Fall 2023, MATH 407, Mid-Term Exam 1

Wednesday, October 4, 2023

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Instructions:

- No books, notes, or help from other people. Think twice (or more) before using a calculator.
- Turn off cell phones.
- Show your work/explain your answers.
- You have 50 minutes to complete the exam.
- There are five problems; 10 points per problem.
- Upload the solutions to GradeScope.

Problem 1. A box contains 5 white balls, 10 blue balls, and 15 green balls [30 total, well mixed]. Eleven (11) balls are taken out of the box, all at once.

Compute the probability that three (3) of those balls are white, two (2) are blue, and six (6) are green. `For your final answer, do not expand/evaluate any of the binomial coefficients.`

`Solution/Answer:` $\frac{\binom{5}{3}\binom{10}{2}\binom{15}{6}}{\binom{30}{11}}$

Problem 2. A fair coin is tossed five times. Compute the probability that there were more heads than tails.

`Solution/Answer:` $P(\mathcal{B}(5,1/2) = 3,4,5) = 2^{-5}\left(\binom{5}{3}+\binom{5}{4}+\binom{5}{5}\right) = (10+5+1)/32 = 1/2$. `Can also argue by symmetry: cannot have equal number of heads and tails, so one is always bigger, and equally likely because the coin is fair.`

Problem 3. In a certain town, there are twice as many cars as trucks, and 15% of trucks and 5% of cars have manual transmission. A vehicle is selected at random, and it has manual transmission. Compute the probability that the vehicle is a truck. `Simplify your answer as much as possible.`

`Solution/Answer: 0.15*(1/3) / (0.15*1/3 + 0.05*(2/3)) = 15/25=0.6.`

Problem 4. A charitable lottery has 10,000 tickets, of which 500 win prizes and the rest win nothing. You buy 60 tickets. Compute, approximately, the number of the prize-winning tickets you expect to find.

`Solution/Answer:` $(500/10000)\cdot 60 = 3$

Problem 5. Let X be a standard normal random variable. In particular, the pdf of X is $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$, $x \in (-\infty, +\infty)$. Define the random variable Y by $Y = \sqrt{|X|}$. In particular, $Y \geq 0$. Compute the probability density function of the random variable Y.

`Solution:` $P(Y \leq x) = P(\sqrt{|X|} \leq x) = P(|X| \leq x^2) = 2P(0 < X < x^2) = 2F_X(x^2) - 1,\ x > 0$. `Then differentiate with respect to` x. `Answer:` $4x\varphi(x^2)I(x>0) = 4x(2\pi)^{-1/2}e^{-x^4/2}I(x>0)$