OSCILLATIONS OF A HANGING CHAIN

MATH 3410, SPRING SEMESTER 2018

http://www.phys.uconn.edu/~rozman/Courses/m3410_18s/

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Equation of motion: Consider a uniform flexible chain (or heavy rope) of length L, fixed at the upper end and free at the lower end (see Fig. 1). We let the x axis be vertical, measured up from the equilibrium position of the free end of the chain. $Y(x,t)$ is the horizontal displacement of the chain at the point with the vertical coordinate x at time t.

We assume that $Y(x,t)$ is small compared to L. Therefore we do not need to consider the difference between distances measured along the chain and distances measured along the x axis, i.e. we can neglect the terms $\sqrt{x^2+Y^2}-x \sim \frac{Y^2}{x}$. For the same reason we can neglect the vertical displacement due to oscillations. We also assume that the angle $\alpha(x)$ between the local direction of the chain and X axis is small, thus

$$\sin\alpha \approx \tan\alpha = \frac{\partial Y}{\partial x}. \tag{1}$$

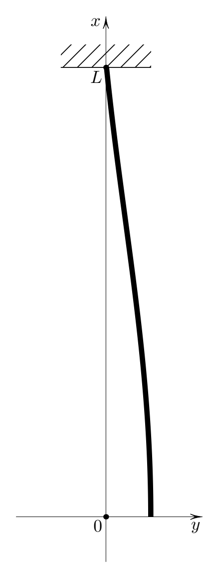

Figure 1: Hanging chain (sketched in bold) and the coordinate axis.

The horizontal component of the net force acting on a segment of the chain of length Δx due to the internal tension $T(x)$ is (see Fig. 2):

$$T(x+\Delta x)\frac{\partial Y(x+\Delta x)}{\partial x} - T(x)\frac{\partial Y(x)}{\partial x} \approx \frac{\partial}{\partial x}\left(T(x)\frac{\partial Y}{\partial x}\right)\Delta x \tag{2}$$

Newton's second law gives the following equation of motion:

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial Y}{\partial x}\right)\Delta x = \rho\,\Delta x\,\frac{\partial^2 Y}{\partial t^2}, \tag{3}$$

where ρ is the chain's linear density (mass per unit length), $\rho\,\Delta x$ is the mass of the segment, and $\frac{\partial^2 Y}{\partial t^2}$ is its acceleration. Canceling common factor Δx in both sides of Eq. (3), we obtain:

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial Y}{\partial x}\right) = \rho\,\frac{\partial^2 Y}{\partial t^2}. \tag{4}$$

For small oscillations of the chain, when we can neglect the vertical displacement due to oscillations, the tension $T(x)$ is the same as for the chain at rest, i.e. the tension at the point with the vertical coordinate x equals to the weight of the part of the chain below x. Therefore,

$$T = \rho g x, \tag{5}$$

where g is the acceleration of gravity.

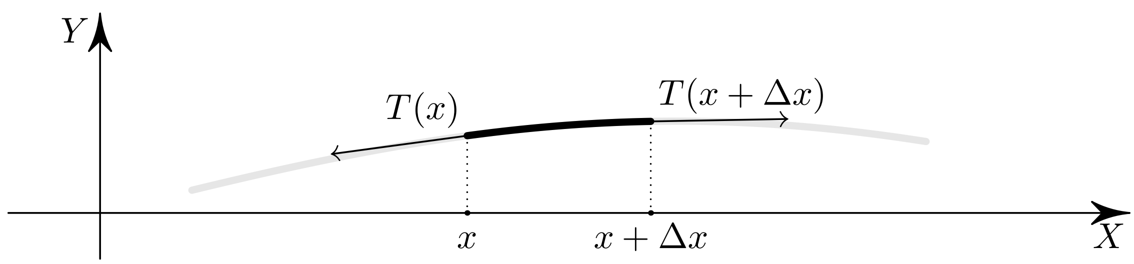

Figure 2: Forces acting on the element of the chain.

The chain is fixed at the top, therefore

$$Y(0,t) = 0. \tag{6}$$

The displacement of the bottom of the chain remain finite at all times, thus

$$|Y(L,t)| < \infty. \tag{7}$$

Separation of variables: We solve the partial differential equation Eq. (3) with the boundary conditions Eqs. (6),(7) separating variables, i.e. assuming that

$$Y(x,t) = y(x)\,u(t). \tag{8}$$

Substituting Eq. (8) into Eq. (4), we obtain:

$$u(t)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \rho\, y(x)\,\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}, \tag{9}$$

or,

$$\frac{1}{y(x)}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \rho\,\frac{1}{u(t)}\,\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}. \tag{10}$$

The right hand side of Eq. (10) is a function of time t. The left hand side is a function of x. The two sides can be equal at all times and coordinates only if they both are equal to the same constant.

As we see shortly, this constant must be real and negative. Indeed, the mechanical system described by Eq. (3) is conservative. Therefore, $u(t)$ cannot grow without a limit or decay to zero. The permissible solutions are only possible for real negative separation constants.

Denoting the constant by $-\omega^2$, we get the following equations:

$$\frac{1}{u(t)}\,\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -\omega^2, \tag{11}$$

$$\frac{1}{y(x)}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\rho\,\omega^2. \tag{12}$$

Equation Eq. (11) can be easily solved:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega^2 u(t) = 0, \tag{13}$$

and

$$u(t) = A\cos(\omega t) + B\sin(\omega t), \tag{14}$$

where A and B are real integration constants. We see that ω is the frequency of the chain's oscillations.

The equation for the amplitude of the oscillations, $y(x)$, is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \rho\,\omega^2\, y(x) = 0. \tag{15}$$

Using the expression for the chain tension, Eq. (5), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\omega^2}{g}\, y(x) = 0. \tag{16}$$

The boundary conditions for Eq. (16) are

$$y(L) = 0, \quad |y(0)| < \infty. \tag{17}$$

To find the solution of Eq. (16) let's change the independent variable from x to z as following:

$$z = 2\sqrt{\frac{\omega^2}{g}x}. \tag{18}$$

$$\frac{dz}{dx} = \sqrt{\frac{\omega^2}{gx}} \quad \longrightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \sqrt{\frac{\omega^2}{gx}}\frac{dy}{dz} \quad \longrightarrow \quad x\frac{dy}{dx} = \sqrt{\frac{\omega^2}{g}x}\frac{dy}{dz} = \frac{z}{2}\frac{dy}{dz}, \tag{19}$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{z}{2}\frac{dy}{dz}\right)\frac{dz}{dx} = \frac{1}{2}\sqrt{\frac{\omega^2}{gx}}\frac{d}{dz}\left(z\frac{dy}{dz}\right) \tag{20}$$

Equation (16) changes to

$$\frac{1}{2}\sqrt{\frac{\omega^2}{gx}}\frac{d}{dz}\left(z\frac{dy}{dz}\right) + \frac{\omega^2}{g}y = 0, \tag{21}$$

or

$$\frac{d}{dz}\left(z\frac{dy}{dz}\right) + zy = 0. \tag{22}$$

Expanding the derivative, we obtain

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + zy = 0. \tag{23}$$

Equation (23) is zero order Bessel equation. The solution that satisfies the boundary conditions Eq. (17) is

$$y(x) = J_0(z) = J_0\left(2\omega\sqrt{\frac{x}{g}}\right). \tag{24}$$

The condition that the top end of the chain is fixed, $y(L) = 0$, determines the characteristic frequencies of the chain:

$$J_0\left(2\omega\sqrt{\frac{L}{g}}\right) = 0. \tag{25}$$

$$\omega_n = \frac{1}{2}\sqrt{\frac{g}{L}}z_n, \tag{26}$$

where z_n are the zeros of Bessel function J_0. For the reference,

$$z_1 \approx 2.4, \quad z_2 \approx 5.5, \quad z_3 \approx 8.7, \ldots \tag{27}$$

The normal modes of the hanging chain are

$$y_n(x) = J_0\left(z_n \sqrt{\frac{x}{L}}\right). \tag{28}$$

The first three lowest-frequency modes are sketched in Fig. 3.

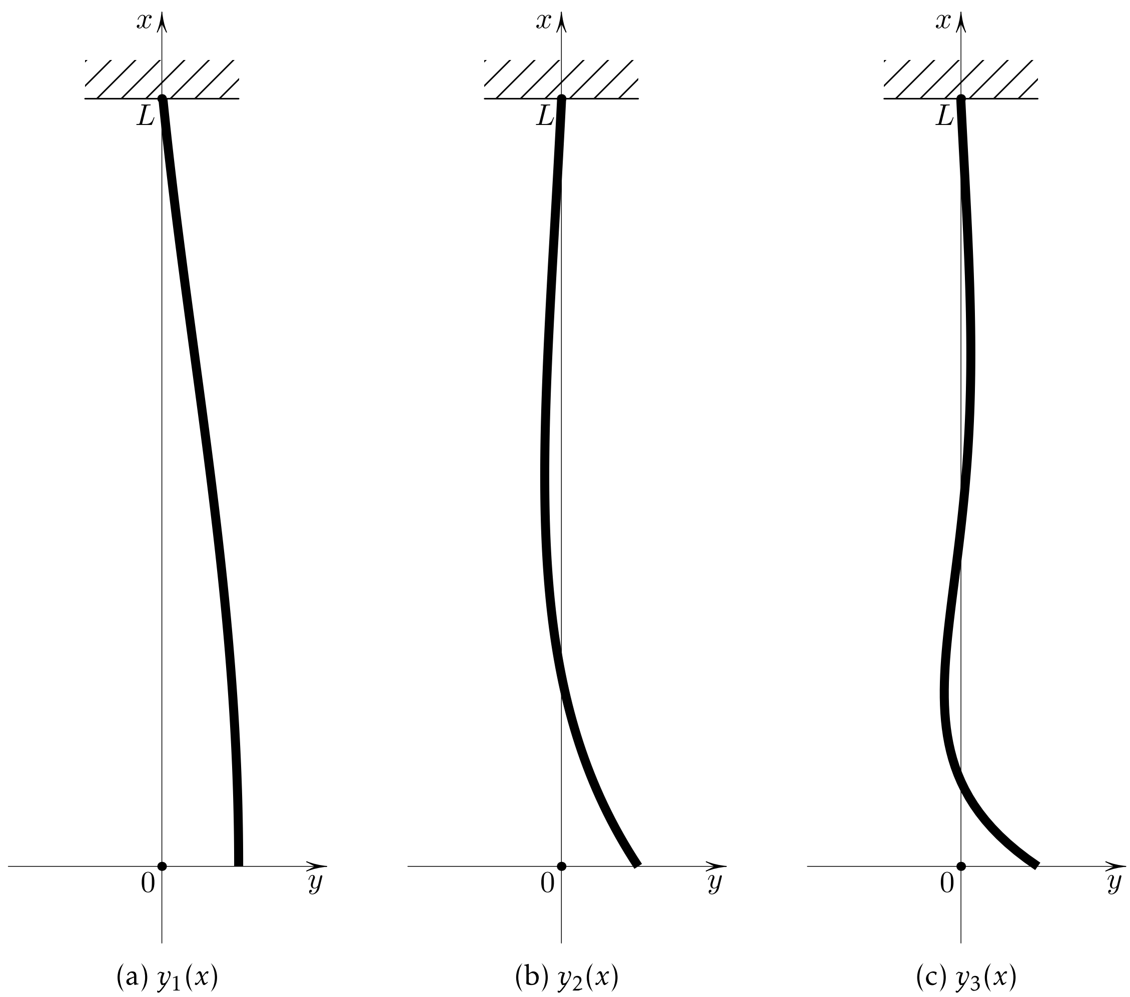

Figure 3: Normal modes of a hanging chain.