Basics of Gamma and Beta functions.[1]

Gamma function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt,\ x>0; \tag{1}$$

$$x\Gamma(x) = \Gamma(x+1),\ \Gamma(n+1) = n! \tag{2}$$

$$\Gamma(1/2) = \sqrt{\pi}; \tag{3}$$

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x); \tag{4}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},\ x \neq 0, \pm 1, \pm 2, \ldots. \tag{5}$$

Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x>0, y>0; \tag{6}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7}$$

Basic Notes

- The first equality in (2) follows from (1) after integration by parts and can be used to define $\Gamma(x)$ for $x<0,\ x\neq -1,-2,-3,\ldots$; the second equality in (2) corresponds to $x=n$.
- Equality (3) is equivalent to
$$\int_{-\infty}^{+\infty} e^{-u^2}\,du = \sqrt{\pi}.$$
- Equality (5) is known as `Euler's reflection formula`. There is no “elementary” proof. When $x\in(0,1)$, one approach is to use (7), with $y=1-x$:
$$\Gamma(x)\Gamma(1-x) = \int_0^1 t^{x-1}(1-t)^{-x}\,dt = \int_0^{+\infty} \frac{s^{x-1}}{1+s}\,ds,$$
where the second equality follows after the substitution $t=s/(s+1)$; the second integral can then be evaluated using contour integration in the complex plane.
- Equality (4) is known as `Legendre's duplication formula` and follows from (7) after setting $x=y$, changing variables in the integral to get
$$\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = 2^{1-2x}B\left(\frac{1}{2},x\right),$$
and using (3).
- To prove equality (7), make substitution $t=u^2$ in (1) to get
$$\Gamma(x) = 2\int_0^{+\infty} u^{2x-1}e^{-u^2}\,du,$$
and make substitution $t=\sin^2\theta$ in (6) to get
$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\ \cos^{2y-1}\theta\,d\theta.$$

[1]Sergey Lototsky, USC. Most recent update on August 4, 2023.

Then

$$\Gamma(x)\Gamma(y) = 4\int_0^{+\infty}\int_0^{+\infty} u^{2x-1}v^{2y-1}e^{-(u^2+v^2)}\,dudv = B(x,y)\Gamma(x+y),$$

where the second equality follows after switching to polar coordinates $u = r\sin\theta,\ v = r\cos\theta$.

FURTHER NOTES

- With some imagination, one can replace real number x in (1) with a complex number $z = x + \mathrm{i}y$ and define Γ as an analytic function in the complex region $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Then (2) extends Γ to a *meromorphic* function in $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots, -n, \ldots$ with corresponding *residue* $(-1)^n/n!$.
- The function $z \mapsto 1/\Gamma(z)$ is *entire* (analytic everywhere in $\mathbb{C}$) and has product representation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}, \tag{8}$$

where

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \ln N\right) \approx 0.577\ldots$$

is the `Euler-Mascheroni constant`. Then

$$-\frac{1}{\Gamma(z)\Gamma(-z)} = z^2\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right),$$

which, together with the equalities $\Gamma(1-z) = -z\Gamma(-z)$ [using (2) with $x = -z$] and

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

leads to Euler's reflection formula (5). Note that an equivalent form of (5) is

$$\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{1}{2}+z\right) = \frac{\pi}{\cos(\pi z)}.$$

- Equality (4) is a particular case of `Gauss's multiplication formula`

$$\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2}\,n^{(1/2)-nz}\Gamma(nz),\ \ n = 2,3,4,\ldots,$$

which can be derived from (8).
- The n-the derivative of the function

$$\psi_0(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is called the `polygamma function` of order n. In particular, ψ_0 itself is known as the `digamma function`, and

$$\psi_0'(1) \equiv \Gamma'(1) \equiv \int_0^{\infty} e^{-t}\,\ln t\,dt = -\gamma.$$

- Various trigonometric and hyperbolic substitutions in the Gamma and Beta integrals lead to a number of remarkable identities, such as

$$\int_0^{+\infty}\frac{\cos(2zt)}{\cosh(\pi t)}\,dt = \frac{1}{2\cosh(z)} \equiv \frac{1}{e^z+e^{-z}},\ \ \mathrm{Im}(z) < \pi/2.$$