2) *is diagonalizable and parabolic,* W^H *is a cylindrical fractional Brownian motion on a separable Hilbert space* $\mathbf{H}$, $\{h_j,\ j \geq 1\}$ *is an orthonormal basis in* $\mathbf{H}$, *and* $H \geq 1/2$.

If $\mathbf{X}$ *is a Hilbert space such that* $\mathbf{H}$ *is a dense subset of* $\mathbf{X}$ *and the inclusion operator* $\mathfrak{j} : \mathbf{H} \to \mathbf{X}$ *is Hilbert–Schmidt, then the process* u *defined by*

$$u(t) = \sum_{j\geq 1} u_j(t)h_j, \tag{2.10}$$

where

$$u_j(t) = \int_0^t e^{-\mu_j(\theta)(t-s)}dW^H_{h_j}(s) \tag{2.11}$$

is a Gaussian process with values in $\mathbf{X}$.

Proof. The argument is similar to the proof of Proposition 2.1(b). The properties of the fractional Brownian motion with $H > 1/2$ imply

$$\mathbb{E}u_j^2(t) = H(2H-1)e^{-2\mu_j(\theta)t}\int_0^t\int_0^t e^{\mu_j(\theta)(s_1+s_2)}|s_1-s_2|^{2H-2}ds_1ds_2;$$

see, for example, Pipiras and Taqqu [13, formulas (4.1), (4.2)]. By direct computations,

$$\begin{aligned}\mathbb{E}u_j^2(t) &= \frac{H(2H-1)}{\mu_j(\theta)}\left(\int_0^t s^{2H-2}e^{-\mu_j(\theta)s}ds - e^{-2\mu_j(\theta)t}\int_0^t s^{2H-2}e^{\mu_j(\theta)s}ds\right)\\ &= \frac{H(2H-1)}{|\mu_j(\theta)|^{2H}}\left(\int_0^{\mu_j(\theta)t} s^{2H-2}e^{-s}ds - e^{-\mu_j(\theta)t}\int_0^t s^{2H-2}e^{-\mu_j(\theta)(t-s)}ds\right).\end{aligned}$$

As $\lim_{j\to\infty}\mu_j(\theta) = +\infty$, we find, for every $t > 0$,

$$\lim_{j\to\infty}\int_0^{\mu_j(\theta)t} s^{2H-2}e^{-s}ds = \int_0^\infty s^{2H-2}e^{-s}ds = \Gamma(2H-1),$$

$$\lim_{j\to\infty}\left|e^{-\mu_j(\theta)t}\int_0^t s^{2H-2}e^{-\mu_j(\theta)(t-s)}ds\right| \leq \lim_{j\to\infty}\frac{e^{-\mu_j(\theta)t}t^{2H-1}}{2H-1} = 0.$$

Hence,

$$\lim_{j\to\infty} |\mu_j(\theta)|^{2H} \mathbb{E}u_j^2(t) = H(2H-1)\Gamma(2H-1), \tag{2.12}$$

and the limit indeed does not depend on t as long as $t > 0$. It follows from (2.12) that $\sup_j \mathbb{E}|u_j(t)|^2 < \infty$ and therefore

$$\mathbb{E}\|u(t)\|_{\mathbf{X}}^2 = \sum_{j=1}^{\infty} \mathbb{E}|u_j(t)|^2 \, \|h_j\|_{\mathbf{X}}^2 < \infty. \tag{2.13}$$

If $H = 1/2$, then $W_{h_j}^H$ is a standard Brownian motion, so that

$$\mathbb{E}u_j^2 = \int_0^t e^{-2\mu_j(\theta)(t-s)} ds \quad \text{and} \quad \lim_{j\to\infty} |\mu_j(\theta)| \, \mathbb{E}u_j^2 = \frac{1}{2};$$

note that passing to the limit $H \to 1/2$ in (2.12) gives the same result. □

Definition 2.4. The process u constructed in Theorem 2.1 is called the solution of Eq. (2.2).

3. The Maximum Likelihood Estimator and its Properties

Consider the diagonalizable equation

$$du(t) + (\mathcal{A}_0 + \theta\mathcal{A}_1)u(t)dt = dW^H(t) \tag{3.1}$$

with solution $u(t) = \sum_{j\geq 1} u_j(t)h_j$ given by (2.11); for simplicity, we assume that $u(0) = 0$. Suppose that the processes $u_1(t), \ldots, u_N(t)$ can be observed for all $t \in [0, T]$; as was mentioned in the Introduction, if u is observable, then each u_k can be computed. The problem is to estimate the parameter θ using these observations.

Recall the notation $\mu_j(\theta) = \rho_j + \nu_j\theta$, where ρ_j and ν_j are the eigenvalues of $\mathcal{A}_0$ and $\mathcal{A}_1$, respectively. Then each u_j is a fractional Ornstein–Uhlenbeck process satisfying

$$du_j(t) = -\mu_j(\theta)u_j(t)dt + dw_j^H(t), \quad u_j(0) = 0, \tag{3.2}$$

and, because of the independence of w_j^H for different j, the processes $u_1, \ldots, u_N$ are (statistically) independent.

Let Γ denote the Gamma-function (see (2.12)). Following Kleptsyna and Le Breton [7], we introduce the notations

$$\kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \quad k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}; \tag{3.3}$$

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}, \quad \mathbf{w}_H(t) = \lambda_H^{-1} t^{2-2H}; \tag{3.4}$$

$$M_j^H(t) = \int_0^t k_H(t,s)dw_j^H(s), \quad Q_j(t) = \frac{d}{d\mathbf{w}_H(t)} \int_0^t k_H(t,s)u_j(s)ds; \tag{3.5}$$

$$Z_j(t) = \int_0^t k_H(t,s)du_j(s). \tag{3.6}$$

In particular, M^H is a martingale with the quadratic characteristic $\mathbf{w}_H$. By a Girsanov-type formula (see, for example, Kleptsyna *et al.* [8, Theorem 3]), there is a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ under which the distribution of the vector $(u_1, \ldots, u_N)$ is the same as the distribution of the vector $(w_1^H, \ldots, w_N^H)$ under the original measure $\mathbb{P}$, and

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\sum_{j=1}^N \mu_j(\theta) \int_0^T Q_j(s)dZ_j(s) + \sum_{j=1}^N \frac{|\mu_j(\theta)|^2}{2} \int_0^T Q_j^2(s)d\mathbf{w}_H(s)\right). \tag{3.7}$$

Note that process Z_j is a semi-martingale [7, Lemma 2.1], and so there is no stochastic integration with respect to fractional Brownian motion in (3.7): $\int_0^T \nu_j Q_j(s)dZ_j(s)$ is an Itô integral. Maximizing the density $d\widetilde{\mathbb{P}}/d\mathbb{P}$ with respect to θ gives the Maximum Likelihood Estimator (MLE) of θ:

$$\widehat{\theta}_N = -\frac{\sum_{j=1}^N \int_0^T \nu_j Q_j(s)\big(dZ_j(s) + \rho_j Q_j(s)d\mathbf{w}_H(s)\big)}{\sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s)d\mathbf{w}_H(s)}. \tag{3.8}$$

Under assumption that $u_1, \ldots, u_N$ are observable, the processes Q_j, Z_j, $j = 1, \ldots, N$ are also observable, and hence the estimator $\widehat{\theta}_N$ defined above is observable. On the other hand, computing (3.8) requires the knowledge of k_H and $\mathbf{w}_H$, and is therefore impossible without the knowledge of the Hurst parameter H.

Notice that, when $H = 1/2$, we have $k_H = 1$, $\mathbf{w}_H(s) = s$, $Q_j(s) = Z_j(s) = u_j(s)$, and (3.8) becomes

$$\widehat{\theta}_N = -\frac{\sum_{j=1}^N \int_0^T \nu_j u_j(s)\big(du_j(s) + \rho_j u_j(s)ds\big)}{\sum_{j=1}^N \int_0^T \nu_j^2 u_j^2(s)du_j(s)}, \tag{3.9}$$

which is the MLE from [3].

The following is the main result of the paper.

Theorem 3.1. *Assume that Eq.* (3.1) *is diagonalizable and parabolic, and* $\theta \in \Theta$. *Then the following conditions are equivalent*:

$$(1) \quad \sum_{j=J}^\infty \frac{\nu_j^2}{\mu_j(\theta)} = +\infty; \tag{3.10}$$

$$(2) \quad \lim_{N\to\infty} \widehat{\theta}_N = \theta \text{ with probability one}, \tag{3.11}$$

where $J = \min\{j : \mu_i(\theta) > 0 \text{ for all } i \geq j\}$.

Proof. Following Kleptsyna and Le Breton [7, Eq. (4.1)], we conclude that

$$\widehat{\theta}_N - \theta = -\frac{\sum_{j=1}^N \int_0^T \nu_j Q_j(s) dM_j^H(s)}{\sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)}. \tag{3.12}$$

Both the top and the bottom on the right-hand side of (3.12) are sums of independent random variables; moreover, it is known from [7, p. 242] that

$$\mathbb{E}\left(\int_0^T Q_j(s) dM_j^H(s)\right)^2 = \mathbb{E}\int_0^T Q_j^2(s) d\mathbf{w}_H(s) ds. \tag{3.13}$$

From the expression for the Laplace transform of $\int_0^T Q_j^2(s) d\mathbf{w}_H(s) ds$ (see [7, Eq. (4.2)]) direct computations show that

$$\lim_{j\to\infty} \mu_j(\theta)\mathbb{E}\int_0^T Q_j^2(s) d\mathbf{w}_H(s) ds = \frac{T}{2} > 0 \tag{3.14}$$

and, with $\mathrm{Var}(\xi)$ denoting the variance of the random variable ξ,

$$\lim_{j\to\infty} \mu_j^3(\theta)\mathrm{Var}\left(\int_0^T Q_j^2(s) d\mathbf{w}_H(s) ds\right) = \frac{T}{2} > 0; \tag{3.15}$$

a detailed derivation of (3.14) and (3.15) is given in the Appendix, Lemmas A.1 and A.2 respectively.

We now see that if (3.10) does not hold, then, by (3.14), the series

$$\sum_{j\geq 1} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)$$

converges with probability one, which, by (3.12), means that (3.11) cannot hold.

On the other hand, if (3.10) holds, then

$$\sum_{n\geq J} \frac{\nu_n^2 \mu_n^{-1}}{\left(\sum_{j=1}^n \nu_j^2 \mu_j^{-1}\right)^2} < \infty. \tag{3.16}$$

Indeed, setting $a_n = \nu_n^2 \mu_n^{-1}$ and $A_n = \sum_{j=1}^n a_j$, we notice that

$$\sum_{n\geq J} \frac{a_n}{A_n^2} \leq \sum_{n\geq J+1} \left(\frac{1}{A_{n-1}} - \frac{1}{A_n}\right) = \frac{1}{A_J}.$$

Then the strong law of large numbers, together with the observation

$$\mathbb{E}\int_0^T Q_j(s) dM_j^H(s) = 0, \quad j \geq 1,$$

implies

$$\lim_{N\to\infty} \frac{\sum_{j=1}^N \int_0^T \nu_j Q_j(s) dM_j(s)}{\sum_{j=1}^N \mathbb{E}\int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)} = 0 \quad \text{with probability one.}$$

Next, it follows from (3.16) and the parabolicity assumption that

$$\sum_{n\geq J} \frac{\nu_n^4 \mu_n^{-3}}{\left(\sum_{j=J}^{n} \nu_j^2 \mu_j^{-1}\right)^2} < \infty, \tag{3.17}$$

because $|\nu_j/\mu_j(\theta)|$ stays bounded for $j \geq J$. Then another application of the strong law of large numbers implies that

$$\lim_{N\to\infty} \frac{\sum_{j=1}^{N} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)}{\sum_{j=1}^{N} \mathbb{E} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)} = 1 \tag{3.18}$$

with probability one, and (3.11) follows. □

Corollary 3.1. *If Eq.* (3.1) *is diagonalizable and parabolic and if* (3.10) *holds, then*

$$\lim_{N\to\infty} \sqrt{\sum_{j=J}^{N} \frac{\nu_j^2}{\mu_j(\theta)}}\ (\widehat{\theta}_N - \theta) = \zeta \tag{3.19}$$

in distribution, where ζ *is a Gaussian random variable with zero mean and variance* $2/T$.

Proof. Consider the sequence of random processes

$$X_N(t) = \frac{\sum_{j=1}^{N} \int_0^t \nu_j Q_j(s) dM_j(s)}{\left(\sum_{j=1}^{N} \mathbb{E} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)\right)^{1/2}},$$

and let $X(t) = w(t)/T$, where w is a standard Brownian motion. It is known from Norros *et al.* [12, Theorem 3.1] that each M_j^H is a Gaussian martingale with independent increments and quadratic variation $\langle M_j^H \rangle(t) = \mathbf{w}_H(t)$. Therefore, each X_N is a square-integrable martingale with quadratic variation

$$\langle X_N \rangle(t) = \frac{\sum_{j=1}^{N} \int_0^t \nu_j^2 Q_j^2(s) d\mathbf{w}_H(t)}{\sum_{j=1}^{N} \mathbb{E} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)}.$$

By (3.18),

$$\lim_{N\to\infty} \langle X_N \rangle(T) = 1 = \langle X \rangle(T).$$

By a limit theorem for martingales (see, for example, Jacod and Shiryaev [6, Theorem VIII.4.17]), $\lim_{N\to\infty} X_N(T) = X(T)$ in distribution. It remains to notice that, by (3.14),

$$\lim_{N\to\infty} \frac{\sum_{j=1}^{N} \nu_j^2/\mu_j(\theta)}{\sum_{j=1}^{N} \mathbb{E} \int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)} = \frac{2}{T}.$$ □

By Corollary 3.1, note that the rate of convergence of the estimators $\widehat{\theta}_N$ does not depend on the Hurst parameter H.

4. Examples

First, we discuss how the presence of the spatial covariance in the noise term affects the model.

Let us consider a more general equation

$$du(t) = (\mathcal{A}_0 + \theta\mathcal{A}_1)u(t)dt + \mathcal{B}dW^H(t),$$

where $\mathcal{B}$ is a linear operator, with $\mathcal{B}\mathcal{B}^\top$ representing the spatial covariance of the noise process. If $\mathcal{B}^{-1}$ exists, the equation is reduced to (3.1) by considering $v = \mathcal{B}^{-1}u$, $\tilde{\mathcal{A}}_0 = \mathcal{B}^{-1}\mathcal{A}_0\mathcal{B}$, $\tilde{\mathcal{A}}_1 = \mathcal{B}^{-1}\mathcal{A}_1\mathcal{B}$:

$$dv(t) = (\tilde{\mathcal{A}}_0 + \theta\tilde{\mathcal{A}}_1)v(t)dt + dW^H(t);$$

to proceed, we certainly need to assume that this equation is diagonalizable and parabolic.

If $\mathcal{B}^{-1}$ does not exist, we have two possibilities:

(1) $(u_0, h_i)_0 = 0$ for every i such that $\mathcal{B}h_i = 0$. In this case, $u_i(t) = 0$ for all $t > 0$, so that we can factor out the kernel of $\mathcal{B}$ and reduce the problem to invertible $\mathcal{B}$.

(2) $(u_0, h_i)_0 \neq 0$ for some i such that $\mathcal{B}h_i = 0$. In this case, $u_i(t) = u_i(0)e^{-\rho_i t - \nu_i \theta t}$ and θ is determined exactly from the observations of $u_i(t)$:

$$\theta = \frac{1}{\nu_i(t-s)} \ln \frac{u_i(s)}{u_i(t)} - \frac{\rho_i}{\nu_i}, \quad t \neq s.$$

Next, let us formulate condition (3.10) in terms of the orders of the operators in the equation. Let $\mathcal{A}_0$, $\mathcal{A}_1$ be differential or pseudo-differential operators, either on a smooth bounded domain in $\mathbb{R}^d$ or on a smooth compact d-dimensional manifold, and let m_0, m_1, be the orders of $\mathcal{A}_0$, $\mathcal{A}_1$ respectively, so that $2m = \max(m_0, m_1)$. Then, under rather general conditions we have

$$\lim_{j\to\infty} |\nu_j| j^{m_1/d} = c_1, \quad \lim_{j\to\infty} \mu_j(\theta) j^{2m/d} = c(\theta) \tag{4.1}$$

for some positive numbers $c_1, c(\theta)$; see, for example, Il'in [5] or Safarov and Vassiliev [15].

If (4.1) holds, then condition (3.10) becomes

$$m_1 \geq m - (d/2), \tag{4.2}$$

which, in the case $H = 1/2$, was established by Huebner and Rozovskii [3]. On the other hand, Theorem 3.1 covers operators with more exotic eigenvalues, such as $\nu_j = j \ln j$ or $\nu_j = e^j$.

Note that, at least as long as $H \geq 1/2$, conditions (3.10) and (4.2) do not involve H.

Let us now look at some concrete examples.

(1) Consider the equation

$$du - \theta u_{xx} dt = dW^H, \quad 0 < t < T, \; x \in (0,1),$$

with zero initial and boundary conditions. This equation was also studied by Prakasa Rao [14].

Clearly, $\nu_j = \pi^2 j^2$, $\rho_j = 0$, $\mu_j(\theta) = \theta\pi^2 j^2$. We take $\Theta = (\theta_0, \theta_1)$ for some $0 < \theta_0 < \theta_1$. Then

$$u_j(t) = \int_0^T e^{-\theta\pi^2 j^2 (t-s)} dw_j^H,$$

$$\widehat{\theta}_N = -\frac{\sum_{j=1}^N j^2 \int_0^T Q_j(t) dZ_j(t)}{\sum_{j=1}^N \int_0^T \pi^2 j^4 Q_j^2(t) d\mathbf{w}_H(t)},$$

$$\lim_{N\to\infty} N^{3/2}(\widehat{\theta}_N - \theta) = \mathcal{N}(0, 6\theta/(\pi^2 T)),$$

where $\mathcal{N}(a,\sigma^2)$ is a normal random variable with mean a and variance σ^2, and the convergence is in distribution.

(2) Consider the equation

$$du - (\mathbf{\Delta} u + \theta u)dt = dW^H, \quad 0 < t < T, \; x \in G \subset \mathbb{R}^d,$$

with zero initial and boundary conditions and $d \geq 2$. Denote by $\lambda_j, \; j \geq 1$ the eigenvalues of the Laplace operator $\mathbf{\Delta}$; recall that $\lambda_j < 0$. Clearly, $\nu_j = -1$, $\rho_j = -\lambda_j$, $\mu_j(\theta) = -\lambda_j - \theta$. We take $\Theta = (\theta_0, \theta_1)$ for some $\theta_0 < \theta_1$; there are no restrictions on the sign of θ. Then

$$u_j(t) = \int_0^T e^{(\lambda_j + \theta)(t-s)} dw_j^H,$$

$$\widehat{\theta}_N = -\frac{\sum_{j=1}^N \int_0^T Q_j(t)(dZ_j(t) - \lambda_j Q_j(t) d\mathbf{w}_H(t))}{\sum_{j=1}^N \int_0^T Q_j^2(t) d\mathbf{w}_H(t)},$$

$$\lim_{N\to\infty} \Psi_d(N)(\widehat{\theta}_N - \theta) = \mathcal{N}(0, \sigma_d^2),$$

where

$$\Psi_d(N) = \begin{cases} \sqrt{\ln N}, & \text{if } d = 2, \\ N^{(d-2)/(2d)}, & \text{if } d > 2, \end{cases} \quad \sigma_d^2 = \begin{cases} \dfrac{2c}{T}, & \text{if } d = 2, \\ \dfrac{2c}{T}\left(1 - \dfrac{2}{d}\right), & \text{if } d > 2, \end{cases}$$

and c is from (2.6).

If $d = 1$, then (4.2) does not hold and $\widehat{\theta}_N$ is not a consistent estimator of θ.

(3) Consider the equation

$$du + (\mathbf{\Delta}^2 u + \theta\mathbf{\Delta} u)dt = dW^H, \quad 0 < t < T, \; x \in G \subset \mathbb{R}^d,$$

with zero initial and boundary conditions:

$$u|_{t=0} = u|_{\partial G} = \mathbf{\Delta} u|_{\partial G} = 0.$$

As before, denote by λ_j the eigenvalues of the Laplacian $\mathbf{\Delta}$. Clearly, $\nu_j = -\lambda_j$, $\rho_j = \lambda_j^2$, $\mu_j(\theta) = \lambda_j^2 + \theta\lambda_j$. We take $\Theta = (\theta_0, \theta_1)$ for some $\theta_0 < \theta_1$; there are no restrictions on the sign of θ. Then, for every $d \geq 1$,

$$u_j(t) = \int_0^T e^{-(\lambda_j^2+\theta\lambda_j)(t-s)} dw_j^H,$$

$$\widehat{\theta}_N = -\frac{\sum_{j=1}^N \int_0^T \lambda_j Q_j(t)(dZ_j(t) - \lambda_j^2 Q_j(t) d\mathbf{w}_H(t))}{\sum_{j=1}^N \int_0^T \lambda_j^2 Q_j^2(t) d\mathbf{w}_H(t)},$$

$$\lim_{N\to\infty} \sqrt{N}(\widehat{\theta}_N - \theta) = \mathcal{N}(0, 2/T).$$

5. Other Possible Estimators

The maximum likelihood estimator (3.8) has three features that are clearly attractive: consistency, asymptotic normality, and absence of stochastic integration with respect to fractional Brownian motion. On the other hand, actual implementation of (3.8) is problematic: when $H > 1/2$, computing the processes Q_j and Z_j is certainly nontrivial. Estimator (3.9) is defined for all $H \geq 1/2$ and contains only the processes u_j, but, when $H > 1/2$, is not an MLE and is even harder to implement because of the stochastic integral with respect to u_j.

With or without condition (3.10), a consistent estimator of θ is possible in the large time asymptotic: for every $j \geq 1$,

$$\lim_{T\to\infty} \frac{\int_0^T \nu_j Q_j(s)\big(dZ_j(s) + \rho_j Q_j(s) d\mathbf{w}_H(s)\big)}{\int_0^T \nu_j^2 Q_j^2(s) d\mathbf{w}_H(s)} = -\theta \tag{5.1}$$

with probability one [7, Proposition 2.2]. For $H > 1/2$, implementation of this estimator is essentially equivalent to the implementation of (3.8).

An alternative to (5.1) was suggested by Maslowski and Pospíšil [11] using the ergodic properties of the OU process. Let us first illustrate the idea on a simple example.

If $a > 0$ and $w = w(t)$ is a standard one-dimensional Brownian motion, then the OU process $dX = -aX(t)dt + dw(t)$ is ergodic and its unique invariant distribution is normal with zero mean and variance $(2a)^{-1}$. In particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X^2(t)dt = \frac{1}{2a} \tag{5.2}$$

with probability one, and so

$$\tilde{a}(T) = \frac{T}{2\int_0^T X^2(t)dt} \tag{5.3}$$

is a consistent estimator of a in the long-time asymptotic. Note that the maximum likelihood estimator in this case is

$$\hat{a}(T) = -\frac{\int_0^T X(t)dX(t)}{\int_0^T X^2(s)ds} \tag{5.4}$$

and is strongly consistent for every $a \in \mathbb{R}$ [10, Theorem 17.4].

Similarly, if $a > 0$, then the fractional OU process

$$dX(t) = -aX(t)dt + dw^H(t), \quad X(0) = 0 \tag{5.5}$$

is Gaussian, and, by (2.12), converges in distribution, as $t \to \infty$, to the Gaussian random variable with zero mean and variance $c(H)a^{-2H}$, where

$$c(H) = H(2H-1)\Gamma(2H-1); \tag{5.6}$$

notice that, in the limit $H \searrow 1/2$, we recover the result for the usual OU process. Further investigation shows that, similar to (5.2),

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^2(s)ds = \frac{c(H)}{a^{2H}}$$

(see [11]). As a result, for every j such that $\theta\nu_j + \rho_j > 0$, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u_j^2(t)dt = \frac{c(H)}{(\theta\nu_j + \rho_j)^{2H}} \tag{5.7}$$

with probability one. Under an additional assumption that $\nu_j \neq 0$, we get an estimator of θ

$$\tilde{\theta}^{(j)}(T) = \frac{1}{\nu_j}\left(\frac{c(H)T}{\int_0^T u_j^2(t)dt}\right)^{\frac{1}{2H}} - \frac{\rho_j}{\nu_j}. \tag{5.8}$$

This estimator is strongly consistent in the long time asymptotic: $\lim_{T\to\infty} |\tilde{\theta}^{(j)}(T) - \theta| = 0$ with probability one [11, Theorem 5.2]. While not a maximum likelihood estimator, (5.8) is easier to implement computationally than (3.8). If, in Theorem 2.1, we have $\mathcal{A}_0 = 0$ and $\nu_j > 0$, then a version of (5.9) exists using all the Fourier coefficients u_j, $j \geq 1$:

$$\tilde{\theta}(T) = \left(\frac{c(H)T\sum_{j=1}^{\infty}\nu_j^{-2H}}{\sum_{j=1}^{\infty}\int_0^T u_j^2(t)dt}\right)^{\frac{1}{2H}}; \tag{5.9}$$

see [11, Theorem 5.2].

An interesting open question related to both (3.8) and (5.8), (5.9) is how to combine estimation of θ with estimation of H.

Appendix A

Below, we prove equalities (3.14) and (3.15).

Lemma A.1. *For every $\theta \in \Theta$ and $H \in [1/2, 1)$,*

$$\lim_{j\to\infty} \mu_j(\theta)\mathbb{E}\int_0^T Q_j^2(s)d\mathbf{w}_H(s) = \frac{T}{2}.$$

Proof. Denote by $\Psi_T^H(a,\mu_j)$ the Laplace transform of $\int_0^T Q_j^2(s)d\mathbf{w}_H(s)$, namely

$$\Psi_T^H(a,\mu_j(\theta)) = \mathbb{E}\exp\left\{-a\int_0^T Q_j^2(s)d\mathbf{w}_H(s)\right\}, \quad a>0. \tag{A.1}$$

We will use the expression for Ψ_T^H from [7, p. 242], and write it as follows

$$\Psi_T^H(a,\mu_j) = \alpha e^{\frac{(\mu_j-\alpha)T}{2}}\left[\Delta_T^H(\mu_j,\alpha)\right]^{-\frac{1}{2}},$$

where $\mu_j = \mu_j(\theta)$, $\alpha := \sqrt{\mu_j^2 + 2a}$,

$$\Delta_T^H(\mu_j,\alpha) = \frac{\pi\alpha T e^{-\alpha T}(\alpha^2-\mu_j^2)}{4\sin(\pi H)} I_{-H}\left(\frac{\alpha T}{2}\right) I_{H-1}\left(\frac{\alpha T}{2}\right) + e^{-\alpha T}\left[\alpha\sinh\left(\frac{\alpha T}{2}\right) + \mu_j\cosh\left(\frac{\alpha T}{2}\right)\right]^2$$

and I_p is the modified Bessel function of the first kind and order p.

Note that

$$\mathbb{E}\int_0^T Q_j^2(s)d\mathbf{w}_H(s) = -\left.\frac{\partial\Psi_T^H(a,\mu_j)}{\partial a}\right|_{a=0}.$$

Direct evaluations (for example, using `Mathematica` computer algebra system) give

$$\left.\frac{\partial\Psi_T^H(a,\mu_j)}{\partial a}\right|_{a=0} = \frac{2+2e^{\mu_j T}(1-\mu_j T) - \mu_j\pi T I_{H-1}\left(\frac{\mu_j T}{2}\right) I_{-H}\left(\frac{\mu_j T}{2}\right)\csc(H\pi)}{4\mu_j^2 e^{\mu_j T}},$$

where $\csc(x) = 1/\sin(x)$. By combining formulas (6.106), (6.155) and (6.162) in [1], we conclude that, for all $p \in (-1,1)$, $p \neq 0$, we have $I_p(x) \sim e^x/\sqrt{2\pi x}$, $x \to \infty$, i.e.

$$\lim_{x\to+\infty} \sqrt{2\pi x}\; e^{-x} I_p(x) = 1. \tag{A.2}$$

Therefore

$$\left.\frac{\partial\Psi_T^H(a,\mu_j)}{\partial a}\right|_{a=0} \sim \frac{2+2e^{\mu_j T}(1-\mu_j T) - e^{\mu_j T}\csc(H\pi)}{4\mu_j^2 e^{\mu_j T}} \sim -\frac{T}{2\mu_j}, \quad j\to\infty,$$

$$\lim_{j\to\infty}\mu_j \left.\frac{\partial\Psi_T^H(a,\mu_j)}{\partial a}\right|_{a=0} = -\frac{T}{2},$$

and the lemma is proved. □

Lemma A.2. *For every $\theta \in \Theta$ and $H \in [1/2, 1)$*

$$\lim_{j\to\infty} \mu_j^3(\theta)\mathrm{Var}\left(\int_0^T Q_j^2(s)d\mathbf{w}_H(s)\right) = \frac{T}{2}.$$

Proof. Note that

$$\mathbf{V} := \mathrm{Var}\left(\int_0^T Q_j^2(s)d\mathbf{w}_H(s)\right) = \left[\frac{\partial^2 \Psi_T^H(a,\mu_j)}{\partial a^2} - \left(\frac{\partial \Psi_T^H(a,\mu_j)}{\partial a}\right)^2\right]_{a=0}, \quad \text{(A.3)}$$

with Ψ_T^H from (A.1). Direct evaluation of the right-hand side of (A.3) (for example, using `Mathematica` computer algebra system) gives

$$\mathbf{V} = \frac{1}{8\mu_j^4 e^{2T\mu_j}}\Bigg(2 - 8e^{\mu_j T}(1+\mu_j T) + 2e^{2\mu_j T}(-5+2\mu_j T)$$
$$+ \pi\mu_j T\csc(\pi H)\Bigg[-2e^{\mu_j T}\mu_j T I_{1-H}\left(\frac{\mu_j T}{2}\right) I_{H-1}\left(\frac{\mu_j T}{2}\right)$$
$$+ I_{-H}\left(\frac{\mu_j T}{2}\right)\Bigg\{4(-1+e^{\mu_j T}(1+\mu_j T))I_{H-1}\left(\frac{\mu_j T}{2}\right)$$
$$- 2e^{\mu_j T}\mu_j T I_H\left(\frac{\mu_j T}{2}\right) + \pi\mu_j T I_{H-1}^2\left(\frac{\mu_j T}{2}\right) I_{-H}\left(\frac{\mu_j T}{2}\right)\csc(H\pi)\Bigg\}\Bigg]\Bigg),$$

where $\csc(x) = 1/\sin(x)$ and I_p is the modified Bessel function of the first kind and order p.

Using (A.2), we conclude that

$$\lim_{j\to\infty} \mu_j^3(\theta)\mathbf{V} = \lim_{j\to\infty} \mu_j^3\Bigg(\frac{-10+4\csc(H\pi)+\csc^2(H\pi)}{8\mu_j^4} + \frac{1}{4\mu_j^4 e^{2\mu_j T}}$$
$$- \frac{\csc(H\pi)+2+2\mu_j T}{2\mu_j^4 e^{\mu_j T}} + \frac{T}{2\mu_j^3}\Bigg) = \frac{T}{2}$$

and complete the proof of the lemma. ∎

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